On Opening Crack Propagation in Viscoelastic Solids

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On opening crack propagation in viscoelastic solids

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Abstract: We show that the Persson-Brener theory of crack propagation in viscoelastic solids gives a viscoelastic fracture energy factor $G/G_0 = 1 + f$ which is nearly the same as the viscoelastic factor obtained using the cohesive-zone model. We also discuss finite-size effects and comment on the use of crack propagation theories for “solids” with a viscoelastic modulus that vanishes at zero frequency.

Introduction

Crack propagation in viscoelastic solids, or at the interface between a viscoelastic solid and a counter surface, have many important applications, e.g., for rubber wear[1], or in adhesion and friction involving rubber-like materials[2–8]. Two different approaches have been applied to crack propagation in viscoelastic solids. One focus on the stress using the cohesive-zone (or Dugdale–Barenblatt) model, and another is based on an energy approach. The first approach was used by Knauss[4] and by Schapery[9, 10] and later by Hui et al[11] and by Greenwood[12, 13]. Since the exact relation between the stress σ and the surface separation u in the cohesive (process) zone, where the bond-breaking is assumed to occur, is not known in general, in the simplest approach it was assumed that the stress is constant and equal to σ_0 for $0 < u < h_0$ and $\sigma = 0$ for $u > h_0$. The second (energy) approach was used by de Gennes[14] in a qualitative way, and by Persson and Brener[15] in a quantitative way. In the latter approach enters a cut-off radius a_0 , which can be interpreted as the radius of curvature of the crack tip in the adiabatic limit, and the stress σ_c at the crack tip (the stress to break the adhesive or cohesive bonds).

Viscoelastic factor G/G_0

The stress σ_0 in the cohesive-zone approach is in general not the same as the stress σ_c in the theory of Persson and Brener. In Ref. [12] it was assumed that the stress in the process zone is a constant $\sigma = \sigma_0$ for $0 < z < h_0$, where σ_0 can be any number as long as the (adiabatic) work of adhesion is given by $G_0 = h_0\sigma_0$. If one use another wall-wall interaction, e.g., based on the Lennard-Jones potential, one gets another relation between G_0 and σ_0 [13]. Since the final result depends on σ_0 this approach is in fact somewhat ill-defined unless the exact relation between the stress and the separation is known (which in general is not the case) and used in the theory. On the contrary in the theory of Persson and Brener only well defined (experimental) quantities occur. Thus the adiabatic crack tip radius a_0 could in principle be measured using, e.g., an electron microscope, or calculated from the adiabatic work of adhesion G_0 and the stress σ_c to break the bonds via $a_0 = E_0 G_0 / (2\pi\sigma_c^2)$. In any case we expect a_0 to be of order the length of the polymer chains

FIG. 1. Three-element viscoelastic model used in model calculation of the crack propagation energy $G(v)$. The low frequency modulus $E(0) = E_0 = E'_0 E_1 / (E'_0 + E_1)$ and the high frequency modulus $E(\infty) = E_1$ and the viscosity $\eta = 1/\tau$ are indicated.

between the cross links, i.e., typically of order 1 nm.

In Ref. [15, 17], it was shown that the Persson–Brener theory gives nearly the same result for the viscoelastic factor $G/G_0 = 1 + f(v)$ as the cohesive-zone model. Thus the numerical results for G/G_0 obtained by Greenwood and by Hui et al for the three-element viscoelastic model (see Fig. 1) is nearly the same as predicted by the Persson–Brener theory if one chooses σ_0 to get the best possible overlap between the two curves (which means using $\sigma_0 \approx 3\sigma_c$), see Fig. 2 (see Appendix A for the equations used in the calculations). Shifting like this is the only meaningful way to compare the factor G/G_0 between the two theories, because the velocity normalization factor in the cohesive-zone approach depends on σ_0 [as $v_0 = E_0 G_0 / (\tau \sigma_0^2)$], and is ill-defined (no interaction force is constant for $0 < z < h_0$ and zero elsewhere).

The fact that the two curves in Fig. 2 agree so well indicate that the viscoelastic factor G/G_0 is not sensitive to the form of the bond-breaking process at the crack tip. This is also consistent with the observation that using a Lennard-Jones type of wall-wall interaction potential[13], or another shorter range potential[16], result in nearly the same result as presented above.

Finite size effect

At high crack-tip speed (or at low temperatures) the main contribution to the viscoelastic energy dissipation comes from a region far from the crack tip. This follows from dimensional arguments: the perturbing deformation frequencies from the moving crack (velocity v) a distance r away from the crack tip must be of order v/r . Thus close to the crack tip the rubber will effectively be in the

FIG. 2. The crack propagation viscoelastic factor $G/G_0 = 1 + f(v)$ as a function of the crack tip speed (log-log scale) for the three level rheology model. The red curve is calculated by Greenwood using the cohesive-zone model with the adiabatic (infinitely slowly) work of adhesion $G_0 = h_0\sigma_0$. The blue curve is from the theory of Persson and Brener assuming a crack (adiabatic) tip cut-off radius a_0 . The reference velocity $v_0 = a_0/\tau$ with $a_0 = E_0G_0/(2\pi\sigma_c^2)$ in the Persson-Brener cut-off model and $v_0 = 1.41b/\tau$ with $b = E_0G_0/\sigma_0^2$ in the cohesive-zone model.

glassy state (elastic response) and the dominant contribution to the energy dissipation will come from regions far from the crack tip. Hence if the solid has a finite extent, say of linear dimension L , at high enough crack tip speed the solid will, from the point of view of viscoelastic dissipation, effectively be in the glassy elastic state everywhere and no viscoelastic energy dissipation will occur during the crack propagation. Thus one may be tempted to claim that for finite solids $f \rightarrow 0$ as $v \rightarrow \infty$. This result was used by de Gennes to argue that for the slab geometry (thickness d_0), as occur for example in pressure sensitive adhesives, for large crack tip speed the crack propagation energy $G(v)$ will decrease with increasing v which may result in mechanical instabilities[14]. However, I will now shown in that this argument is in fact not correct and there exist no finite size effect (see also Ref. [18]).

Fig. 3(a) shows a fast moving opening crack in a thin viscoelastic slab (thickness d_0) under tension. The slab is assumed to be infinite long in the crack propagation direction. The slab is elongated by $d_0\epsilon_0$, and we wait until a fully relaxed state is formed before inserting the crack. Thus the elastic energy stored in the strip A of width Δx and volume $\Delta V = w d_0 \Delta x$ (where w is the width of the solid in the y -direction) is

$$U_0 = \frac{1}{2} \sigma_0 \epsilon_0 \Delta V = \frac{\sigma_0^2}{2E_0} \Delta V.$$

This energy is partly used to break the interfacial bonds and partly dissipated due to the material viscoelastic-

FIG. 3. Crack propagation in a viscoelastic slab clamped between two rigid flat surfaces. (a) The slab is infinite long ($L = \infty$) in the crack propagation direction (x -direction). (b) The slab is of finite length L . In (a) the segment (of width Δx) at A is stretched with the strain ϵ . At B the strain is reduced (high frequency relaxation) by $\Delta\epsilon = \sigma_0/E_1$ due to the abrupt decrease in the tensile stress from σ_0 to 0 when the crack tip pass the segment. At C, far away from the crack tip, the strain vanish due to (slow) viscoelastic relaxation.

ity. The crack propagation energy $G = U_0/(w\Delta x) = \sigma_0^2 d_0/(2E_0)$.

Consider now the slab Δx as it moves from one side of the crack to the other side. During this transition it will experience an (elongation) stress $\sigma(t)$ which for a very fast moving crack can be considered as a step function where $\sigma = \sigma_0$ for $t < 0$ and $\sigma = 0$ for $t > 0$, where $t = 0$ correspond to the case where the segment Δx is at the crack tip. The viscoelastic material will respond to this step-like change in the stress with its high frequency modulus E_1 so the strain in the segment Δx will abruptly drop by $\Delta\epsilon = \sigma_0/E_1$ as the crack pass the segment. The drop in the elastic energy

$$\Delta U = \frac{1}{2} \sigma_0 \Delta\epsilon \Delta V = \frac{\sigma_0^2}{2E_1} \Delta V,$$

is used to break the interfacial bonds (energy $\Delta\gamma = G_0$ per unit surface area), i.e. $G_0 = \Delta U/(w\Delta x)$ or

$$G_0 = \frac{\sigma_0^2}{2E_1} d_0 = \frac{\sigma_0^2 d_0}{2E_0} \frac{E_0}{E_1} = G \frac{E_0}{E_1},$$

so that $G = G_0 E_1/E_0$. The remaining elastic energy stored in the segment Δx ,

$$U = U_0 - \Delta U = \frac{1}{2} \sigma_0^2 \left(\frac{1}{E_0} - \frac{1}{E_1} \right) \Delta V,$$

is dissipated in the slow viscoelastic relaxation occurring far away from the crack tip so that finally the material reach its fully relaxed state (zero strain and stress).

FIG. 4. The real part of the viscoelastic modulus as a function of frequency (log-log scale) for a physically crosslinked polymer, which behaves as a fluid for small frequencies, corresponding to long times (schematic). The modulus E_0 in the rubbery region $\omega_a < \omega < \omega_b$ is approximately constant and much smaller than the modulus E_1 in the glassy region (high frequencies).

For a solid with a finite extent in all directions the fast relaxation process at the crack tip is the same as above so the result $G = G_0 E_1 / E_0$ still holds. However, for a fast moving crack the time it takes for the crack to fracture the whole interface, $\Delta t = L/v$, is so short that, in accordance with the discussion presented earlier, negligible viscous energy relaxation has occurred during the crack propagation act. Thus when the crack separate the two solids the viscoelastic solid is still in a strained state. Only after a possible long time period will it return to a strain (and stress) free state. Thus in this case the viscoelastic energy dissipation occur in a process separated from the actual crack propagation [see Fig. 3(b)], and this fact was overlooked in the earlier energy based discussions of finite size effects. We conclude there exist no finite size effects and the theory developed in Ref. [19–21] is not relevant for crack propagation in viscoelastic solids.

Application to “solids” with $E(\omega = 0) = 0$

Schapery applied his viscoelastic crack propagation theory to “solids”, with has a viscoelastic modulus that vanish for zero frequency (i.e., a relaxation modulus which vanish for long times)[9, 10]. Such a “solid” is really a liquid with a non-Newtonian and possible complex rheology, and in this case no rigorous crack propagation theory can be developed. Thus, for example, the Johnson, Kendall, and Roberts (JKR) adhesion theory[22] assumes that the solid deformations far from the crack tip (here the line separating the contact area from the non-contact area), is characterized by the low frequency (fully relaxed state) elastic modulus $E_0 = E(\omega = 0)$, which would vanish in the present case. We note, however,

that physically crosslinked polymer materials may have a rubbery plateau (see Fig. 4 and Ref. [23, 24]) for $\omega_a < \omega < \omega_b$, and behave as an elastic solid for all practical time scales $1/\omega_b < t < 1/\omega_a$, in which case the Persson-Brener (or cohesive-zone) theory could still be applied but with E_0 being the modulus in the rubbery plateau region.

If the adiabatic crack tip radius a_0 is treated as a constant as the low-frequency modulus E_0 is varied then the viscoelastic factor G/G_0 is independent of E_0 for low enough velocities and hence well-defined even in the limit $E_0 = 0$ (see Appendix B). Nevertheless, the quantities a_0 and G_0 are not well defined in the limit $E_0 = 0$ since they both refer to the adiabatic limit where the “solid” responds as a fluid, where no crack-like defects can occur.

Discussion

In a recent series of papers Ciavarella et al [25, 26] have criticized the Persson-Brener theory and claimed that the Persson-Brener theory[15] give a viscoelastic factor $G/G_0 = 1 + f$ which differ strongly from what Ciavarella et al denote as the exact viscoelastic factor obtained using the cohesive-zone model[11, 12]. We have shown above that both theories gives nearly the same result for G/G_0 if the ill-defined quantity σ_0 in the cohesive-zone theory is chosen appropriately!

In another paper Popov[27] claim that viscoelasticity increases the JKR pull-off force with a factor E_1/E_0 in the quasi-static limit. This result is incorrect: in the quasi-static case the work of adhesion is not influenced by the viscoelasticity and only in the limit $v \rightarrow \infty$ is the work of adhesion increased by the factor of E_1/E_0 . The result of Popov would hold if there would be no shortest length scale in the problem but in reality there is a short distance cut-off denoted by a_0 above.

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Ethics declarations

Conflict of interest

The authors declare that they have no conflicts of interest.

Appendix A

Here we give the equations for viscoelastic crack propagation used in the calculations in Fig. 2.

When a strip of a viscoelastic material is exposed to an oscillating strain $\epsilon(\omega)\exp(-i\omega t)$ the amplitude of the oscillating stress

$$\sigma(\omega) = E(\omega)\epsilon(\omega)$$

This equation define the viscoelastic modulus $E(\omega)$, which is a complex quantity, where the imaginary part is

related to energy dissipation (transfer of mechanical energy into the disordered heat motion). In the calculation in Fig. 2 we used the three-element rheological model illustrated in Fig. 1. For this model the viscoelastic modulus

$$E = \frac{E_0 E_1 (1 - i\omega\tau)}{E_1 - i\omega\tau E_0} \quad (A1)$$

The ratio between the high frequency and low frequency modulus, E_1/E_0 , is typically very large, e.g., ~ 1000 .

In the Persson-Brener theory the crack propagation energy $G(v) = G_0 a(v)/a_0$, where $a(v)$ is the velocity dependent effective crack tip radius, and where $a_0 = E_0 G_0 / (2\pi\sigma_c^2)$. The fracture energy $G(v)$ satisfy[15]

$$\frac{G_0}{G} = 1 - E_0 \frac{2}{\pi} \int_0^{\omega_c} d\omega F(\omega) \frac{1}{\omega} \text{Im} \frac{1}{E(\omega)} \quad (A2)$$

where

$$F(\omega) = \left[1 - \left(\frac{\omega}{\omega_c} \right)^2 \right]^{1/2}$$

where $\omega_c = 2\pi v/a$. For numerical calculations it is more convenient to reformulate (A2) into[6, 19]

$$\frac{G_0}{G} = 1 - \frac{E_1 \frac{2}{\pi} \int_0^{\omega_c} d\omega \frac{1}{\omega} F(\omega) \text{Im} \frac{1}{E(\omega)}}{1 + E_1 \frac{2}{\pi} \int_0^{\infty} d\omega \frac{1}{\omega} \text{Im} \frac{1}{E(\omega)}}. \quad (A3)$$

Since ω_c depends on a (and hence on G since $G = G_0 a/a_0$) this is an implicit equation for $G(v)$. Thus the theory gives both crack propagation energy $G(v)$ and the (velocity-dependent) radius of the crack tip,

$$a(v) = a_0 \frac{G(v)}{G_0} = \frac{E_0 G}{2\pi\sigma_c^2}. \quad (A4)$$

Since $E(\omega)$ typically varies with ω over very many decades in frequency, for the numerical evaluation of the integrals in (A3) it is convenient to write (see Ref. [6]) $\omega = \omega_0 e^x$, so that if ω varies over ~ 30 decades, x varies only by a factor ~ 100 .

In the cohesive-zone model assuming that the stress is constant and equal to σ_0 for $0 < u < h_0$ and $\sigma = 0$ for $u > h_0$, and assuming the viscoelastic modulus (A1), one get[12]:

$$\frac{G_0}{G} = \frac{E_0}{E_1} + \frac{1}{2} \left(1 - \frac{E_0}{E_1} \right) \alpha \int_0^1 dx H(x) e^{-\alpha(1-x)}, \quad (A5)$$

where $\alpha = l/(v\tau)$ and

$$H(x) = 2x^{1/2} - (1-x) \ln \left(\frac{1+x^{1/2}}{1-x^{1/2}} \right),$$

and the width l of the crack tip process zone

$$l = \frac{\pi E_0 G}{4 \sigma_0^2} \quad (A6)$$

FIG. 5. The viscoelastic factor G/G_0 as a function of the crack tip speed (log-log scale) for the case $E_1 = 100$, $\tau = 1$ and $E_0 = 1$ (green curve), $E_0 = 0.1$ (blue curve) and $E_0 = 0$ (red dashed curve). In the calculation we have used the adiabatic crack tip radius $a_0 = 0.03$.

Here we have assumed plain stress; for plain strain E must be replaced with $E/(1-\nu^2)$, where ν is the Poisson ratio. Since l depends on G , (A5) is an implicit equation for $G(v)$.

Appendix B

Using the viscoelastic modulus (A1), which we can also write as

$$\frac{1}{E} = \frac{1}{E_1} + \left(\frac{1}{E_0} - \frac{1}{E_1} \right) \frac{1}{1 - i\omega\tau},$$

in (A2) and denoting $\omega = \omega_c x$ gives

$$\frac{G_0}{G} = 1 - \left(1 - \frac{E_0}{E_1} \right) \frac{2}{\pi} \int_0^1 dx (1-x^2)^{1/2} \frac{\xi}{1 + \xi^2 x^2},$$

where $\xi = 2\pi v\tau/a = 2\pi[v\tau/a_0]G_0/G$. Hence G/G_0 depends only on $v\tau/a_0$ and on E_0/E_1 and is well-defined also in the fluid limit $E_0 = 0$.

Fig. 5 shows the viscoelastic factor G/G_0 as a function of the crack tip speed on a log-log scale. We have used $E_1 = 100$, $\tau = 1$ and $E_0 = 1$ (green curve), $E_0 = 0.1$ (blue curve) and $E_0 = 0$ (red dashed curve), and the adiabatic crack tip radius $a_0 = 0.03$. Note that for low crack tip speed G/G_0 is independent of E_0 . In fact, for $E_0 = 0$ the curve extend linearly (on the log-log scale) towards infinite v with the slope $1/2$ corresponding to $G \sim v^{1/2}$ (dashed line in the figure). For a more realistic viscoelastic modulus the exponent α in $G \sim v^\alpha$ will be different from $1/2$ as discussed elsewhere[15].

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Figures

Figure 1

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Figure 2

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Figure 3

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Figure 4

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Figure 5

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