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Abstract According to economics, prices are determined by the relationship between supply and demand: they correspond to the equilibrium in which supply is exactly equal to the demand, i.e., for which the optimal amount that the seller is willing to sell at this price is exactly equal to the optimal amount that the buyer is willing to pay. In many situations, the corresponding prices are indeed uniquely determines by the supply-demand relation. From the purely mathematical viewpoint, there are situations when the equilibrium is not unique, but most economists believe that in practical situations, equilibrium prices are usually uniquely determined. In this paper, we provide a simple but realistic example in which the equilibrium is not unique.

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1 Formulation of the Problem

Why equilibrium is important in economics. At first glance, the notion of equilibrium should be a taboo in economic research – equilibrium means stability, means that the situation remains the same. It may be a good feature for a mechanical structure, but an economy is supposed to grow: if it stays the same, this is called stagnation, not a good phenomenon.

This negative meaning is correct if we are talking about well-established economy, where prices are set. But in the current economy – which is largely market economy – how do we come up with prices? Prices come as an equilibrium between supply and demand.

- If for a certain price, the supply exceeds demand, this means that the stores cannot sell all the items at the original price – so they decrease the price, they start a sale, and demand increases.
- Vice versa, if demand exceeds supply – e.g., if we are looking for fashionable toys for Christmas – people cannot buy their toy at the current price, so they offer larger prices on the web – and the price effectively increases.

It is important to estimate the equilibria. Of course, the above-described fluctuations are rather rare: in a big store, there are some products on sale, but for most products, the price is reasonably stable. The reason for this stability is that companies try to avoid such fluctuations – which are not good neither for the customers nor for the manufacturers and sellers – by carefully analyzing the supply and demand and, thus, by estimating the equilibrium prices.

Methods of estimating equilibria are described, e.g., in [1–3, 6–13, 16].

Uniqueness is important. Ideally, the corresponding system should have a unique equilibrium. Situations where there are several equilibria are less stable – the system can continuously oscillate between two equilibria, which is also not good from the economic viewpoint. This cyclic behaviors happened in the past, with:

- periods of boom – when both prices and salaries would grow,
- followed by crisis periods, when both prices and salaries would decrease.

From the purely mathematical viewpoint, it is possible to have situations with two or more equilibria – simply because there are mathematical functions with two or more maxima. However, in most realistic economic models, equilibrium is unique. So maybe there is a general theorem according to which in reasonable economic situations, equilibrium is unique? Unfortunately, no.

What we do in this paper. We started our research exactly by trying to prove that equilibrium is always unique – but that did not work. Instead, we found a simple example of a reasonable economic situation in which there are two different equilibria. This example is what we describe in this paper.

2 Description of the Example

Let us make this example as simple as possible. We started with a very complex example, that required complex theorems to prove that the two states are indeed equilibria. Good news is that, by analyzing this original complex example from the economic viewpoint, we succeeded in making it as simple as possible: with only two agents, two products, a simple situation, and easy-to-prove equilibria.

Let us describe this simple example.

Starting situation. Let us consider a situation in which we have two agents: we will call them Agent 1 and Agent 2.

We will consider the situation when we have only 2 products: two stocks. In the beginning:

- Agent 1 has a certain amount of Stock A; let us denote its current overall cost by a.
- Agent 2 has a similar amount of some other Stock B.

Let us also assume that:

- Agent 1 considers Stock A too risky, and Stock B much less risky, and that
- in contrast, Agent 2 consider Stock B too risky and stick A less risky.

For simplicity, let us assume that:

- according to Agent 1, the standard deviation of Stock A is $\sigma_A > 0$, while the standard deviation of B is negligible – so it can be taken as 0;
- similarly, according to Agent 2, the standard deviation of Stock B is $\sigma_B > 0$, while the standard deviation of A is negligible – so it can be taken as 0.

As a result of this different in opinions:

- Agent 1 wants to exchange his shares in Stock A in for Agent 2's shares in Stock B, and
- Agent 2 is also willing to make this exchange.

So, here, we have supply and we have demand. We need to find out how this will change the prices of the two stocks.

Comments.

- The fact that two agents have different predictions about the same stock makes perfect sense. If all agents had the same predictions about the stock behavior, there will no incentive for stock trading. So, the very fact that there is a large amount of stocks changing hands all the time is a good indication that different agents have different predictions of stock prices.
- Why exchange stocks and not just buy and/or sell? Well, we mortals keep some of our money in cash. However, from the economic viewpoint, this makes no sense: if we invest money into some financial instruments, they will likely grow, while if we keep them as cash, they don't. Because of this, both agents do not have spare money to simply buy additional stocks for their portfolios. What they can do, however, is exchange stocks.

How people make decisions: reminder. To find out how different price changes will affect the agent's behavior, let us recall how people make rational decisions in the first place. According to decision theory (see, e.g., [4, 5, 14, 15, 17–19]), a rational agent maximizes the expected value of his/her utility u. In general, utility u non-linearly depends on the overall amount of money m.

If the original amount of money was m_0, then the utility corresponding to additional amount x is equal to $u(m_0 + x)$. We are not considering situations when agents gamble their whole estates, we are considering a routine financial transaction. In such a transaction, the expected gain x is much smaller than the overall amount of money m_0 that the agent has. Since $x \ll m_0$, we can expand the dependence $u(m_0+x)$ in Taylor series and keep only the first few terms in this expansion.

The simplest such approximation is linear, but since we would like to take into account the non-linearity of the function $u(m)$, let us also keep the quadratic term. Then, we get

$$u(m_0+x) = u(m_0) + u'(m_0)\cdot x + \frac{u''(m_0)}{2}\cdot x^2, \quad (1)$$

where, as usual, $u'(m_0)$ denotes the derivative and $u''(m_0)$ denotes the second derivative.

The more money we have, the better, so the function $u(m)$ is increasing, and $u'(m_0) > 0$. On the other hand, the increase in our utility caused by having one additional dollar is:

– much higher when the original amount was low and
– much lower when this dollar is added to the originally large amount.

Thus, the value $u'(m)$ decreases with m and thus, the derivative of this value is negative: $u''(m_0) < 0$.

Because of the formula (1), the expected value $E[u(m_0+x)]$ of utility has the form

$$E[u(m_0+x)] =$$

$$u(m_0) + u'(m_0)\cdot E[x] + \frac{u''(m_0)}{2}\cdot E[x^2], \quad (2)$$

Let us denote the expected monetary gain by

$$\mu \stackrel{\text{def}}{=} E[x].$$

In these terms, the second moment $E[x^2]$ can be represented as $\mu^2+\sigma^2$, where σ is the standard deviation, so the formula (2) takes the form

$$E[u(m_0+x)] =$$

$$u(m_0) + u'(m_0)\cdot \mu + \frac{u''(m_0)}{2}\cdot (\mu^2+\sigma^2). \quad (3)$$

For stocks, the expected gain is usually small – at least when we consider short-term gains – while fluctuations are much higher. Thus, $\mu \ll \sigma$, so we can safely ignore the term μ^2 in the sum $\mu^2+\sigma^2$, and get a simplified formula

$$E[u(m_0+x)] = u(m_0) + u'(m_0)\cdot \mu + \frac{u''(m_0)}{2}\cdot \sigma^2. \quad (4)$$

If we add the same constant $u(m_0)$ to all the values of the objective function, this does not change which values of this function are larger and which are smaller. Thus, maximizing the function (4) and maximizing the same function with the constant $u(m_0)$ subtracted select exactly the same optimal decisions. So, for simplicity, we can subtract this constant and consider a simplified objective function

$$u'(m_0)\cdot \mu + \frac{u''(m_0)}{2}\cdot \sigma^2. \quad (5)$$

Similarly, if we multiply all the values of the objective function by the same positive number $u'(m_0)$, this does not change which values of this function are larger and which are smaller. Thus, maximizing the function (5) and maximizing the same function divided by $u'(m_0)$ select exactly the same optimal decisions. So, for simplicity, we can divide by this constant and consider the following simplified objective function:

$$U \stackrel{\text{def}}{=} \mu - \alpha\cdot\sigma^2, \quad (6a)$$

where we denoted

$$\alpha \stackrel{\text{def}}{=} -\frac{u''(m_0)}{2u'(m_0)}. \quad (6b)$$

We will call this function *re-scaled expected utility*, or simply *utility*, for short.

Let us apply this to our situation. According to our analysis, to describe an agent's decision making, we need to know the value α describing this agent. Let us denote the values of the quantity α corresponding to the two agents by α_1 and α_2.

We want to find the relative prices p_A and p_B – i.e., how much we will pay next year for what costs \$1 today – that lead to an equilibrium, i.e., for which both agents are interested in trading. At present, the stocks they exchange have the same price, but they do exchange because they have different expectations on how the prices will change next year. Let us estimate the corresponding utilities of both agents.

For Agent 1:

– the next year's value of his/her $c\cdot a$ amount of Stock B will be $p_B\cdot c\cdot a$,
– the value of his/her $(1-c)\cdot a$ amount of Stock A will be $p_A\cdot(1-c)\cdot a$, and
– the variance of the value – which, according his/her belief, comes exclusively from Stock A – is equal to

$$(1-c)^2\cdot\sigma_A^2.$$

Thus, the overall utility $U_1(c)$ of Agent 1 is equal to

$$U_1(c) = p_B\cdot c\cdot a + p_A\cdot(1-c)\cdot a - \alpha_A\cdot\sigma_A^2\cdot(1-c)^2. \quad (7)$$

Similarly, for Agent 2:

– the next year's value of his/her $c\cdot a$ amount of Stock A will be $p_A\cdot c\cdot a$,

– the value of his/her $(1-c)\cdot a$ amount of Stock B will be $p_B \cdot (1-c)\cdot a$, and
– the variance of the value – which, according his/her belief, comes exclusively from Stock B – is equal to

$$(1-c)^2 \cdot \sigma_B^2.$$

Thus, the overall utility $U_2(c)$ of Agent 2 is equal to

$$U_2(c) = p_A \cdot c \cdot a + p_B \cdot (1-c)\cdot a - \alpha_B \cdot \sigma_B^2 \cdot (1-c)^2. \quad (8)$$

In this situation, what does it mean to have an equilibrium. In our example:

– Agent 1 finds the value c from the interval $[0,1]$ by maximizing the expression (7), while
– Agent 2 finds the value c by maximizing the expression (8).

Equilibrium is when both these optimizations lead to the same optimal value c.

So, we need to find the (relative) prices p_A and p_B for which the two optimization problems:

– maximizing the utility (7) of Agent 1 and
– maximizing the utility (8) of Agent 2

lead to the same value c.

Let us show that these equilibrium prices are not uniquely determined – there are different pairs (p_A, p_B) for which we have an equilibrium.

First case of the equilibrium. The first idea is to keep the prices as is, i.e., to take $p_A = p_B = 1$. In this case, the expression (7) takes the form

$$U_1(c) = c \cdot a + (1-c)\cdot a - \alpha_A \cdot \sigma_A^2 \cdot (1-c)^2 =$$
$$a - \alpha_A \cdot \sigma_A^2 \cdot (1-c)^2, \quad (9)$$

and the expression (8) takes the form

$$U_2(c) = c \cdot a + (1-c)\cdot a - \alpha_B \cdot \sigma_B^2 \cdot (1-c)^2 =$$
$$a - \alpha_B \cdot \sigma_B^2 \cdot (1-c)^2. \quad (10)$$

In both cases, the utility is the largest when the negative term is the smallest, i.e., when $1-c=0$ and $c=1$.

In both optimization problems, we have the same optimal value, so this is indeed an equilibrium.

Second case of the equilibrium. But what is both prices increase, i.e., we have $p_A = p_B > 1$? In this case, similar arguments show that the exact same value $c = 1$ will be attained.

So, for the new pair of prices, we also have an equilibrium.

This is exactly what we were looking for. Thus, in this simple situation, we indeed have a non-unique equilibrium.

Comment. Of course, a similar "non-uniqueness" can be observed in all the situations in which the objective functions are linear. In such situations, if we simply multiply all the prices by a constant, all the equalities and inequalities will remain valid, and thus, what was an equilibrium remains an equilibrium.

However, in our case, the situation is different: the objective functions are non-linear. In this case, in general, simply multiplying all the prices by a constant will not lead to an equilibrium – and even if it does lead to an equilibrium, the resulting optimal arrangement is, in general, different.

Author contributions All the authors contributed equally to this research paper.

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal ethical standards This article does not contain any studies with human participants or animals performed by any of the authors.

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