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Abstract In this paper, we study a stochastic competition model with time delay and harvesting. We first simplify it through the stochastic center manifold reduction principle and stochastic averaging method as a one-dimensional *Markov* diffusion process. Singular boundary theory and an invariant measure are applied to analyze stochastic stability and bifurcation. The T-S fuzzy model of the system is constructed, and the H_∞ fuzzy controller is designed to eliminate the bifurcation phenomenon through a linear matrix inequality approach. Numerical simulation is used to demonstrate our results.

Keywords Stochastic competition system · Stage-structured · Hopf bifurcation · Discrete time-delay · T-S fuzzy system · H_∞ control

1 Model description

In nature, competition among populations is widespread because of limited natural resources. Early studies mainly considered deterministic models [1, 2]. However, the population must

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be disturbed by realistic environmental noise [3, 4], which is important in the study of bio-mathematical models, and research of stochastic models is inevitable. Individual organisms experience a growth process. Young individuals have a weaker ability to cope with environmental disturbances, predators, and competitors' survival pressure. The survival ability of adult individuals is strong, and they are able to conceive the next generation. Hence, the stage-structured model is popular among scholars [5–8]. Because the impact of competition between populations is not immediate, it is necessary to consider the time delay [7–9]. We discuss the following stage-structured competition model with time-delay:

$$\begin{cases} \dot{x}_1 = a_{11}x_2 - a_{12}x_1^2 - sx_1, \\ \dot{x}_2 = a_{21}x_1 - a_{22}x_2^2 - d_1x_2x_3(t - \tau_1) - \beta x_2, \\ \dot{x}_3 = x_3(r(1 - \frac{x_3}{k_3}) - d_2x_2(t - \tau_2) - qE), \end{cases} \quad (1)$$

x_i is the density of the i th species, $i = 1, 2, 3$, where x_1, x_2 are respectively the juveniles and adults of one of two species. a_{11} is the birth rate of juveniles and a_{21} is the transformation rate from juveniles to adults. a_{12}, a_{22} denote interspecific competitive coefficients of x_1 and x_2 . Considering that x_1 is young and not competitive, we assume that only x_2 and x_3 are competitive, and d_1 and d_2 are the rates of loss of x_2 and x_3 populations in competition. r and k_3 are respectively the intrinsic growth rate and environmental capacity of species x_3 . The sum of the death and the conversion rate of the juveniles x_1 and the sum of the death rate of the adults x_2 are expressed by s and β , respectively. q is the catchability coefficient of species x_3 . E denotes the effort used to harvest population x_3 . τ_1, τ_2 is the time delay in competition. All the parameters are positive constants.

Considering environmental noise [7] such as lightning, rain, and drought, we have the following stochastic differential system:

$$\begin{cases} \dot{x}_1 = a_{11}x_2 - a_{12}x_1^2 - sx_1 + \sigma_1 x_1 \xi_1(t), \\ \dot{x}_2 = a_{21}x_1 - a_{22}x_2^2 - d_1x_2x_3(t - \tau_1) - \beta x_2 + \sigma_2 x_2 \xi_2(t), \\ \dot{x}_3 = x_3(r(1 - \frac{x_3}{k_3}) - d_2x_2(t - \tau_2) - qE) + \sigma_3 x_3 \xi_3(t), \end{cases} \quad (2)$$

where $\xi_i(t)$, $i = 1, 2$, is independent white noise with intensity σ_i^2 .

Throughout this article, we make the following assumption.

Assumption 1 *Because of limited environmental supply and interspecific and intraspecific constraints, there must be environmental capacity k_i of species x_i .*

2 Stochastic stability

$P(x_1^*, x_2^*, x_3^*)$ is the positive equilibrium point of the deterministic system (2). Let $y_1 = x_1 - x_1^*$, $y_2 = x_2 - x_2^*$, $y_3 = x_3 - x_3^*$. Then

$$\begin{cases} \dot{y}_1 = b_{11}y_1 + b_{12}y_2 - a_{12}y_1^2 + \sigma_1(y_1 + x_1^*)\xi_1(t), \\ \dot{y}_2 = b_{21}y_1 + b_{22}y_2 + b_{23}y_3(t - \tau_1) - a_{22}y_2^2 - d_1y_2y_3(t - \tau_1) + \sigma_1(y_2 + x_2^*)\xi_1(t), \\ \dot{y}_3 = b_{32}y_2(t - \tau_2) + b_{33}y_3 - \frac{r}{k_3}y_3^2 - d_2y_2(t - \tau_2)y_3 + \sigma_2(y_3 + x_3^*)\xi_2(t), \end{cases} \quad (3)$$

where

$$\begin{aligned} b_{11} &= -(2a_{12}x_1^* + s), b_{12} = a_{11}, b_{21} = a_{21}, b_{22} = -(2a_{22}x_2^* + d_1x_3^* + \beta), \\ b_{23} &= -d_1x_2^*, b_{32} = -d_2x_3^*, b_{33} = r - 2\frac{r}{k_3}x_3^* - d_1x_2^* - qE, \end{aligned}$$

For simplicity, writing system (3) in the following matrix form:

$$\dot{y}(t) = Ay(t) + B_1y(t - \tau_1) + B_2y(t - \tau_2) + F(y(t), y(t - \tau_1), y(t - \tau_2), \xi_1(t), \xi_2(t)), \quad (4)$$

For simplicity, we write system (3) in matrix form:

$$\begin{aligned} A &= \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} & B_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix} & B_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{32} & 0 \end{bmatrix} \\ F &= \begin{bmatrix} -a_{12}y_1^2 + \sigma_1(y_1 + x_1^*)\xi_1(t) \\ -a_{22}y_2^2 - d_1y_2y_3(t - \tau_1) + \sigma_1(y_2 + x_2^*)\xi_1(t) \\ -\frac{r}{k_3}y_3^2 - d_2y_2(t - \tau_2)y_3 + \sigma_2(y_3 + x_3^*)\xi_2(t) \end{bmatrix} \end{aligned}$$

The characteristic equation of linear system $y'(t) = Ay(t) + B_1y(t - \tau_1) + B_2y(t - \tau_2)$ of system (4) is

$$p(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3\lambda e^{-\lambda(\tau_1+\tau_2)} + p_4e^{-\lambda(\tau_1+\tau_2)} + p_5 = 0, \quad (5)$$

where $p_1 = -(b_{11} + b_{22} + b_{32})$, $p_2 = b_{11}b_{33} + (b_{11} + b_{22})b_{32} - b_{12}b_{21}$, $p_3 = -b_{23}b_{32}$, $p_4 = b_{11}b_{23}b_{32}$,

$p_5 = b_{12}b_{21}b_{32} - b_{11}b_{22}b_{32}$ equation (5) has no zero characteristic root when $p_3 + p_4 + p_5 \neq 0$. Thus, the trivial equilibrium $y = 0$ becomes unstable only when equation (5) has at least a pair of purely imaginary roots $\lambda = \pm i\omega$ (i is the imaginary unit), at which Hopf bifurcation occurs. Substituting $\lambda = i\omega$ in equation (5), one can obtain that

$$p(i\omega) = G(i\omega, \tau_1, \tau_2) + iM(i\omega, \tau_1, \tau_2) = 0,$$

$$G = -p_1\omega^2 + p_3\omega \sin \omega(\tau_1 + \tau_2) + p_4 \cos \omega(\tau_1 + \tau_2) + p_5 = 0,$$

$$M = -\omega^3 + p_2\omega + p_3\omega \cos \omega(\tau_1 + \tau_2) - p_4 \sin \omega(\tau_1 + \tau_2) = 0.$$

While studying the critical infinite-dimensional problem on a two-dimensional center manifold, we express the delay equation as an abstract evolution equation on complete probability space $C([- \tau, 0], R^3)$:

$$\dot{y}(t) = Dy_t(\theta) + F(t, y_t(\theta), \xi_1(t), \xi_2(t)), \quad (6)$$

where $y_t(\theta) = y(t + \theta)$, $-\tau < \theta < 0$, and D is a linear operator for the critical case, expressed for the initial condition $u(\theta) \in C$ by

$$Du(\theta) = \begin{cases} \frac{du(\theta)}{d\theta}, & \theta \in [-\tau, 0) \\ Au(0) + B_1u(-\tau_1) + B_2u(-\tau_2), & \theta = 0. \end{cases} \quad (7a)$$

$$(7b)$$

The nonlinear operator F has the form

$$F = \begin{cases} 0, & \theta \in [-\tau, 0) \\ F(0, u(-t), \xi_1(0), \xi_2(0)), & \theta = 0. \end{cases}$$

Similarly, the continuously differentiable function v is on the adjoint space $\hat{C}([0, \tau], R^3)$ with dual operator

$$D^*v(\theta) = \begin{cases} \frac{dv(\theta)}{d\theta}, & \theta \in [0, \tau) \\ A^*u(0) + B_1^*u(\tau_1) + B_2^*u(\tau_2), & \theta = 0 \end{cases}$$

The case $p(\lambda) = 0$ has a single pair of purely imaginary eigenvalues $\lambda = \pm i\omega$ through Hopf bifurcation. C can be split into two subspaces as $C = P_\Lambda \oplus Q_\Lambda$, where P_Λ is a two-dimensional space spanned by the eigenvectors of the operator D corresponding to the eigenvalues $\pm i\omega$, and Q_Λ is the infinite-dimensional subspace corresponding to remaining eigenvalues of $p(\lambda) = 0$. Then, for $u \in C, v \in \hat{C}$, we can define a bilinear operator,

$$(u, v) = (v(0), u(0)) - b_{23} \int_{-\tau_1}^0 v(\zeta + \theta) u(\xi) d\zeta - b_{32} \int_{-\tau_2}^0 v(\zeta + \theta) u(\xi) d\zeta \quad (8)$$

Corresponding to the critical characteristic root $i\omega$, the complex eigenvector $q(\theta) \in C$ satisfies

$$\frac{dq(\theta)}{d\theta} = i\omega q(\theta), \quad \theta \in [-\tau, 0), \quad (9)$$

$$Aq(0) + B_1q(-\tau_1) + B_2q(-\tau_2) = i\omega q(0), \quad \theta = 0.$$

The general solution of equation (9) is $q(\theta) = \tilde{c}e^{i\omega\theta}$. From the boundary condition equation (7b), we find the following matrix equation:

$$\begin{bmatrix} b_{11} - i\omega & b_{12} & 0 \\ b_{21} & b_{22} - i\omega & b_{23}e^{-i\omega\tau_1} \\ 0 & b_{32}e^{-i\omega\tau_2} & b_{33} - i\omega \end{bmatrix} \tilde{c} = 0$$

Let $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)^T$. We choose $\tilde{c}_1 = 1$; then $\tilde{c}_2, \tilde{c}_3$ can be determined uniquely, i.e.,

$$\tilde{c} = \begin{bmatrix} 1 & \frac{i\omega - b_{11}}{b_{12}} & \frac{b_{12}b_{21} + \omega^2 - b_{11}b_{22} + (b_{11} + b_{22})i\omega}{-b_{12}b_{23}} e^{i\omega\tau_1} \end{bmatrix}^T.$$

Thus, the real bases in $P \in C$ and $\hat{P} \in \hat{C}$ are respectively as follows:

$$\Phi(\theta) = (\phi_1, \phi_2) = \begin{bmatrix} \cos \omega\theta & \sin \omega\theta \\ -\frac{b_{11}\cos \omega\theta + \omega \sin \omega\theta}{b_{12}} & \frac{\omega \cos \omega\theta - b_{11}\sin \omega\theta}{b_{12}} \\ H_1 \cos \omega(\theta + \tau_1) - H_2 \sin \omega(\theta + \tau_1) & H_1 \sin \omega(\theta + \tau_1) + H_2 \cos \omega(\theta + \tau_1) \end{bmatrix}, \quad (10)$$

$$\Psi(s) = (\psi_1, \psi_2) = \begin{bmatrix} L_1 \cos \omega(\tau_2 - s) + L_2 \sin \omega(\tau_2 - s) & L_2 \cos \omega(\tau_2 - s) - L_1 \sin \omega(\tau_2 - s) \\ -\frac{b_{33} \cos \omega(\tau_2 - s) + \omega \sin \omega(\tau_2 - s)}{b_{32}} & \frac{\omega \cos \omega(\tau_2 - s) - b_{33} \sin \omega(\tau_2 - s)}{b_{32}} \\ \cos \omega s & -\sin \omega s \end{bmatrix}, \quad (11)$$

where $H_1 = -\frac{b_{12}b_{21}-b_{11}b_{22}+\omega^2}{b_{12}b_{23}}$, $H_2 = -\frac{(b_{11}+b_{22})\omega}{b_{12}b_{23}}$, $L_1 = \frac{\omega^2-b_{11}b_{33}}{b_{12}b_{32}}$, $L_2 = -\frac{(b_{11}+b_{33})\omega}{b_{12}b_{32}}$.

the inner product matrix $(\Psi(s), \Phi(\theta)) = (\psi_j(s), \phi_k(\theta))$, $j, k = 1, 2$ has the following expression:

$$\begin{aligned} (\Psi(s), \Phi(\theta)) &= \begin{bmatrix} \psi_1^T(s) \\ \psi_2^T(s) \end{bmatrix} \begin{bmatrix} \phi_1(\theta) & \phi_2(\theta) \end{bmatrix} = \begin{bmatrix} (\psi_1^T(s)\phi_1(\theta)) & (\psi_1^T(s)\phi_2(\theta)) \\ (\psi_2^T(s)\phi_1(\theta)) & (\psi_2^T(s)\phi_2(\theta)) \end{bmatrix}, \\ (\psi_1^T(s)\phi_1(\theta)) &= (L_1 \cos \omega(\tau_2 - s) + L_2 \sin \omega(\tau_2 - s)) \cos \omega \theta + \frac{b_{33} \cos \omega(\tau_2 - s) + \omega \sin \omega(\tau_2 - s)}{b_{32}} \\ &\quad \frac{b_{11} \cos \omega \theta + \omega \sin \theta}{b_{12}} - \cos \omega s (H_1 \cos \omega(\tau_1 + \theta) - H_2 \sin \omega(\tau_1 + \theta)), \\ (\psi_1^T(s)\phi_2(\theta)) &= (L_1 \cos \omega(\tau_2 - s) + L_2 \sin \omega(\tau_2 - s)) \sin \omega \theta + \frac{b_{33} \cos \omega(\tau_2 - s) + \omega \sin \omega(\tau_2 - s)}{b_{32}} \\ &\quad \frac{b_{11} \sin \omega \theta - \omega \cos \theta}{b_{12}} - \cos \omega s (H_1 \cos \omega(\tau_1 + \theta) + H_2 \sin \omega(\tau_1 + \theta)), \\ (\psi_2^T(s)\phi_1(\theta)) &= (L_2 \cos \omega(\tau_2 - s) - L_1 \sin \omega(\tau_2 - s)) \cos \omega \theta + \frac{\omega \cos \omega(\tau_2 - s) - b_{33} \sin \omega(\tau_2 - s)}{b_{32}} \\ &\quad \frac{b_{11} \cos \omega \theta + \omega \sin \omega \theta}{b_{12}} + \sin \omega s (H_1 \cos \omega(\tau_1 + \theta) - H_2 \sin \omega(\tau_1 + \theta)), \\ (\psi_2^T(s)\phi_2(\theta)) &= (L_2 \cos \omega(\tau_2 - s) - L_1 \sin \omega(\tau_2 - s)) \sin \omega \theta + \frac{\omega \cos \omega(\tau_2 - s) - b_{33} \sin \omega(\tau_2 - s)}{b_{32}} \\ &\quad \frac{b_{11} \sin \omega \theta - \omega \cos \theta}{b_{12}} + \sin \omega s (H_1 \cos \omega(\tau_1 + \theta) + H_2 \sin \omega(\tau_1 + \theta)). \end{aligned}$$

The substitution of the elements of (Ψ, Φ) in the bilinear relation (8) yields the nonsingular matrix

$$(\Psi(s), \Phi(\theta))_{nsg} = \begin{bmatrix} (\psi_1, \phi_1) & (\psi_1, \phi_2) \\ (\psi_2, \phi_1) & (\psi_2, \phi_2) \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \quad (12)$$

Then the basis $\Psi \in \hat{C}$ is normalized to

$$\tilde{\Psi}(s) = ((\Psi, \Phi)_{nsg}^{-1}(s) \Psi^T(s))^T = \begin{bmatrix} \tilde{\Psi}_{11} & \tilde{\Psi}_{12} \\ \tilde{\Psi}_{21} & \tilde{\Psi}_{22} \\ \tilde{\Psi}_{31} & \tilde{\Psi}_{32} \end{bmatrix}$$

$$\begin{aligned}
\tilde{\Psi}_{11} &= \frac{1}{\det(\Psi, \Phi)_{nsg}^{-1}(s)} \{N_4(L_1 \cos \omega(\tau_2 - s)) + L_2 \sin \omega(\tau_2 - s) \\
&\quad - N_2(L_2 \cos \omega(\tau_2 - s)) - L_1 \sin \omega(\tau_2 - s)\}, \\
\tilde{\Psi}_{21} &= \frac{1}{\det(\Psi, \Phi)_{nsg}^{-1}(s)} \{N_4 \frac{b_{33} \cos \omega(\tau_2 - s) + \omega \sin \omega(\tau_2 - s)}{b_{32}} \\
&\quad - N_2 \frac{\omega \cos \omega(\tau_2 - s) - b_{33} \sin \omega(\tau_2 - s)}{b_{32}}\}, \\
\tilde{\Psi}_{31} &= \frac{1}{\det(\Psi, \Phi)_{nsg}^{-1}(s)} \{N_4 \cos \omega s - N_2 \sin \omega s\}, \\
\tilde{\Psi}_{12} &= -\frac{1}{\det(\Psi, \Phi)_{nsg}^{-1}(s)} \{N_3(L_1 \cos \omega(\tau_2 - s)) + L_2 \sin \omega(\tau_2 - s) \\
&\quad - N_1(L_2 \cos \omega(\tau_2 - s)) - L_1 \sin \omega(\tau_2 - s)\}, \\
\tilde{\Psi}_{22} &= -\frac{1}{\det(\Psi, \Phi)_{nsg}^{-1}(s)} \{N_3 \frac{b_{33} \cos \omega(\tau_2 - s) + \omega \sin \omega(\tau_2 - s)}{b_{32}} \\
&\quad - N_1 \frac{\omega \cos \omega(\tau_2 - s) - b_{33} \sin \omega(\tau_2 - s)}{b_{32}}\}, \\
\tilde{\Psi}_{32} &= -\frac{1}{\det(\Psi, \Phi)_{nsg}^{-1}(s)} \{N_3 \cos \omega s - N_1 \sin \omega s\},
\end{aligned}$$

Substituting $(\tilde{\Psi}(s), \phi_k(\theta))$, generated by the new inner product matrix $(\tilde{\Psi}(s), \Phi(\theta))$ in the bilinear pairing expression (8), the available identity matrix is

$$(\tilde{\Psi}, \Phi)_{id} = \frac{1}{N_1 N_4 + N_2^2} \begin{bmatrix} N_1 N_4 + N_2^2 & 0 \\ 0 & N_1 N_4 + N_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next, by defining $z = (z_1, z_2) = (\Psi, u_t)$, which actually represents the local coordinate system on the two-dimensional center manifold, induced by the basis Ψ , with the aid of equation (10) and (11), one can decompose u_t into two parts to obtain

$$u_t = u_t^{P^\wedge} + u_t^{Q^\wedge} = \Phi(\Psi, u_t) + u_t^{Q^\wedge} = \Phi z + u_t^{Q^\wedge} \quad (13)$$

which implies that the projection of u_t on the center manifold is Φz . Then, applying (13) and (6) results in

$$(\Psi, \Phi z + u_t^{Q^\wedge}) = (\Psi, D(\Phi z + u_t^{Q^\wedge})) + (\Psi, F(\Phi z + u_t^{Q^\wedge}))$$

Thus,

$$(\Psi, \Phi) \dot{z} = (\Psi, D\Phi)z + (\Psi, F(t, \Phi z + u_t^{Q^\wedge}))$$

Finally, we obtain the equation of the stochastic center manifold:

$$\dot{z} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} z + \mathcal{N}(z, \xi_1, \xi_2)$$

where $\mathcal{N}(z, \xi_1, \xi_2)$ represents the nonlinear terms contributed from the original system to the stochastic center manifold.

By the change of variable $y_i^{P^\wedge}(\theta) = \Phi(\theta)z(t) + y_i^{Q^\wedge}(\theta)$ with $z(t) = (\tilde{\Psi}(s), \Phi^{P^\wedge}(\theta))$, a first-order approximation for $\theta = -\tau_1, -\tau_2$ is as follows:

$$\begin{aligned} \begin{bmatrix} y_1(t - \tau_1) \\ y_2(t - \tau_1) \\ y_3(t - \tau_1) \end{bmatrix} &= \Phi(-\tau_1)z(t) = \begin{bmatrix} z_1 \cos \omega \tau_1 - z_2 \sin \omega \tau_1 \\ \frac{z_1(\omega \sin \omega \tau_1 - b_{11} \cos \omega \tau_1) + z_2(\omega \cos \omega \tau_1 + b_{11} \sin \omega \tau_1)}{b_{12}} \\ z_1 H_1 + z_2 H_2 \end{bmatrix}, \\ \begin{bmatrix} y_1(t - \tau_2) \\ y_2(t - \tau_2) \\ y_3(t - \tau_2) \end{bmatrix} &= \Phi(-\tau_2)z(t) \\ &= \begin{bmatrix} z_1 \cos \omega \tau_2 - z_2 \sin \omega \tau_2 \\ \frac{z_1(\omega \sin \omega \tau_2 - b_{11} \cos \omega \tau_2) + z_2(\omega \cos \omega \tau_2 + b_{11} \sin \omega \tau_2)}{b_{12}} \\ z_1(H_1 \cos \omega(\tau_1 - \tau_2) - H_2 \sin \omega(\tau_1 - \tau_2)) + z_2(H_1 \sin \omega(\tau_1 - \tau_2) + H_2 \cos \omega(\tau_1 - \tau_2)) \end{bmatrix}, \\ \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} &= \Phi(0)z(t) = \begin{bmatrix} z_1 \\ \frac{-z_1 b_{11} + z_2 \omega}{b_{12}} \\ z_1 H_1 + z_2 H_2 \end{bmatrix}. \end{aligned}$$

The lowest-order nonlinear terms of the center manifold needed to determine the solutions are $\mathcal{N}(z) = \tilde{\Psi}^T(0)F(\Phi z, \Phi \xi_1, \Phi \xi_2)$. Therefore, one can have the stochastic center manifold

$$\begin{cases} \dot{z}_1 = -\omega z_2 + e_{10}z_1 + e_{100}z_2 + e_{11}z_1^2 + e_{12}z_1z_2 + e_{13}z_2^2 + \sigma_1(e_{14}z_1 + e_{15}z_2 + e_{16})\xi_1(t) \\ \quad + \sigma_2(e_{17}z_1 + e_{18}z_2 + e_{19})\xi_2(t), \\ \dot{z}_2 = \omega z_1 + e_{20}z_1 + e_{200}z_2 + e_{21}z_1^2 + e_{22}z_1z_2 + e_{23}z_2^2 + \sigma_1(e_{24}z_1 + e_{25}z_2 + e_{26})\xi_1(t) \\ \quad + \sigma_2(e_{27}z_1 + e_{28}z_2 + e_{29})\xi_2(t), \end{cases} \quad (14)$$

Setting the coordinate transformation $z_1 = r \cos \theta$, $z_2 = r \sin \theta$ for system (14),

$$\begin{aligned}
\dot{r}(t) = & r e_{10} \cos^2 \theta + r(e_{100} + e_{20}) \sin \theta \cos \theta + r e_{200} \sin^2 \theta + r^2(e_{13} \cos^3 \theta + e_{21} \sin^3 \theta \\
& + (e_{11} + e_{22}) \sin^2 \theta \cos \theta + (e_{12} + e_{23}) \sin \theta \cos^2 \theta) + \sigma_1(e_{16} \cos \theta + e_{26} \sin \theta \\
& + r(e_{15} \cos^2 \theta + (e_{25} + e_{14}) \sin \theta \cos \theta + e_{24} \sin^2 \theta)) \xi_1(t) \\
& + \sigma_2(e_{19} \cos \theta + e_{29} \sin \theta + r(e_{18} \cos^2 \theta + (e_{28} + e_{17}) \sin \theta \cos \theta + e_{27} \sin^2 \theta)) \xi_2(t) \\
\dot{\theta}(t) = & \omega + e_{21} \cos^2 \theta + (e_{20} + e_{200}) \sin \theta \cos \theta + e_{200} \sin^2 \theta + r(e_{23} \cos^3 \theta - e_{11} e_{21} \sin^3 \theta \\
& + (e_{21} - e_{12}) \sin^2 \theta \cos \theta + (e_{22} - e_{13}) \sin \theta \cos \theta) + \sigma_1\left(\frac{1}{r}(e_{26} \cos \theta - e_{16} \sin \theta) \right. \\
& + e_{25} \cos^2 \theta + (e_{24} - e_{15}) \sin \theta \cos \theta - e_{14} \sin^2 \theta) \xi_1(t) + \sigma_2\left(\frac{1}{r}(e_{29} \cos \theta - e_{19} \sin \theta) \right. \\
& \left. + e_{28} \cos^2 \theta + (e_{27} - e_{18}) \sin \theta \cos \theta - e_{17} \sin^2 \theta) \xi_2(t) \right.
\end{aligned} \tag{15}$$

According to the Khasminskii limit theorem [10, 11], if σ_1, σ_2 are small enough, then the response process $\{r(t), \theta(t)\}$ weakly converges to the two-dimensional *Markov* diffusion process. Through the stochastic averaging method [10], we obtain Itô's stochastic differential equation (19), and

$$\begin{cases} dr = m_r dt + \sigma_{11} dW_r + \sigma_{12} dW_\theta, \\ d\theta = m_\theta dt + \sigma_{21} dW_r + \sigma_{22} dW_\theta, \end{cases} \tag{16}$$

where W_r, W_θ , respectively, are the independent and standard Wiener processes. $\begin{bmatrix} m_r \\ m_\theta \end{bmatrix}$ is a

drift vector, and $\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ is a diffusion coefficient matrix. Set the parameters as follows:

$$\begin{aligned} m_r &= \frac{r}{2}(e_{10} + e_{200}) + \frac{\sigma_1^2}{8}r(5e_{15}^2 + 5e_{24}^2 + 3(e_{14} + e_{25})^2 - 2e_{15}e_{24}) \\ &\quad + \frac{\sigma_2^2}{8}r(5e_{18}^2 + 5e_{27}^2 + 3(e_{17} + e_{28})^2 - 2e_{18}e_{27}) + \frac{r}{2}(\sigma_1^2(e_{16}^2 + e_{26}^2) + \sigma_2^2(e_{19}^2 + e_{29}^2)), \\ \sigma_{11}^2 &= \frac{1}{2}(\sigma_1^2(e_{16}^2 + e_{26}^2) + \sigma_2^2(e_{19}^2 + e_{29}^2)) + \frac{\sigma_1^2}{8}(3e_{15}^2 + 2e_{15}e_{24} + (e_{14} + e_{25})^2 + 3e_{24}^2) \\ &\quad + \frac{\sigma_2^2}{8}(3e_{18}^2 + 2e_{18}e_{27} + (e_{17} + e_{28})^2 + 3e_{27}^2), \\ \sigma_{12}^2 &= \sigma_{21}^2 = \frac{r}{4}\{\sigma_1^2(e_{25} - e_{14})(e_{24} + e_{15}) + \sigma_2^2(e_{28} - e_{17})(e_{18} + e_{27})\}. \end{aligned}$$

When $\sigma_{12}^2 = \sigma_{21}^2 = 0$, i.e., $e_{25} - e_{14} = 0$ or $e_{28} - e_{17} = 0$, $r(t)$ is a one-dimensional Markov diffusion process. Thus, we have

$$dr = [(\mu_{10} + \frac{\mu_1}{8})r + \frac{\mu_2}{r}]dt + (\mu_2 + \frac{\mu_3}{8}r^2)^{\frac{1}{2}}dW_r, \quad (17)$$

where

$$\begin{aligned} \mu_{10} &= \frac{1}{2}(e_{10} + e_{200}), \quad \mu_{20} = \omega + \frac{1}{2}(e_{20} + e_{100}), \\ \mu_1 &= \sigma_1^2(5e_{15}^2 + 5e_{24}^2 + 3(e_{14} + e_{25})^2 - 2e_{15}e_{24}) + \sigma_2^2(5e_{18}^2 + 5e_{27}^2 + 3(e_{17} + e_{28})^2 - 2e_{18}e_{27}), \\ \mu_2 &= \frac{1}{2}(\sigma_1^2(e_{16}^2 + e_{26}^2) + \sigma_2^2(e_{19}^2 + e_{29}^2)), \\ \mu_3 &= \sigma_1^2(3e_{15}^2 + 2e_{15}e_{24} + (e_{14} + e_{25})^2 + 3e_{24}^2) + \sigma_2^2(3e_{18}^2 + 2e_{18}e_{27} + (e_{17} + e_{28})^2 + 3e_{27}^2), \end{aligned}$$

Next, the stability of the system is analyzed according to singular boundary theory. In fact, μ_2 is not equal to 0. One can find $r = 0$ is a nonsingular boundary of the system (17) if $\mu_2 + \frac{\mu_3}{8}r^2 \neq 0$ at $r = 0$. Through calculation, we can find that $r = 0$ is a regular boundary. If $r = +\infty$, $m_r = \infty$, then $m_r = +\infty$, $r = +\infty$ is the second singular boundary.

$$\alpha_r = 2, \quad \beta_r = 1,$$

$$c_r = -\lim_{r \rightarrow +\infty} \frac{2m_r r^{\alpha_r - \beta_r}}{\sigma_{11}^2} = -\lim_{r \rightarrow +\infty} \frac{2[(\mu_{10} + \frac{\mu_1}{8})r + \frac{\mu_2}{r}]r}{\mu_2 + \frac{\mu_3}{8}r^2} = -\frac{2(8\mu_{10} + \mu_1)}{\mu_3}.$$

So, if $c_r > -1$, i.e., $\frac{8\mu_{10}+\mu_1}{\mu_3} < \frac{1}{2}$, then the boundary $r = +\infty$ is exclusively natural;

if $c_r < -1$, i.e., $\frac{8\mu_{10}+\mu_1}{\mu_3} > \frac{1}{2}$, then the boundary $r = +\infty$ is attractively natural;

if $c_r = -1$, i.e., $\frac{8\mu_{10}+\mu_1}{\mu_3} = \frac{1}{2}$, then the boundary $r = +\infty$ is strictly natural.

Remark 1 If $c_r < -1$, then the boundary $r = +\infty$ is attractively natural, i.e., the system is unstable at the equilibrium point, and *Hopf* bifurcation is possible.

3 Hopf bifurcation

Based on (17), we can obtain the *FPK* equation as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r} \left\{ \left[\left(\mu_{10} + \frac{\mu_1}{8} \right) r + \frac{\mu_2}{r} \right] p(r) \right\} + \frac{1}{2} \frac{\partial^2}{\partial r^2} \left\{ \left(\mu_2 + \frac{\mu_3}{8} r^2 \right) p(r) \right\} \quad (18)$$

with the initial value condition

$$p(r, t | r_0, t_0) \rightarrow \delta(r - r_0), t \rightarrow t_0,$$

where $p(r, t | r_0, t_0)$ is the transition probability density of diffusion process $r(t)$. The invariant measure of $r(t)$ is the steady-state probability density $p_{st}(r)$, which is the solution of the degenerate system, as follows:

$$-\frac{\partial}{\partial r} \left\{ \left[\left(\mu_{10} + \frac{\mu_1}{8} \right) r + \frac{\mu_2}{r} \right] p(r) \right\} + \frac{1}{2} \frac{\partial^2}{\partial r^2} \left\{ \left(\mu_2 + \frac{\mu_3}{8} r^2 \right) p(r) \right\} = 0.$$

Solving the above equation, one can obtain

$$p_{st}(r) = 4 \sqrt{\frac{2}{\pi}} 2^{-3h} \mu_3^{2-h} \left(\frac{\mu_3}{\mu_2} \right)^{\frac{3}{2}} \Gamma(2-h) \left(\Gamma\left(\frac{1}{2}-h\right) \right)^{-1} r^2 (\mu_3 r^2 + 8\mu_2)^{h-2},$$

where $h = \frac{8\mu_{10}+\mu_1}{\mu_3}$, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. We now calculate the most possible amplitude r^* of system (17), i.e., $p_{st}(r)$ has a maximum value at r^* . So, we have

$$\frac{dp_{st}(r)}{dr} \Big|_{r=r^*} = 0, \quad \frac{d^2 p_{st}(r)}{dr^2} \Big|_{r=r^*} < 0,$$

and the solution is $r = 0$ or $r = \tilde{r} = \sqrt{\frac{-8\mu_2}{8\mu_{10}+\mu_1-\mu_3}} \left(\frac{-8\mu_2}{8\mu_{10}+\mu_1-\mu_3} < \frac{1}{2} \right)$. The relations

$$\begin{aligned} \frac{d^2 p_{st}(r)}{dr^2} \Big|_{r=0} &= 2^{7+3(8\mu_{10}+\mu_1-\mu_3)\mu^{-1}} \mu_2^{2+(8\mu_{10}+\mu_1-\mu_3)\mu^{-1}} > 0, \\ \frac{d^2 p_{st}(r)}{dr^2} \Big|_{r=\tilde{r}} &= \frac{(8\mu_{10}+\mu_1-\mu_3)^3 (8\mu_2 - \frac{8\mu_2\mu_3}{8\mu_{10}+\mu_1-\mu_3})^{\frac{8\mu_{10}+\mu_1}{\mu_3}}}{-16(8\mu_{10}+\mu_1-\mu_3)} < 0, \end{aligned}$$

show that $r = \tilde{r}$. $p_{st}(r)$ is minimum at $r = 0$. Stochastic system (17) is almost unstable at $r = 0$. Hence, system (2) may show *Hopf* bifurcation at $r = \tilde{r}$. Then

$$x_1^2 + x_2^2 + x_3^2 = \frac{-8\mu_2}{8\mu_{10}+\mu_1-\mu_3}, (r = \tilde{r}).$$

4 H_∞ control

Hopf bifurcation indicates the instability of the system, which is not what we want to see. T-S fuzzy control [12] approximates or represents a global nonlinear system model using several local linear system models, and solves the control problem of the whole local nonlinear system by means of analysis and control of the linear system. H_∞ control based on T-S fuzzy control [13–16] is widely used in various fields. To eliminate this bifurcation phenomenon, the application of T-S fuzzy H_∞ control to solve the problem of nonlinear biological systems will be a general trend in the field of biological control [17, 18].

Next, introducing the state feedback control for system (3), one can obtain the controlled system (19):

$$\begin{cases} \dot{y}_1 = b_{11}y_1 + b_{12}y_2 - a_{12}y_1^2 + \sigma_1(y_1 + x_1^*)\xi_1(t), \\ \dot{y}_2 = b_{21}y_1 + b_{22}y_2 + b_{23}y_3(t - \tau_1) - a_{22}y_2^2 - d_{12}y_2y_3(t - \tau_1) + \sigma_1(y_2 + x_2^*)\xi_1(t), \\ \dot{y}_3 = b_{32}y_2(t - \tau_2) + b_{33}y_3 - \frac{r}{k_3}y_3^2 - d_{22}y_2(t - \tau_2)y_3 + u(t) + \sigma_2(y_3 + x_3^*)\xi_2(t), \end{cases} \quad (19)$$

where $u(t)$ is the control variable.

We suppose that $y_i(t) \in [-k_i, k_i]$, $i = 1, 2, 3$. A system of fuzzy equations is given as follows, which can describe system (19) when $y_i(t) \in [-k_i, k_i]$, $i = 1, 2, 3$:

Rule 1: If $y_1(t)$ is M_{11} , $y_2(t)$ is M_{21} , and $y_3(t)$ is M_{31} , then

$$\dot{y}(t) = A_{11}y(t) + A_{11}y(t - \tau_1) + A_{21}y(t - \tau_2) + B_{11}\xi(t) + B_{21}\bar{\xi}(t) + C_1u(t).$$

Rule 2: if $y_1(t)$ is M_{12} , $y_2(t)$ is M_{22} , $y_3(t)$ is M_{32} , then

$$\dot{y}(t) = A_2 y(t) + A_{12} y(t - \tau_1) + A_{22} y(t - \tau_2) + B_{12} \xi(t) + B_{22} \bar{\xi}(t) + C_2 u(t),$$

where

$$\begin{aligned} y(t) &= \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \end{bmatrix}^T, \quad \xi(t) = \begin{bmatrix} \xi_1(t) & \xi_1(t) & \xi_2(t) \end{bmatrix}^T, \\ \bar{\xi}(t) &= \begin{bmatrix} y_1(t)\xi_1(t) & y_2(t)\xi_1(t) & y_3(t)\xi_2(t) \end{bmatrix}^T, \\ M_{11} &= \frac{1}{2}(1 - y_1(t)/k_1), \quad M_{21} = \frac{1}{2}(1 - y_2(t)/k_2), \quad M_{31} = \frac{1}{2}(1 - y_3(t)/k_3), \\ M_{12} &= \frac{1}{2}(1 + y_1(t)/k_1), \quad M_{22} = \frac{1}{2}(1 + y_2(t)/k_2), \quad M_{32} = \frac{1}{2}(1 + y_3(t)/k_3), \\ A_1 &= \begin{bmatrix} b_{11} + a_{12}k_1 & b_{12} & 0 \\ b_{21} & b_{22} + a_{22}k_2 & 0 \\ 0 & 0 & b_{33} + r \end{bmatrix}, \quad A_2 = \begin{bmatrix} b_{11} - a_{12}k_1 & b_{12} & 0 \\ b_{21} & b_{22} - a_{22}k_2 & 0 \\ 0 & 0 & b_{33} - r \end{bmatrix}, \\ A_{11} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23} + d_1k_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & d_2k_3 + b_{32} & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23} - d_1k_2 \\ 0 & 0 & 0 \end{bmatrix}, \\ A_{22} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 - d_2k_3 + b_{32} & 0 & 0 \end{bmatrix}, \quad B_1 = B_{11} = B_{12} = \begin{bmatrix} \sigma_1 x_1^* & 0 & 0 \\ 0 & \sigma_1 x_2^* & 0 \\ 0 & 0 & \sigma_2 x_3^* \end{bmatrix}, \\ B_2 = B_{21} = B_{22} &= \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}, \quad C = C_1 = C_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$h_i(y(t))$ are the membership functions for $y(t) \in M_{ij}(i, j = 1, 2)$, $\beta_i(y(t)) = \prod_{j=1}^3 M_{ij}(y_j(t)) \geq 0$, $h_i(y(t)) = \beta_i(y(t)) / \sum_{i=1}^2 \beta_i(y(t))$, and $\sum_{i=1}^2 h_i(y(t)) = 1$. The global model can be described as:

$$\dot{y}(t) = \sum_{i=1}^2 h_i(y(t)) [A_i y(t) + A_{1i} y(t - \tau_1) + A_{2i} y(t - \tau_2) + B_1 \xi(t) + B_2 \bar{\xi}(t) + C u(t)] \quad (20)$$

Making a transformation $y(t - \tau_1) = y(t) - \int_{-\tau_1}^0 \dot{y}(t + \alpha) d\alpha$ from the Newton-Leibniz formula, system (20) is equivalent to

$$\dot{y}(t) = \sum_{i=1}^2 h_i(y(t)) \left[\bar{A}_i y(t) - A_{1i} \int_{-\tau_1}^0 \dot{y}(t + \alpha) d\alpha - A_{2i} \int_{-\tau_2}^0 \dot{y}(t + \alpha) d\alpha + B_1 \xi(t) + B_2 \bar{\xi}(t) + Cu(t) \right] \quad (21)$$

where $\bar{A}_i = A_i + A_{1i} + A_{2i}$.

Lemma 1 [19] *For any $u, v \in \mathbb{R}^n$, and any matrix $Z > 0$ with appropriate dimensions, the following inequality holds:*

$$u^T v + v^T u \leq u^T Z^{-1} u + v^T Z v.$$

Theorem 1 *For a given $\gamma > 0$, if there exist matrices $Q > 0$, $P_i > 0$, $R_i > 0$, $M_i > 0$ which satisfy the linear matrix inequalities*

$$\begin{bmatrix} \bar{A}_i Q + Q \bar{A}_j^T + C M_j + M_j^T C & 0 & 0 & 0 & B_1 & B_2 & \frac{A_{1i} + A_{1j}}{2} & \frac{A_{2i} + A_{2j}}{2} \\ * & -R_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \tau_1 P_1 + \tau_2 P_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & -\frac{1}{\tau_1} P_1 & 0 \\ * & * & * & * & * & * & * & -\frac{1}{\tau_2} P_2 \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

$$\begin{bmatrix}
\frac{(N_i M_j + N_j M_i)^T}{2} & Q & Q \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-I & 0 & 0 \\
* & -R_1 & 0 \\
* & * & -R_2
\end{bmatrix} < 0 \quad (22)$$

where $*$ indicates a symmetric term, then $u(t) = \sum_{j=1}^2 h_j(y(t))K_j y(t)$ makes the H_∞ norm of system (23) less than γ . The closed-loop system is quadratically stable,

$$\begin{cases}
\dot{y}(t) = \sum_{j=1}^2 \sum_{i=1}^2 h_i(y(t))h_j(y(t))((\bar{A}_i + CK_j)y(t) - A_{1i} \int_{-\tau_1}^0 \dot{y}(t+\alpha) d\alpha \\
\quad - A_{2i} \int_{-\tau_2}^0 \dot{y}(t+\alpha) d\alpha + B_1 \xi(t) + B_2 \bar{\xi}(t)), \\
z(t) = \sum_{j=1}^2 \sum_{i=1}^2 h_i(y(t))h_j(y(t))N_i K_j y(t),
\end{cases} \quad (23)$$

Proof Construct the singular Lyapunov function $V(t) = y^T(t)Py(t) + V_1(t) + V_2(t)$,

$$V_1(t) = \int_{t-\tau_1}^t y^T(s)S_1 y(s)ds + \int_{-\tau_1}^0 \int_{t+\alpha}^t \dot{y}^T(s)P_1 \dot{y}(s)dsd\alpha$$

$$V_2(t) = \int_{t-\tau_2}^t y^T(s)S_2 y(s)ds + \int_{-\tau_2}^0 \int_{t+\alpha}^t \dot{y}^T(s)P_2 \dot{y}(s)dsd\alpha,$$

where $P = Q^{-1}$ and $S_i = R_i^{-1}$. By Lemma 1, let $u = 2y^T(t)PA_{1i}$, $v = \dot{y}(t+\alpha)$, $Z = P_1$. Then

$$\begin{aligned}
& - \sum_{j=1}^2 \sum_{i=1}^2 \int_{-\tau_1}^0 2y^T(t)PA_{1i}\dot{y}(t+\alpha)d\alpha \leq \sum_{j=1}^2 \sum_{i=1}^2 \tau_1 y^T(t)P \frac{A_{1i} + A_{1j}}{2} P_1^{-1} \frac{(A_{1i} + A_{1j})^T}{2} P y(t) \\
& + \int_{-\tau_1}^0 \dot{y}^T(t+\alpha)P_1 \dot{y}(t+\alpha)d\alpha
\end{aligned}$$

similarly,

$$-\sum_{j=1}^2 \sum_{i=1}^2 \int_{-\tau_2}^0 2y^T(t) P A_{2i} \dot{y}(t+\alpha) d\alpha \leq \sum_{j=1}^2 \sum_{i=1}^2 \tau_2 y^T(t) P \frac{A_{2i} + A_{2j}}{2} P_2^{-1} \frac{(A_{2i} + A_{2j})^T}{2} P y(t) \\ + \int_{-\tau_2}^0 \dot{y}^T(t+\alpha) P_2 \dot{y}(t+\alpha) d\alpha$$

After calculation,

$$(\dot{y}^T(t) P y(t))' = \dot{y}^T(t) P y(t) + y^T(t) P \dot{y}(t) \\ \leq \sum_{j=1}^2 \sum_{i=1}^2 [y^T(t) (P(\bar{A}_i + C K_j) + (\bar{A}_i + C K_j)^T P + \tau_1 P \frac{A_{1i} + A_{1j}}{2} P_1^{-1} \frac{(A_{1i} + A_{1j})^T}{2} P \\ + \tau_2 P \frac{A_{2i} + A_{2j}}{2} P_2^{-1} \frac{(A_{2i} + A_{2j})^T}{2} P) y(t) + \xi^T(t) B_1^T P y(t) + y^T(t) P B_1 \xi(t) \\ + \bar{\xi}^T(t) B_2^T P y(t) + y^T(t) P B_2 \bar{\xi}(t) + \int_{-\tau_1}^0 \dot{y}^T(t+\alpha) P_1 \dot{y}(t+\alpha) d\alpha \\ + \int_{-\tau_2}^0 \dot{y}^T(t+\alpha) P_2 \dot{y}(t+\alpha) d\alpha]$$

$$\dot{V}_1(t) = y^T(t) S_1 y(t) - y^T(t - \tau_1) S_1 y(t - \tau_1) + \tau_1 \dot{y}^T(t) P_1 \dot{y}(t) - \int_{-\tau_1}^0 \dot{y}^T(t+\alpha) P_1 \dot{y}(t+\alpha) d\alpha$$

$$\dot{V}_2(t) = y^T(t) S_2 y(t) - y^T(t - \tau_2) S_2 y(t - \tau_2) + \tau_2 \dot{y}^T(t) P_2 \dot{y}(t) - \int_{-\tau_2}^0 \dot{y}^T(t+\alpha) P_2 \dot{y}(t+\alpha) d\alpha$$

Let $\eta(t) = \begin{bmatrix} y(t) & y(t - \tau_1) & y(t - \tau_2) & \dot{y}(t) & \xi(t) & \bar{\xi}(t) \end{bmatrix}^T$. Then the derivative of the Lyapunov function is given from the above equations by

$$\dot{V}(t) \leq \sum_{j=1}^2 \sum_{i=1}^2 h_j h_i \eta^T(t) \begin{bmatrix} R_{ij} & 0 & 0 & 0 & P B_1 & P B_2 \\ 0 & -S_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -S_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_1 P_1 + \tau_2 P_2 & 0 & 0 \\ B_1^T P & 0 & 0 & 0 & 0 & 0 \\ B_2^T P & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \eta(t) \\ \leq \sum_{j=1}^2 \sum_{i=1}^2 h_j h_i \eta^T(t) Q_{ij} \eta(t) - z^T z + \gamma^2 \xi^T(t) \xi(t),$$

where $R_{ij} = P(\bar{A}_i + C K_j) + (\bar{A}_i + C K_j)^T P + S_1 + S_2 + \tau_1 P \frac{A_{1i} + A_{1j}}{2} P_1^{-1} \frac{(A_{1i} + A_{1j})^T}{2} P \\ + \tau_2 P \frac{A_{2i} + A_{2j}}{2} P_2^{-1} \frac{(A_{2i} + A_{2j})^T}{2} P$.

$$Q_{ij} = \begin{bmatrix} R_{ij} + \frac{[N_i K_j + N_j K_i]^T [N_i K_j + N_j K_i]}{4} & 0 & 0 & 0 & PB_1 & PB_2 \\ 0 & -S_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -S_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_1 P_1 + \tau_2 P_2 & 0 & 0 \\ B_1^T P & 0 & 0 & 0 & -\gamma^2 I & 0 \\ B_2^T P & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Thus, $Q_{ij} < 0$ implies $\|z\|_2^2 \leq \gamma^2 \|\xi\|_2^2$. Therefore, when $Q < 0$, the system (23) is quadratically stable with an H_∞ norm bound γ . Applying the Schur complement theorem, $Q_{ij} < 0$ can be transformed to Eq. (22).

Remark 2 The fuzzy controller is designed to represent the government's management of resource development. The bifurcation phenomenon in biological system can be eliminated by state feedback control to ensure the stability of the system. With the purpose of maintain sustainable development and market stability, managers can take certain measures in daily life, such as, adjusting the tax rate, controlling catches, reduction of environmental pollution and so on.

5 Numerical analysis

We select some parameters to fully reflect the relationship among the steady-state probability density and position of stochastic *Hopf* bifurcation with the value of μ_2 in Section (3), and further demonstrate the effect of the controller in Section (4), with the following parameters:

$$\begin{cases} \dot{x}_1 = 0.15x_2 - 0.04x_1^2 - 0.6x_1 + 0.5x_1\xi_1(t), \\ \dot{x}_2 = 0.5x_1 - 0.001x_2^2 - 0.003x_2x_3 - 0.12x_2 + 0.5x_2\xi_1(t), \\ \dot{x}_3 = x_3(1.68(1 - \frac{x_3}{k_3}) - 0.024x_2 - 1.5 * 1) + 0.7x_3\xi_2(t), \end{cases}$$

We know by calculation that when $\mu_{10} = -0.6720$, $\mu_1 = 2.2717$, $\mu_3 = 0.7710$, the four curves in Fig. 1 correspond to cond 1, 2, 3, 4, respectively, in Table 1. It can be seen from

Table 1 The probabilities and the positions of the Hopf bifurcation occurrence.

Condition	Parameters	$r = \tilde{r}$	$P_{st}(\tilde{r})$
Cond1	$\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.0040, \mu_3 = 0.7710$	0.0059	7.6800
Cond2	$\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.0095, \mu_3 = 0.7710$	0.01441	4.9850
Cond3	$\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.0255, \mu_3 = 0.7710$	0.3792	3.042
Cond4	$\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.1635, \mu_3 = 0.7710$	1.786	1.201

Fig. 1 The steady-state probability density and position of stochastic *Hopf* bifurcation at $\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.0040, 0.0095, 0.0255, 0.1635, \mu_3 = 0.7710$.

Fig. 1 that the position of stochastic *Hopf* bifurcation increases with the increase of μ_2 , while the steady-state probability density of stochastic *Hopf* bifurcation decreases.

The H_∞ controller is designed for system (19), so that the H_∞ norm of the closed-loop system (19) is less than $\gamma = 0.01$. Using LMI toolbox in MATLAB, the parameters satisfying Theorem 1 can be obtained as follows:

$$P = \begin{bmatrix} -28.9455 & -44.8728 & 0.2033 \\ -44.8728 & -59.6085 & 1.0517 \\ 0.2033 & 1.0517 & 12.0072 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.2227 & 0 & 0 \\ 0 & 0.2058 & 0 \\ 0 & 0 & 0.2232 \end{bmatrix},$$

Fig. 2 Responses of $y_1(t)$, $y_2(t)$, $y_3(t)$, $z(t)$ in system (19).

$$\begin{aligned}
 P_2 &= \begin{bmatrix} 0.6404 & 0 & 0 \\ 0 & 0.7156 & 0 \\ 0 & 0 & 0.6402 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 3.2168 & 0.1595 & -0.0245 \\ 0.1595 & 3.3274 & 0.0165 \\ -0.0245 & 0.0165 & 3.2893 \end{bmatrix}, \\
 S_2 &= \begin{bmatrix} 3.2168 & 0.1595 & -0.0245 \\ 0.1595 & 3.3274 & 0.0165 \\ -0.0245 & 0.0165 & 3.2893 \end{bmatrix}, \\
 K_1 &= \begin{bmatrix} -0.0049 & 0.0027 & -0.0650 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.0094 & 0.0065 & -0.0115 \end{bmatrix}.
 \end{aligned}$$

Fig. 2 shows that the output and state variables of the closed-loop system (19) gradually tend to zero over time. This fully demonstrates the feasibility of the controller.

6 Conclusion

This paper studied the *Hopf* bifurcation phenomenon of a stochastic competition population with stage structure under the influence of white noise. By using the stochastic center

manifold and stochastic average method to reduce dimension, and through stability analysis and bifurcation position research, we can take μ_2 as a bifurcation parameter, and the position of stochastic *Hopf* bifurcation will increase with the increase of μ_2 , while the steady-state probability density of stochastic *Hopf* bifurcation will decrease. To eliminate the bifurcation phenomenon, a fuzzy state feedback controller was designed using the T-S fuzzy system control method to ensure the stability of the closed-loop system.

Conflict of Interest

The authors declare that they have no conflict of interest.

Founding

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Figures

Figure 1

The steady-state probability density and position of stochastic Hop f bifurcation at $\mu_{10} = -0.6720$; $\mu_1 = 2.27170$; $\mu_2 = :0040; 0.0095; 0.0255; 0.1635$; $\mu_3 = 0.7710$

Figure 2

Responses of $y_1(t)$, $y_2(t)$, $y_3(t)$, $z(t)$ in system (19).