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On Fuzzy Contractive Mappings in Fuzzy Metric Spaces

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Abstract

In this paper we introduce a new fuzzy contraction mapping and prove that such mappings have fixed point in τ -complete fuzzy metric spaces. As an application, we shall utilize the results obtained to show the existence and uniqueness of random solution for the following random linear random operator equation. Moreover, we shall show that the existence and uniqueness of the solutions for nonlinear Volterra integral equations on a kind of particular fuzzy metric space.

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Keywords: random fixed point; random operator; random operator equation; Contractive mapping; fixed point; t -norm; fuzzy metric space; Non-Archimedean fuzzy metric space.

1 INTRODUCTION

The concept of fuzzy metric spaces was obtained in different ways [7, 13, 14, 24–26]. To obtain a Hausdorff topology of fuzzy metric spaces, [7, 8, 13] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [14] and showed that every ordinary metric induces a fuzzy metric in the sense of George and Veeramani [7].

Fixed point theorems play a fundamental role in demonstrating the existence of solutions to a wide variety of problems arising in mathematics, physics, engineering, medicine and social sciences. The study of fixed point theorems in fuzzy mathematics was instigated by Weiss [32], Butnariu [3], Singh and Talwar [30], Estruch and Vidal [5], Wang et al. [33], Mihet [22], Qiu et al. [23] and Beg and Abbas [2]. Heilpern [11] introduced the concept of fuzzy contraction mappings and established the fuzzy Banach contraction principle on a complete metric linear spaces with the d_∞ -metric for fuzzy sets. Azam and Beg [1] proved common fixed point theorems for a pair of fuzzy mappings satisfying Edelstein, Alber and Guerr-Delabriere type contractive conditions in ametric linear space. Rashid [25] give a generalization of Hicks type contractions and SB-type contractions in fuzzy metric spaces. He proved some fixed point theorems for these new type contraction mappings on fuzzy metric spaces.

In this paper we introduce a new fuzzy contraction mapping and prove that such mappings have fixed point in complete fuzzy metric spaces. As an application, we shall utilize the results obtained to show the existence and uniqueness of random solution for the following random linear random operator equation. Moreover, we shall show that the existence and uniqueness of the solutions for nonlinear Volterra integral equations on a kind of particular fuzzy metric space. The structure of this paper is as follows. In Section 2, we recall some definitions and the uniform structure of fuzzy metric spaces. In Section 3, we give some fixed point theorems for several

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contraction type mappings in FM-spaces and we shall utilize the results obtained to show the existence and uniqueness of random solution for the following random linear random operator equation. Section 4 is devoted to give some common fixed point theorems for Contraction type mappings. Our results generalize and extend many known results in fuzzy metric spaces and probabilistic metric spaces, see [13, 18, 22, 24, 25, 31, 32].

2 PRELIMINARIES

First we recall some of the basic concepts, which will be used in the sequel.

Definition 1. [12, 27] Let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$. Then T is said to be t -norm if and only if for all $x, y, z \in [0, 1]$, we have

- (T1) $T(x, y) = T(y, x)$ (commutativity),
- (T2) $T(x, y) \leq T(x, z)$, if $y \leq z$ (monotonicity),
- (T3) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
- (T4) $T(x, 1) = x$.

Definition 2. [27] A binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t -norm if $([0, 1], T)$ is a topological monoid with unit 1 such that $T(a, b) \leq T(c, d)$ whenever $a \leq c, b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 3. [14] A triple (X, M, T) is called a fuzzy metric space (briefly, a FM-space) if X is an arbitrary (non-empty) set, T is a continuous t -norm and M is a fuzzy set on $X \times X \times [0, \infty)$ such that the following axioms hold:

- (FM-1) $M(x, y, 0) = 0$ for all $x, y \in X$,
- (FM-2) $M(x, y, t) = 1$ for every $t > 0$ if and only if $x = y$,
- (FM-3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$,
- (FM-4) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous for all $x, y \in X$,
- (FM-5) $M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s))$ for all $x, y, z \in X$ and for all $t, s \in [0, \infty)$.

We will refer to the fuzzy metric spaces in the sense of Kramosil and Michalek as KM-fuzzy metric spaces. If, in the above definition, the condition (FM-5) is replaced by the condition

- (FM-5A) $M(x, z, \max\{t, s\}) \geq T(M(x, y, t), M(y, z, s))$ for all $x, y, z \in X$ and for all $t, s \in [0, \infty)$,

then (X, M, T) is called a non-Archimedean fuzzy metric space. It is easy to check that (FM-5A) implies (FM-5), that is, every a non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 1. Let (X, d) be an ordinary metric space and let β be a nondecreasing and continuous function from $(0, \infty)$ into $(0, 1)$ such that $\lim_{t \rightarrow \infty} \beta(t) = 1$. Some examples of these functions are $\beta(t) = \frac{t}{t+1}$, $\beta(t) = 1 - e^{-t}$, and $\beta(t) = e^{-\frac{1}{t}}$. Let $T(a, b) \leq ab$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define $M(x, y, t) = [\beta(t)]^{d(x, y)}$ for all $x, y \in X$. It is easy to see that is a non-Archimedean fuzzy metric space.

Definition 4. [7] A triple (X, M, T) is called a fuzzy metric space (briefly, a FM-space) if X is an arbitrary (non-empty) set, T is a continuous t -norm and M is a fuzzy set on $X \times X \times [0, \infty)$ such that the following axioms hold:

- (FM-1) $M(x, y, t) > 0$ for all $x, y, z \in X$,

(FM-2) $M(x, y, t) = 1$ for every $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$,

(FM-4) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous for all $x, y \in X$,

(FM-5) $M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s))$ for all $x, y, z \in X$ and for all $t, s \in [0, \infty)$.

We will refer to the fuzzy metric spaces in the sense of George and Veeramani as GV-fuzzy metric spaces.

Let (X, M, Δ) be a fuzzy metric space. The open ball $B_M(x, r, t)$ for $t > 0$ with centre $x \in X$ and radius r , $0 < r < 1$ is defined by:

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1r\}$$

A subset A of a fuzzy metric space (X, M, Δ) is said to be open if given any point $x \in A$, there exists $0 < r < 1$, and $t > 0$ such that $B(x, r, t) \subseteq A$.

The family :

$$\tau_M = \{A \subset X : x \in A \text{ if only if there exist } t > 0 \text{ and } 0 < r < 1 \text{ such that } B(x, r, t) \subset A\}$$

is a topology on X , τ_M is called the topology on X induced by the fuzzy metric M . (X, τ_M) is a Hausdorff first countable topological space (see [7]).

Definition 5. ([27]) Let (X, M) be a fuzzy metric space. Then

- (i) A sequence $\{x_n\}$ in X is said to be τ -convergent to $x \in X$, denoted it by $x_n \xrightarrow{\tau} x$, if for any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $N_0 = N_0(\epsilon, \lambda)$ such that

$$M(x_n, x, \epsilon) > 1 - \lambda,$$

whenever $n \geq N_0$.

- (ii) A sequence $\{x_n\}$ in X is said to be τ -Cauchy if for any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $N_0 = N_0(\epsilon, \lambda)$ such that

$$M(x_n, x_m, \epsilon) > 1 - \lambda,$$

whenever $n, m \geq N_0$.

- (iii) (X, M) is said to be complete if every τ -Cauchy sequence in X is τ -convergent to some point in X .

Definition 6 ([27]). A mapping $T : X \rightarrow X$ is said to be τ -continuous if, for any sequence $\{x_n\}$ in X such that $x_n \xrightarrow{\tau} x$, $Tx_n \xrightarrow{\tau} Tx$.

Definition 7 ([25]). Let (X, M, Δ) be a fuzzy metric space with continuous t -norm Δ . A mapping $T : X \rightarrow X$ is a contraction mapping (or a SB -contraction mapping) if and only if there is an $\mu \in (0, 1)$ such that

$$(1) \quad M(Tx, Ty, t) \geq M\left(x, y, \frac{t}{\mu}\right) \quad \text{for all } x, y \in X \quad t > 0.$$

Proposition 1 ([25]). Let (X, M, Δ) be a fuzzy metric space with continuous t -norm Δ . Let $T : X \rightarrow X$ be a contraction mapping satisfying condition 1. Then either

- (i) T has a unique fixed point, or
- (ii) For every $p_0 \in X$, $\sup\{G_{p_0}(t) : t \geq 0\} < 1$, where

$$G_{p_0}(t) = \inf\{M(p_0, p_m, t) : p_m = Tp_{m-1}, m = 1, 2, \dots\}.$$

3 Fixed point theorems for Contraction type mappings

In this section, we give some fixed point theorems for several contraction type mappings in FM-spaces.

Definition 8 ([27]). A set A in FM-space (X, M, Δ) is said to be fuzzy bounded if

$$(2) \quad \sup_{s>0} \sup_{t<s} \inf_{p,q \in A} M(p, q, t) = 1.$$

Clearly, it follows that the condition (2) is equivalent to the following condition:

$$\sup_{s>0} \inf_{p,q \in A} M(p, q, t) = 1.$$

In what follows, we shall adopt the following notations:

Let (X, M) be a FM-space and T be a self-mapping on (X, M) . For any $x, y \in X$ and positive integer i , we denote

- (i) $O_T(x; 0, \infty) = \{T^n x\}_{n=0}^\infty$,
- (ii) $O_T(x; i, \infty) = \{T^n x\}_{n=i}^\infty$,
- (iii) $O_T(x, y; 0, \infty) = O_T(x; 0, \infty) \cup O_T(y; 0, \infty)$,
- (iv) $O_T(x, y; i, \infty) = O_T(x; i, \infty) \cup O_T(y; i, \infty)$, respectively,

and $O_T(x; 0, \infty) = \{T^n x\}_{n=0}^\infty$ is called the orbit of x under T .

Definition 9 ([27]). A function $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to satisfy the condition (Φ) if it is strictly increasing, $\Phi(0) = 0$, and $\lim_{n \rightarrow \infty} \Phi^n(t) = \infty$ for all $t > 0$, where $\Phi^n(t)$ denotes the n -th iteration of $\Phi(t)$.

Lemma 1 ([27]). Let $\Phi(t)$ satisfy the condition (Φ) . Then

- (a) $\Phi(t) > t$ for all $t > 0$,
- (b) for any positive integer $n = 1, 2, \dots$,

$$\Phi^n(t) > \Phi^{n-1}(t) > \dots > \Phi(t) > t, \quad t > 0.$$

Throughout this section, we always assume that (X, M, Δ) is a τ -complete Menger FM-space, Δ is a continuous t -norm, and $\Phi(t)$ is a function satisfying the condition (Φ) .

Theorem 1. Let T be a τ -continuous self-mapping on (X, M, Δ) . Suppose that for each $x \in X$, the orbit $O_T(x; 0, \infty)$ is fuzzy bounded, and that for any $x \in X$, there exists a positive integer $m(x)$ such that, for all $t \geq 0$,

$$(3) \quad \inf_{p,q \in O_T(x; m(x), \infty)} M(p, q, t) \geq \inf_{p,q \in O_T(x; 0, \infty)} M(p, q, \Phi(t)).$$

Then T has a fixed point in X , and for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = T^n x_0$, $n \in \mathbb{N}$, τ -converges to a fixed point of T .

Proof. For any $x_0 \in X$, by assumptions, there exists a positive integer $m(x_0)$ such that

$$\inf_{p,q \in O_T(x; m(x_0), \infty)} M(p, q, t) \geq \inf_{p,q \in O_T(x_0; 0, \infty)} M(p, q, \Phi(t)), \quad t \geq 0.$$

Now we choose a sequence $\{m(j)\}$ of positive integers defined as follows:

$$m(0) = m(x_0), m(j+1) = m(j) + m(x_{m(j)}), \quad j = 0, 1, \dots,$$

where $x_{m(j)} = T^{m(j)}x_0$.

By the induction, we can prove the following inequality holds:

$$\begin{aligned}
(4) \quad \inf_{p,q \in O_T(x_0; m(j+1), \infty)} M(p, q, t) &\geq \inf_{p,q \in O_T(x_0; m(j), \infty)} M(p, q, \Phi(t)) \\
&\geq \dots \\
&\geq \inf_{p,q \in O_T(x_0; m(0), \infty)} M(p, q, \Phi^{j+1}(t)) \\
&\geq \inf_{p,q \in O_T(x_0; 0, \infty)} M(p, q, \Phi^{j+2}(t)) \\
&\geq \sup_{s < \Phi^{j+2}(t)} \inf_{p,q \in O_T(x_0; 0, \infty)} M(p, q, s).
\end{aligned}$$

By using Lemma 1

$$\lim_{j \rightarrow \infty} \inf_{p,q \in O_T(x_0; m(j+1), \infty)} M(p, q, t) = H(t), \quad t \geq 0,$$

which implies that $\{x_n\}$ is a τ -Cauchy sequence in X . By the τ -completeness of (X, M, Δ) , we can suppose $x_n \xrightarrow{\tau} x_* \in X$. By the τ -continuity of T , it is easy to see that x_* is a fixed point of T . This completes the proof. $\blacksquare$

Theorem 2. Let T be a τ -continuous self-mapping on (X, M, Δ) . Suppose that for each $x \in X$, the orbit $O_T(x; 0, \infty)$ is fuzzy bounded, and suppose that there exists a positive integer m such that, for any $x \in X$, the following holds:

$$(5) \quad \inf_{p,q \in O_T(x; m, \infty)} M(p, q, t) \geq \inf_{p,q \in O_T(x; 0, \infty)} M(p, q, \Phi(t)), \quad t \geq 0.$$

Then the conclusion of Theorem 1 still holds.

Lemma 2. Let T be a τ -continuous self-mapping on (X, M, Δ) . Then the following conditions are equivalent:

(i) There exists a positive integer m such that, for any $x \in X$,

$$\inf_{p,q \in O_T(x; m, \infty)} M(p, q, t) \geq \inf_{p,q \in O_T(x; 0, \infty)} M(p, q, \Phi(t)), \quad t \geq 0.$$

(ii) There exists a positive integer m such that, for any $x \in X$ and any non-negative integer k ,

$$M(T^m x, T^{m+k} x, t) \geq \inf_{p,q \in O_T(x; 0, \infty)} M(p, q, \Phi(t)), \quad t \geq 0.$$

Proof. The proof is straightforward and so is omitted. $\blacksquare$

Theorem 3. Let T be a τ -continuous self-mapping on (X, M, Δ) . Suppose that for each $x \in X$, the orbit $O_T(x; 0, \infty)$ is fuzzy bounded, and that the condition (i) or (ii) in Lemma 2 is satisfied. Then the conclusion of Theorem 1 still holds.

Theorem 4. Let T be a τ -continuous self-mapping on (X, M, Δ) . Suppose that there exist positive integers m, n such that for any $x, y \in X$ and any $t \geq 0$,

$$(6) \quad M(T^m x, T^n x, t) \geq \inf_{p,q \in O_T(x, y; 0, \infty)} M(p, q, \Phi(t)), \quad t \geq 0.$$

Then T has a unique fixed point in X , and for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = T^n x_0$, $n = 1, 2, \dots$, τ -converges to this fixed point.

Proof. Without loss of generality, we can assume that $m \geq n$. For any $x \in X$ and a non-negative integer k , put $y = T^{m-n+k}x$. It follows from (6) that, for all $t \geq 0$, we have

$$\begin{aligned} M(T^m x, T^{m+k} x, t) &\geq \inf_{p, q \in O_T(x, T^{m-n+k} x; 0, \infty)} M(p, q, \Phi(t)) \\ &= \inf_{p, q \in O_T(x; 0, \infty)} M(p, q, \Phi(t)). \end{aligned}$$

By Theorem 4, T has a fixed point in X , and for any $x_0 \in X$, the sequence $\{x_n = T^n x_0\}$ τ -converges to a fixed point x_* of T .

Suppose that y_* is another fixed point of T . Then for all $t \geq 0$, we have

$$\begin{aligned} M(x_*, y_*, t) &= M(T^m x_*, T^n y_*, t) \geq \inf_{p, q \in O_T(x_*, y_*; 0, \infty)} M(p, q, \Phi(t)) \\ &= M(x_*, y_*, \Phi(t)). \end{aligned}$$

Repeating the above procedure by the condition (Φ) , we obtain

$$M(x_*, y_*, t) \geq M(x_*, y_*, \Phi^n(t)) \rightarrow H(t), \quad n \rightarrow \infty$$

which implies that $x_* = y_*$. This completes the proof. $\blacksquare$

Theorem 5. Let (X, M, Δ) be a τ -complete Menger FM-space, Δ be a continuous t -norm satisfying the condition $\Delta(t, t) \geq t$ for all $t \in [0, 1]$. Suppose $T : X \rightarrow X$ is a contraction mapping, i.e., there exists a constant $k \in (0, 1)$ such that, for all $t \geq 0$ and $x, y \in X$,

$$(7) \quad M(Tp, Tq, t) \geq M\left(p, q, \frac{t}{k}\right).$$

Then T has a unique fixed point x_* , and for each $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = T^n x_0$, $n = 1, 2, \dots$, τ -converges to the point x_* in X .

Proof. Take $\Phi(t) = \frac{t}{k}$, $t \geq 0$, and then it is easy to see that $\Phi(t)$ satisfies the condition (Φ) . Since T is a contraction mapping, obviously, T is τ -continuous. We now prove that under the condition (7) and $\Delta(t, t) \geq t$, $t \in [0, 1]$, for each $x \in X$, $O_T(x; 0, \infty)$ is fuzzy bounded.

In fact, for each $x_0 \in X$ and any $\lambda \in (0, 1)$, by the property of distribution function, there exists a number $t_1 = t_1(\lambda) > 0$ such that for all $t \geq t_1$,

$$(8) \quad M(x_0, T x_0, t) > 1 - \lambda.$$

If $k_1 \in (k, 1)$, for any positive integer $n > 1$, it follows from (7) that

$$\begin{aligned} (9) \quad M(x_0, T^n x_0, t) &\geq \Delta(M(x_0, T x_0, (k_1 - k)t), M(T x_0, T^n x_0, kt)) \\ &\geq \Delta(M(x_0, T x_0, (k_1 - k)t), M(x_0, T^{n-1} x_0, t)) \\ &\geq \min\{M(x_0, T x_0, (k_1 - k)t), M(x_0, T^{n-1} x_0, t)\}. \end{aligned}$$

From (9), we have

$$(10) \quad M(x_0, T^{n-1} x_0, t) \geq \min\{M(x_0, T x_0, (k_1 - k)k_1^{-1}t), M(x_0, T^{n-2} x_0, k_1^{-1}t)\}.$$

Substituting (10) into (9), simplifying the expression and noting $k_1^{-1} > 1$, we have

$$(11) \quad M(x_0, T^n x_0, k_1 t) \geq \min\{M(x_0, T x_0, (k_1 - k)t), M(x_0, T^{n-2} x_0, t)\}.$$

Repeating the above procedures, we have

$$(12) \quad M(x_0, T^n x_0, k_1 t) \geq \min\{M(x_0, T x_0, (k_1 - k)t), M(x_0, x_0, t)\} = M(x_0, T x_0, (k_1 - k)t).$$

Therefore, we have

$$(13) \quad M(x_0, T^n x_0, t) \geq M(x_0, T x_0, (k_1 - k)k_1^{-1}t), \quad n > 1.$$

Combining (8) and (13), since $t \geq \max\{t_1, \frac{k_1 t_1}{k_1 - k}\} = \frac{k_1 t_1}{k_1 - k}$, we have $M(x_0, T^n x_0, t) > 1 - \lambda$ for all $n = 0, 1, 2, \dots$. Hence we have

$$(14) \quad \inf_{q \in O_T(x_0; 0, \infty)} M(x_0, q, t) > 1 - \lambda$$

for all $t \geq \frac{k_1 t_1}{k_1 - k}$. In view of arbitrariness of $\lambda \in (0, 1)$, we have

$$(15) \quad \sup_{t > 0} \inf_{q \in O_T(x_0; 0, \infty)} M(x_0, q, t) = 1.$$

However, for any given $p_0, q_0 \in O_T(x; 0, \infty)$, we have

$$(16) \quad \begin{aligned} M(p_0, q_0, t) &\geq \Delta \left(M \left(p_0, x_0, \frac{t}{2} \right), M \left(x_0, q_0, \frac{t}{2} \right) \right) \\ &\geq \Delta \left(\inf_{q \in O_T(x_0; 0, \infty)} M \left(p_0, x_0, \frac{t}{2} \right), \inf_{q \in O_T(x_0; 0, \infty)} M \left(x_0, q_0, \frac{t}{2} \right) \right) \\ &\geq \inf_{q \in O_T(x_0; 0, \infty)} M \left(p_0, q, \frac{t}{2} \right). \end{aligned}$$

By the arbitrariness of $p_0, q_0 \in O_T(x_0; 0, \infty)$, it follows from (15) and (16) that

$$\sup_{t > 0} \inf_{p, q \in O_T(x_0; 0, \infty)} M(p, q, t) \geq \sup_{t > 0} \inf_{q \in O_T(x_0; 0, \infty)} M \left(x_0, q, \frac{t}{2} \right) = 1.$$

This shows that, for any $x_0 \in X$, the orbit $O_T(x_0; 0, \infty)$ is fuzzy bounded. Thus the conclusion of Theorem 5 can be obtained from Theorem 4. This completes the proof. $\blacksquare$

Theorem 6. Let (X, M, Δ) be a τ -complete Menger FM-space with a t -norm Δ satisfying $\sup_{0 \leq t \leq 1} \Delta(t, t) = 1$. Let $T : X \rightarrow X$ be a mapping satisfying the following conditions:

- (i) for each $x \in X$, $\sup_{t \in \mathbb{R}} G_x(t) = 1$, where $G_x(t) = \inf_{n \in \mathbb{Z}^+} M(x, T^n x, t)$,
- (ii) for each $x \in X$, there exists a positive integer $m(x) \in \mathbb{Z}^+$ such that for any $y \in X$ and $t \in \mathbb{R}^+$,

$$(17) \quad M(T^{m(x)}x, T^{m(x)}y, t) \geq \min\{M(x, y, \Phi(t)), M(x, T^{m(x)}x, \Phi(t)), M(x, T^{m(x)}y, \Phi(t))\},$$

where $\Phi(t)$ is a left-continuous function satisfying the condition (Φ) . Then

- (a) for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = T^{m(x_{n-1})}x_{n-1}$, $n = 1, 2, \dots$, τ -converges to some point x_* in X ,
- (b) In addition, if there exists $t^* \in \mathbb{R}^+$ such that $M(x_*, T^{m(x_*)}x_*, t^*) = 1$, then x_* is the unique fixed point of T in X , and the iterative sequence $\{T^n x_0\}$ τ -converges to x_* .

Proof. (a) We prove that $\{x_n\}$ is a τ -Cauchy sequence in X . Letting $m(x_i) = m_i$, $i = 1, 2, \dots$, for any $n \in \mathbb{Z}^+$, by (17), we have

$$\begin{aligned} M(x_i, T^n x_i, t) &\geq \min\{M(x_{i-1}, T^n x_{i-1}, \Phi(t)), M(x_{i-1}, T^{m_{i-1}+n}x_{i-1}, \Phi(t)), \\ &\quad M(x_{i-1}, T^{m_{i-1}+n}x_{i-1}, \Phi(t))\} \\ &\geq G_{x_{i-1}}(\Phi(t)). \end{aligned}$$

Therefore, $G_{x_i}(t) \geq G_{x_{i-1}}(\Phi(t)) \geq \dots \geq G_{x_0}(\Phi(t))$ and so we have

$$(18) \quad M(x_i, x_{i+j}, t) = M(x_i, T^{m_{i+j-1}+\dots+m_i} x_i, t) \geq G_{x_i}(t) \geq G_{x_0}(\Phi(t)).$$

Since the distribution function is nondecreasing, $\sup_{t \in \mathbb{R}} G_{x_0}(t) = 1$ and $\Phi^i(t) \rightarrow \infty$ as $i \rightarrow \infty$, for all $t > 0$, we know that $G_{x_0}(\Phi^i(t)) \rightarrow 1$ as $i \rightarrow \infty$. This implies that $\{x_n\}$ is a τ -Cauchy sequence in X . Since X is τ -complete, let $x_n \rightarrow x_*$ in X .

(b) First we prove that x_* is a fixed point of $T^{m(x_*)}$. Letting $m(x_*) = m_*$, by the assumption, there exist $t_* \in \mathbb{R}$ such that $M(x_*, T^{m_*} x_*, t_*) = 1$. Denote

$$(19) \quad t_0 = \inf\{t : M(x_*, T^{m_*} x_*, t) = 1\}.$$

It is obvious that $t_0 \leq t_*$. Next we prove that $t_0 = 0$. In fact, if $t_0 > 0$, by the left-continuity of Φ , there exist $t_1, t_2 \in \mathbb{R}^+$, $0 < t_2 < t_1 < t_0$, such that $\Phi(t_2) > t_0$. From (19), it follows that

$$M(x_*, T^{m_*} x_*, t_1) < 1, \quad M(x_*, T^{m_*} x_*, \Phi(t_2)) = 1.$$

On the other hand, it follows from

$$\begin{aligned} M(x_*, T^{m_*} x_*, \Phi(t_2)) &\geq \Delta(M(x_*, x_i, \Phi(t_2) - t_0), M(x_i, T^{m_*} x_i, t_0)) \\ &\geq \Delta(M(x_*, x_i, \Phi(t_2) - t_0), G_{x_0}(\Phi^i(t_0))) \\ &\rightarrow 1 \quad (i \rightarrow \infty) \end{aligned}$$

that

$$\begin{aligned} M(T^{m_*} x_i, T^{m_*} x_*, t_2) &\geq \min\{M(x_*, x_i, \Phi(t_2)), M(x_*, T^{m_*} x_*, \Phi(t_2)), M(x_*, T^{m_*} x_i, \Phi(t_2))\} \\ &\rightarrow 1 \quad (i \rightarrow \infty). \end{aligned}$$

Since

$$\begin{aligned} M(T^{m_*} x_i, T^{m_*} x_*, t_1 - t_2) &\geq \Delta\left(M\left(x_*, x_i, \frac{t_1 - t_2}{2}\right), G_{x_0}\left(\Phi^i\left(\frac{t_1 - t_2}{2}\right)\right)\right) \\ &\rightarrow 1 \quad (i \rightarrow \infty), \end{aligned}$$

we have

$$\begin{aligned} M(T^{m_*} x_i, T^{m_*} x_*, t_1) &\geq \Delta(M(x_*, T^{m_*} x_*, \Phi(t_2)), M(T^{m_*} x_i, T^{m_*} x_*, t_2)) \\ &\rightarrow 1 \quad (i \rightarrow \infty), \end{aligned}$$

which contradicts $M(x_*, T^{m_*} x_*, t_1) < 1$. Hence we have $t_0 = 0$ and so $M(x_*, T^{m_*} x_*, t) = 1$ for all $t > 0$, i.e., $T^{m_*} x_* = x_*$.

Next we prove that x_* is the unique fixed point of T^{m_*} . Then, for any $t > 0$, we have

$$\begin{aligned} M(x_*, y_*, t) &= M(T^{m_*} x_*, T^{m_*} y_*, t) \\ &\geq \min\{M(x_*, y_*, \Phi(t)), M(x_*, x_*, \Phi(t)), M(x_*, y_*, \Phi(t))\} \\ &= M(x_*, y_*, \Phi(t)) \\ &\geq \dots \\ &\geq M(x_*, y_*, \Phi^i(t)) \\ &\rightarrow 1 \quad (i \rightarrow \infty), \end{aligned}$$

i.e., $x_* = y_*$. Thus, the fixed point of T^{m_*} is unique.

Finally, we prove that x_* is also the unique fixed point of T and $T^n x_0 \xrightarrow{\tau} x_*$. In fact, since $T^{m_*} x_* = x_*$, $T^{m_*} T x_* = T x_*$. Noting that x_* is the unique fixed point of T^{m_*} , thus we have

$x_* = Tx_*$. The uniqueness of the fixed point x_* is obvious.

For any $n \in \mathbb{Z}^+$, $n > m_*$, we may write it by $n = km_* + s$, $0 \leq s < m_*$. For any $t > 0$, by (17), we have, for all $t > 0$,

$$\begin{aligned} M(x_*, T^n x_0, t) &= M(T^{m_*} x_*, T^{m_*k+s} x_0, t) \\ &\geq \min\{M(x_*, T^{(k-1)m_*+s} x_*, \Phi(t)), M(x_*, T^n x_0, \Phi(t))\} \\ &\geq \dots \\ &\geq \min\{M(x_*, T^{(k-1)m_*+s} x_0, \Phi(t)), M(x_*, T^n x_0, \Phi^i(t))\} \\ &\rightarrow 1 \quad (k \rightarrow \infty). \end{aligned}$$

Letting $i \rightarrow \infty$, we obtain, for all $t > 0$,

$$\begin{aligned} M(x_*, T^n x_0, t) &\geq M(x_*, T^{(k-1)m_*+s} x_0, \Phi(t)) \\ &\geq \dots \\ &\geq M(x_*, T^s x_0, \Phi^k(t)) \\ &\rightarrow 1 \quad (k \rightarrow \infty). \end{aligned}$$

This implies that $T^n x_0 \rightarrow x_*$ as $k \rightarrow \infty$. This completes the proof. $\blacksquare$

Theorem 7. Let (X, M, Δ) be a τ -complete Menger FM-space and the t -norm satisfy the condition that for any $t_0 \in (0, 1]$ is continuous at $t = 1$. Let $T : X \rightarrow X$ be a mapping satisfying the following conditions:

(i) for each $x \in X$, $\sup_{t \in \mathbb{R}} G_x(t) = 1$,

(ii) for each $x \in X$, there exists $m(x) \in \mathbb{Z}^+$ such that for all $y \in X$ and $t \in \mathbb{R}^+$,

$$M(T^{m(x)} x, T^{m(x)} y, t) \geq \min \left\{ M \left(x, y, \frac{t}{k} \right), M \left(x, T^{m(x)} x, \frac{t}{k} \right), M \left(x, T^{m(x)} y, \frac{t}{k} \right) \right\},$$

where $k \in (0, 1)$ is a constant.

Then T has a unique fixed point $x_* \in X$, and for any $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ τ -converges to the fixed point x_* .

Proof. First we note that, under the assumption on Δ , $\sup_{0 < t < 1} \Delta(t, t) = 1$. Letting $\Phi(t) = t/k$, then it satisfies the conditions in Theorem 6.

Next, we prove that, for all $t > 0$,

$$M(x_*, T^{m_*} x_*, t) = 1.$$

In fact, for any $t > 0$, we have

$$M(x_*, T^{m_*} x_i, t) \geq \Delta \left(M \left(x_*, x_i, \frac{t}{2} \right), G_{x_0} \left(\frac{t}{2k^i} \right) \right) \rightarrow 1 \quad (i \rightarrow \infty).$$

Taking $k_1 \in (k, 1)$, we have $0 < \frac{k}{k_1} < 1$ and

$$M \left(T^{m_*} x_i, T^{m_*} x_*, \frac{kt}{k_1} \right) \geq \min \left\{ M \left(x_*, x_i, \frac{t}{k_1} \right), M \left(x_*, T^{m_*} x_*, \frac{t}{k_1} \right), M \left(x_*, T^{m_*} x_i, \frac{t}{k_1} \right) \right\}$$

for all $t > 0$. Since $M(x_*, x_i, t/k_1) \rightarrow 1$ and $M(x_*, T^{m_*} x_i, t/k_1) \rightarrow 1$ for all $t > 0$ ($i \rightarrow \infty$), there exists $n \in I^+$ such that, for $i > N_0$, we have

$$M \left(T^{m_*} x_i, T^{m_*} x_*, \frac{kt}{k_1} \right) \geq M \left(x_*, T^{m_*} x_*, \frac{t}{k_1} \right), \quad t > 0.$$

On the other hand, we have

$$\begin{aligned} M(x_*, T^{m_*} x_*, t) &\geq \Delta \left(M \left(x_*, T^{m_*} x_i, \left(1 - \frac{k}{k_1} \right) t \right), M \left(T^{m_*} x_i, T^{m_*} x_*, \frac{kt}{k_1} \right) \right) \\ &\geq \Delta \left(M \left(x_*, T^{m_*} x_i, \left(1 - \frac{k}{k_1} \right) t \right), M \left(x_*, T^{m_*} x_*, \frac{t}{k_1} \right) \right). \end{aligned}$$

Letting $i \rightarrow \infty$ in the preceding expression, we have, for all $t > 0$,

$$\begin{aligned} M(x_*, T^{m_*} x_*, t) &\geq M \left(x_*, T^{m_*} x_*, \frac{t}{k_1} \right) \\ &\geq \dots \\ &\geq M \left(x_*, T^{m_*} x_*, \frac{t}{k_1^j} \right) \\ &\rightarrow 1 \quad (j \rightarrow \infty). \end{aligned}$$

Hence $M(x_*, T^{m_*} x_*, t) = 1$ for all $t > 0$. By Theorem 6, the conclusion is obtained. This completes the proof. $\blacksquare$

By the same way as stated above, we can prove the following:

Theorem 8. *Let T be a self-mapping on a τ -complete Menger FM-space (X, M, Δ) . Suppose that for each $x \in X$, the orbit $O_T(x; 0, \infty)$ is fuzzy bounded, and that for each $x \in X$ there exists an integer $m(x) \geq 1$ such that for all $y \in X$ and $t \geq 0$,*

$$M(T^{m(x)} x, T^{m(x)} y, t) \geq \min \left\{ M \left(x, y, \frac{t}{k} \right), M \left(x, T^{m(x)} y, \frac{t}{k} \right), M \left(x, T^{m(x)} y, \frac{t}{k} \right) \right\},$$

where k is a constant with $k \in (0, 1)$. Then T has a unique fixed point x_* in X and $T^n x_0 \rightarrow x_*$ for each $x_0 \in X$.

As an application, in the following, we shall utilize the results obtained above to show the existence and uniqueness of random solution for the following random linear random operator equation

$$(20) \quad x(\omega) = T(\omega, x(\omega)) + y(\omega).$$

In the sequel, we shall always assume that $(\Omega, \mathcal{A}, P)$ is a complete probability measure space, (X, d) is a separable linear metric space, and $(X, \mathcal{B})$ is a measurable space, where $\mathcal{B}$ is the σ -algebra of all Borel subsets of X .

Let T be a mapping from $\Omega \times X$ into X . For any positive integer i and $x, y \in X$, we denote

$$\begin{aligned} O_T(\omega, x; 0, \infty) &= \{x = x_0(\omega), x_1(\omega) = T(\omega, x_0(\omega)), \dots, x_n(\omega) = T(\omega, x_{n-1}(\omega)), \dots\} \\ O_T(\omega, x, y; 0, \infty) &= O_T(\omega, x; 0, \infty) \cup O_T(\omega, y; 0, \infty) \\ O_T(\omega, x; i, \infty) &= \{x_i(\omega) = T(\omega, x_{i-1}(\omega)), \dots, x_n(\omega) = T(\omega, x_{n-1}(\omega)), \dots\} \\ O_T(\omega, x, y; i, \infty) &= O_T(\omega, x; i, \infty) \cup O_T(\omega, y; i, \infty) \end{aligned}$$

Let $A \subset X$ and denote $\delta(A) = \sup_{x, y \in A} d(x, y)$. The proof of the following lemmas are straightforward and are omit here.

Lemma 3. *Let $x(\omega)$ be an X -valued random variable. Let $T(., .)$ be a d -continuous random operator. Then $T(\omega, x(\omega))$ is also an X -valued random variable.*

Definition 10. Let (X, d) be a metric space. The space (X, M, Δ) is called an induced Menger FM-space if $\Delta = \min$ and M is defined by

$$M(x, y, t) = H(t - d(x, y)) = \begin{cases} 0, & \text{if } d(x, y) \geq t; \\ 1, & \text{if } d(x, y) < t. \end{cases}$$

It is known that if (X, d) is a complete metric space, then the induced Menger FM-space (X, M, Δ) is a τ -complete Menger FM-space and that a sequence $\{x_n\}$ in X τ -converges to a point $x_* \in X$ if and only if $\{x_n\}$ in X converges in the metric d to x_* .

Lemma 4. Let (X, d) be a complete linear metric space and (X, M, Δ) be an induced Menger FM-space from (X, d) . Let $\{x_n\}$ be a sequence in X . Then we have

$$x_n \xrightarrow{\tau} x \text{ if and only if } x_n \xrightarrow{d} x.$$

Theorem 9. Let (X, d) be a complete separable linear metric space and (X, M, Δ) the induced τ -complete Menger FM-space. Suppose that $T(., .) : \Omega \times X \rightarrow X$ is a d -continuous random operator, and that for any $x \in X$ and any $\omega \in \Omega$, $O_s(\omega, x; 0, \infty)$ is d -bounded. Suppose further that there exist positive integers m, n satisfying the following:

$$(21) \quad P\{\omega \in \Omega : d(S^n(\omega, x), S^m(\omega, y)) = \Phi(\omega, \delta(O_s(\omega, x, y; 0, \infty)))\} = P(E_{x,y}) = 1$$

for all $x, y \in X$, where the random operator $S : \Omega \times X \rightarrow X$ is defined as follows:

$$S(\omega, x) = T(\omega, x) + y(\omega),$$

and $\Phi : \Omega \times [0, \infty) \rightarrow [0, \infty)$ is a random function satisfying the following conditions (Φ_1) and (Φ_2) :

(Φ_1) for any $\omega \in \Omega$, $\Phi(\omega, 0) = 0$ and $\Phi(\omega, .)$ is strictly increasing respect to $t \in [0, \infty)$.

(Φ_2) for any $\omega \in \Omega$ and any $t > 0$, $\lim_{n \rightarrow \infty} \Phi^{-n}(\omega, t) = \infty$, where $\Phi^{-n}(\omega, .)$ denotes the inverse function of $\Phi(\omega, .)$, and $\Phi^{-n}(\omega, .)$ is the n -th iterative of $\Phi^{-1}(\omega, .)$.

Then for any given X -valued random variable $\tilde{y}(\omega)$, there exists a unique random solution $x_*(\omega)$ of the following nonlinear random operator equation:

$$(22) \quad x(\omega) = T(\omega, x(\omega)) + \tilde{y}(\omega),$$

and for any X -valued random variable $x_0(\omega)$, the sequence

$$x_n(\omega) = T(\omega, x_{n-1}(\omega)) + \tilde{y}(\omega), \quad n = 1, 2, \dots$$

of X -valued random variable converges almost surely to $x_*(\omega)$.

Proof. By the separability of X and the continuity of $T(\omega, .)$, it is easy to see that

$$G = \bigcap_{x \in X} \bigcap_{y \in X} E_{x,y} \in \mathcal{A} \quad \text{and} \quad P(G) = 1,$$

where the set $E_{x,y}$ is defined by (21). For any fixed $\tilde{\omega} \in G$, it follows from (21) that

$$d(S^n(\tilde{\omega}, x), S^m(\tilde{\omega}, y)) \leq \Phi(\tilde{\omega}, \delta(\tilde{\omega}, x, y; 0, \infty)), \quad x, y \in X.$$

Hence for all $x, y \in X$ and any $t \geq 0$, we have, by condition (Φ_1) ,

$$\begin{aligned} M(S^n(\tilde{\omega}, x), S^m(\tilde{\omega}, y), t) &= H(t - d(S^n(\tilde{\omega}, x), S^m(\tilde{\omega}, y))) \\ &\geq H(t - \Phi(\tilde{\omega}, \delta(\tilde{\omega}, x, y; 0, \infty))) \\ &= H(\Phi(\tilde{\omega}, \Phi^{-1}(\tilde{\omega}, t)) - \Phi(\tilde{\omega}, \delta(\tilde{\omega}, x, y; 0, \infty))) \\ &= H(\Phi^{-1}(\tilde{\omega}, t) - \delta(\tilde{\omega}, x, y; 0, \infty)) \\ &= \inf_{p, q \in O_s(\tilde{\omega}, x, y; 0, \infty)} M(p, q, \Phi^{-1}(\tilde{\omega}, t)). \end{aligned}$$

Next since $\Phi(\tilde{\omega}, t)$ satisfies the conditions (Φ_1) and (Φ_2) , we have that $\Phi^{-1}(\tilde{\omega}, t)$ satisfies the condition (Φ) . From Lemma (4), it follows that $S(\tilde{\omega}, \cdot)$ is τ -continuous. Since each d -bounded set in X is fuzzy bounded, it follows that for any $x \in X$, the orbit $O_s(\tilde{\omega}, x; 0, \infty)$ is fuzzy bounded. Hence all the conditions of Theorem 4 are satisfied. By using Theorem 4, there exist a unique fixed point $x_*(\tilde{\omega}) \in X$ and for any X -valued random variable $x_0(\omega)$, when $\omega = \tilde{\omega}$ the sequence $\{x_n(\tilde{\omega})\}_{n=0}^\infty$ converges in τ (hence in d) to $x_*(\tilde{\omega})$. By the arbitrariness of $\tilde{\omega} \in G$, this implies that $x_n(\omega) \rightarrow x_*(\omega)$ a.s. By Lemma 3, it is easy to know that $\{x_n(\omega)\}$ is a sequence of X -valued random variables, so that its strong limit $x_*(\omega)$ is also an X -valued random variable, and $x_*(\omega)$ is the unique random fixed point of $S(\cdot, \cdot)$. Accordingly, $x_*(\omega)$ is the unique random solution of the equation (22). This completes the proof. $\blacksquare$

4 Common fixed point Theorems for Contraction type mappings

Throughout this section, we always assume that (X, M, Δ) is a τ -complete non-Archimedean FM -space, Δ is a continuous t -norm which is stronger than $\Delta_m = \max\{\text{sum} - 1, 0\}$, i.e., $\Delta(a, b) \geq \Delta_m(a, b)$ for all $a, b \in [0, 1]$, and the function $\Phi(t)$ satisfies the condition (Φ) .

Theorem 10. *Let (X, M, Δ) , Δ , and Φ be the same as above. Let $\{T_n\}_{n=1}^\infty$ be a sequence of self-adjoint on (X, M, Δ) . Suppose that there exists a functional sequence $\{m_n(x)\}_{n=1}^\infty : (X, M, \Delta) \rightarrow \mathbb{Z}^+$ such that for each $n \in \mathbb{Z}^+$ and each $x \in X$, $m_n(x) | m_b(T_n x)$ and that, for any $i, j \in \mathbb{Z}^+$, $i \neq j$ and $x, y \in X$, the following holds:*

$$(23) \quad M(T_i^{m_i(x)} x, T_j^{m_j(y)} y, t) \geq \min_{p, q \in \{x, y, T_i^{m_i(x)} x, T_j^{m_j(y)} y\}} M(p, q, \Phi(t))$$

for all $t \geq 0$. Suppose further that there exists some $x_0 \in X$ such that the set $\{x_n\}_{n=0}^\infty \subset X$ defined by

$$(24) \quad x_n = T_n^{m_n(x_{n-1})} x_{n-1}$$

for all $n = 1, 2, \dots$ is fuzzy bounded. Then there exists an unique common fixed point $x^* \in X$ and $x_n \xrightarrow{\tau} x^*$.

Proof. For given $x_0 \in X$, we prove that the sequence $\{x_n\}$ defined by (24) is a τ -Cauchy sequence in X . In fact, for any $i, j \in \mathbb{Z}^+$, it follows from (23) that, for all $t \geq 0$,

$$(25) \quad \begin{aligned} M(x_i, x_j, t) &= M(T_i^{m_i(x_{i-1})} x_{i-1}, T_j^{m_j(x_{j-1})} x_{j-1}, t) \\ &\geq \min_{p, q \in \{x_{i-1}, x_{j-1}, x_i, x_j\}} M(p, q, \Phi(t)). \end{aligned}$$

Therefore, for any $m, n \in \mathbb{Z}^+$ ($m < n$), from (25) we have

$$\inf_{m \leq i, j \leq n} M(x_i, x_j, t) \geq \inf_{m-1 \leq i, j \leq n} M(x_i, x_j, \Phi(t)).$$

By the arbitrariness of $n \in \mathbb{Z}^+$ ($n > m$), we have

$$(26) \quad \inf_{m \leq i, j} M(x_i, x_j, t) \geq \inf_{m-1 \leq i, j} M(x_i, x_j, \Phi(t)) \geq \sup_{u < \Phi(t)} \inf_{i, j \geq m-1} M(x_i, x_j, \Phi(t)).$$

By the induction, it is easy to prove that

$$(27) \quad \inf_{m \leq i, j} M(x_i, x_j, t) \geq \inf_{0 \leq i, j} M(x_i, x_j, \Phi(t)) \geq \sup_{u < \Phi^m(t)} \inf_{i, j \geq 0} M(x_i, x_j, u).$$

for all $m = 1, 2, \dots$ and $t \geq 0$. In view of condition (Φ) and the fuzzy boundedness of the set $\{x_n\}$ defined by (24), it follows that for all $t \geq 0$,

$$\begin{aligned}
\lim_{m \rightarrow \infty} \inf_{m \leq i, j} M(x_i, x_j, t) &\geq \lim_{m \rightarrow \infty} \left(\sup_{u < \Phi^m(t)} \inf_{i, j \geq 0} M(x_i, x_j, u) \right) \\
&= \begin{cases} 0, & \text{if } t = 0; \\ \sup_{u > 0} \inf_{i, j \geq 0} M(x_i, x_j, u), & \text{if } t > 0. \end{cases} \\
(28) \quad &= \begin{cases} 0, & \text{if } t = 0; \\ 1, & \text{if } t > 0. \end{cases}
\end{aligned}$$

Consequently, for any given $\epsilon > 0$ and $0 < \lambda < 1$, there exists $N = N(\epsilon, \lambda) \in \mathbb{Z}^+$ such that for all $m \geq N$,

$$\inf_{m \leq i, j} M(x_i, x_j, \epsilon) > 1 - \lambda.$$

Therefore, we have

$$(29) \quad M(x_i, x_j, \epsilon) > 1 - \lambda, \quad i, j \geq N.$$

This implies that the sequence $\{x_n\}$ defined by (24) is a τ -Cauchy sequence in X . By the τ -completeness of X , we can suppose that $x_n \xrightarrow{\tau} x^* \in X$.

Now we prove that x^* is a common periodic point of $\{T_n\}$, i.e.,

$$T_n^{m_n(x^*)} x^* = x^*, \quad n = 1, 2, \dots$$

Indeed, for any $i \in \mathbb{Z}^+$, it follows from (23) that, for any $n > i$, we have

$$\begin{aligned}
M(x_n, T_i^{m_i(x^*)} x^*, t) &= M(T_n^{m_n(x_{n-1})} x_{n-1}, T_i^{m_i(x^*)} x^*, t) \\
(30) \quad &\geq \min_{p, q \in \{x_{n-1}, x^*, x_n, T_i^{m_i(x^*)} x^*\}} M(p, q, \Phi(t)) \quad \text{for all } t \geq 0.
\end{aligned}$$

Let G_0 be the set of all discontinuity points of $M(x^*, T_i^{m_i(x^*)} x^*, t)$. Since Φ^m is strictly increasing, we know that $\Phi^{-m}(G_0)$ is the set of discontinuity points of $M(x_n, T_i^{m_i(x^*)} x^*, \Phi^m(t))$, $m = 1, 2, \dots$. Moreover, $G_0, \Phi^{-m}(G_0)$, $m = 1, 2, \dots$ are the countable sets and so

$$G = G_0 \cup \left(\sum_{m=1}^{\infty} \Phi^{-m}(G_0) \right)$$

is also countable. Let $\tilde{G} = \mathbb{R}^+ \setminus G$. When $t = 0$ or $t \in \tilde{G}$ (i.e., t is a common continuity point of $M(x^*, T_i^{m_i(x^*)} x^*, t)$ and $M(x^*, T_i^{m_i(x^*)} x^*, \Phi^m(t))$, $m = 1, 2, \dots$), it follows from (30) that

$$M(x^*, T_i^{m_i(x^*)} x^*, t) = \lim_{n \rightarrow \infty} M(x_n, T_i^{m_i(x^*)} x^*, t) \geq M(x^*, T_i^{m_i(x^*)} x^*, \Phi(t)).$$

Repeating this procedure, we can prove that

$$M(x^*, T_i^{m_i(x^*)} x^*, t) \geq M(x^*, T_i^{m_i(x^*)} x^*, \Phi(t)) \geq \dots \geq M(x^*, T_i^{m_i(x^*)} x^*, \Phi^n(t)).$$

Letting $n \rightarrow \infty$ and noting the condition (Φ) , for all $t \in \tilde{G}$ or $t = 0$, we have

$$(31) \quad M(x^*, T_i^{m_i(x^*)} x^*, t) = H(t) = \begin{cases} 0, & \text{if } t = 0; \\ 1, & \text{if } t > 0. \end{cases}$$

When $t \in G$ with $t > 0$, by the density of real numbers, there exist $t_1, t_2 \in \tilde{G}$ such that $0 < t_1 < t < t_2$. Since their distribution function is nondecreasing, we have, from (31),

$$1 = H(t_1) = M(x^*, T_i^{m_i(x^*)} x^*, t_1) \leq M(x^*, T_i^{m_i(x^*)} x^*, t) \leq M(x^*, T_i^{m_i(x^*)} x^*, t_2) = 1.$$

This shows that, for all $t \in G$ with $t > 0$,

$$(32) \quad M(x^*, T_i^{m_i(x^*)} x^*, t) = H(t).$$

Combining (31) and (32), we have

$$M(x^*, T_i^{m_i(x^*)} x^*, t) = H(t), \quad t \geq 0, \text{ i.e., } x^* = T_i^{m_i(x^*)} x^*, \quad i = 1, 2, \dots.$$

To prove that x^* is the unique periodic point of $\{T_n\}_{n=1}^\infty$, we proceed as follows:
Suppose that $y^* \in X$ is another periodic point of some T_j , i.e.,

$$y^* = T_j^{m_j(y^*)} y^*.$$

Then for any $i \in \mathbb{Z}^+$, $i \neq j$, we have

$$\begin{aligned} M(x^*, y^*, t) &= M(T_i^{m_i(x^*)} x^*, T_j^{m_j(y^*)} y^*, t) \geq \min_{p, q \in \{x^*, y^*\}} M(p, q, \Phi(t)) \\ &= M(x^*, y^*, \Phi(t)), \quad t \geq 0. \end{aligned}$$

Repeating this procedure, for all $n = 1, 2, \dots$ and $t \geq 0$, we can prove

$$M(x^*, y^*, t) \geq M(x^*, y^*, \Phi^n(t)).$$

Letting $n \rightarrow \infty$ and noting the condition (Φ) , we have

$$M(x^*, y^*, t) = H(t), \quad t \geq 0, \quad \text{i.e., } x^* = y^*.$$

Furthermore, by the assumption, since $m_n(x) | m_n(T_n x)$ for all $x \in X$ and for all $n \in \mathbb{Z}^+$, there exists, for each $i = 1, 2, \dots$ some $k_i \in \mathbb{Z}^+$ such that

$$(33) \quad m_i(T_i x^*) = k_i m_i(x^*).$$

Hence we have, for each $i = 1, 2, \dots$,

$$\begin{aligned} T_i x^* &= T_i T_i^{m_i(x^*)} x^* = T_i T_i^{2m_i(x^*)} x^* = \dots = T_i T_i^{k_i m_i(x^*)} x^* \\ &= T_i T_i^{m_i(T_i x^*)} x^* = T_i^{m_i(T_i x^*)} T_i x^*. \end{aligned}$$

This implies that $T_i x^*$ is also a periodic point of T_i . Since x^* is the unique periodic point of T_i , we have $x^* = T_i x^*$, $i = 1, 2, \dots$, which means that x^* is the desired unique common fixed point of $\{T_n\}_{n=1}^\infty$. This completes the proof. $\blacksquare$

Corollary 1. Let $\{T_n\}_{n=1}^\infty$, (X, M, Δ) and $\{m_n(x)\}_{n=1}^\infty$ be the same as in Theorem 10. Suppose that for any $i, j \in \mathbb{Z}^+$, and any $x, y \in X$, the following holds:

$$M(T_i^{m_i(x)} x, T_j^{m_j(y)} y, t) \geq \min_{p, q \in \{x, y, T_i^{m_i(x)} x, T_j^{m_j(y)} y\}} M\left(p, q, \frac{t}{h}\right)$$

for all $t \geq 0$. Suppose further that there exists $x_0 \in X$ such that the set $\{x_n\}_{n=1}^\infty$ defined by (24) is fuzzy bounded. Then there exists a unique common fixed point $x^* \in X$ and $x_n \xrightarrow{\tau} x^*$.

Proof. Taking $\Phi(t) = t/h$, $h \in (0, 1)$, it is easy to see that $\Phi(t)$ satisfies the condition (Φ) . So the result follows now from Theorem 10. $\blacksquare$

Corollary 2. Let $\{T_n\}_{n=1}^\infty$ be a sequence of self-mappings on (X, M, Δ) . Suppose that there exists a sequence $\{m_n\}_{n=1}^\infty$ in $\mathbb{Z}^+$ such that for any $x, y \in X$ and any $i, j \in \mathbb{Z}^+$, $i \neq j$, the following holds:

$$(34) \quad M(T_i^{m_i} x, T_j^{m_j} y, t) \geq \min_{p, q \in \{x, y, T_i^{m_i} x, T_j^{m_j} y\}} M\left(p, q, \frac{t}{h}\right)$$

for all $t \geq 0$. Suppose further that there exists $x_0 \in X$ such that the set $\{x_n\}_{n=1}^\infty$ defined by (24) is fuzzy bounded. Then there exists a unique common fixed point $x^* \in X$ and $x_n \xrightarrow{\tau} x^*$.

Throughout this section, we always assume that (X, M, Δ) is a τ -complete non-Archimedean FM -space, Δ is a continuous t -norm, and the function $\Phi(t)$ satisfies the condition (Φ) . Suppose further that S, T are self-mappings on (X, M, Δ) and that they are commutative and τ -continuous. Furthermore, we denote

$$O_{S,T}(x; 0, \infty) = \{S^i T^j x\}_{i,j=0}^\infty, \quad x \in X.$$

Theorem 11. *Let (X, M, Δ) , S, T and Φ be the same as above. Suppose that, for each $x \in X$, the set $O_{S,T}(x; 0, \infty)$ is fuzzy bounded. Suppose further that there exist $m, m' \in \mathbb{Z}^+$, $m + m' \geq 1$, such that for each $x \in X$ and $t \geq 0$, the following holds:*

$$(35) \quad \inf_{p,q \in O_{S,T}(S^m T^{m'} x; 0, \infty)} M(p, q, t) \geq \inf_{p,q \in O_{S,T}(x; 0, \infty)} M(p, q, \Phi(t)).$$

Then for each $x_0 \in X$, the sequence $\{S^n T^n\}_{n=0}^\infty$ converges in τ to some common fixed point $x^* \in X$ of S and T .

Proof. Letting $h = \max\{m, m'\}$, for any given $x_0 \in X$, from (35), we have

$$\begin{aligned} \inf_{p,q \in O_{S,T}(S^h T^h x_0; 0, \infty)} M(p, q, t) &\geq \inf_{p,q \in O_{S,T}(S^m T^{m'} x_0; 0, \infty)} M(p, q, t) \\ &\geq \inf_{p,q \in O_{S,T}(x_0; 0, \infty)} M(p, q, \Phi(t)) \\ &\geq \sup_{u < \Phi(t)} \inf_{p,q \in O_{S,T}(x_0; 0, \infty)} M(p, q, u) \end{aligned}$$

for all $t \geq 0$. By the induction, we can prove the following inequality holds:

$$\begin{aligned} \inf_{p,q \in O_{S,T}(S^{nh} T^{nh} x_0; 0, \infty)} M(p, q, t) &\geq \inf_{p,q \in O_{S,T}(S^{(n-1)h} T^{(n-1)h} x_0; 0, \infty)} M(p, q, \Phi(t)) \\ (36) \quad &\geq \dots \\ &\geq \inf_{p,q \in O_{S,T}(x_0; 0, \infty)} M(p, q, \Phi^n(t)) \\ &\geq \sup_{u < \Phi^n(t)} \inf_{p,q \in O_{S,T}(x_0; 0, \infty)} M(p, q, u) \end{aligned}$$

for all $t \geq 0$. By the condition (Φ) , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \inf_{p,q \in O_{S,T}(S^{nh} T^{nh} x_0; 0, \infty)} M(p, q, t) &\geq \lim_{n \rightarrow \infty} \sup_{u < \Phi^n(t)} \inf_{p,q \in O_{S,T}(x_0; 0, \infty)} M(p, q, u) \\ &= \begin{cases} \sup_{u > 0} \inf_{p,q \in O_{S,T}(x_0; 0, \infty)} M(p, q, u), & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases} \\ &= H(t), \quad t \geq 0. \end{aligned}$$

Therefore, for any given $\epsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that

$$\inf_{p,q \in O_{S,T}(S^{nh} T^{nh} x_0; 0, \infty)} M(p, q, \epsilon) > 1 - \lambda, \quad n \geq N.$$

This implies that the subsequences of $\{S^i T^j x_0\}_{i,j=0}^\infty$ in which the indexes i and j are both convergent to ∞ , are τ -Cauchy sequences in X . By the τ -completeness of X , they all converge in τ to the same limit $x^* \in X$. In particular, the following three subsequences converge in τ to the limit x^* :

$$\{S^n T^n x_0\}_{n=0}^\infty, \quad \{S(S^n T^n) x_0\}_{n=0}^\infty, \quad \{T(S^n T^n x_0)\}_{n=0}^\infty.$$

By the τ -continuity of S and T , we have

$$x^* = Sx^* = Tx^*,$$

that is, x^* is a common fixed point of S and T . This completes the proof. $\blacksquare$

Taking $S = I_X$ (the identity mapping on X), we have

Corollary 3. *Let (X, M, Δ) , and Φ be the same in Theorem 11. Suppose that T is a τ -continuous self-mapping on (X, M, Δ) and that for each $x \in X$ the set $O_T(x; 0, \infty) = \{T^n x\}_{n=0}^\infty$ is fuzzy bounded. If there exists $m \in \mathbb{Z}^+$ such that for all $t \geq 0$ and $x \in X$,*

$$\inf_{p, q \in O_T(T^m x; 0, \infty)} M(p, q, t) \geq \inf_{p, q \in O_T(x; 0, \infty)} M(p, q, \Phi(t)).$$

Then for any $x_0 \in X$, the sequence $\{T^n x_0\}_{n=0}^\infty$ τ -converges to some fixed point of T in X .

Theorem 12. *Let (X, M, Δ) , S, T and Φ be the same in Theorem 11. Suppose that, for each $x \in X$, the set $O_{S,T}(x; 0, \infty)$ is fuzzy bounded. Suppose further that there exist $m, m', n, n' \in \mathbb{Z}^+$, $m + m' \geq 1, n + n' \geq 1$, such that for each $x, y \in X$ and $t \geq 0$, the following holds:*

$$(37) \quad \inf_{\substack{p, q \in O_{S,T}(S^m T^{m'} x; 0, \infty) \\ \cup O_{S,T}(S^n T^{n'} y; 0, \infty)}} M(p, q, t) \geq \inf_{\substack{p, q \in O_{S,T}(x; 0, \infty) \\ \cup O_{S,T}(y; 0, \infty)}} M(p, q, \Phi(t)).$$

Then there exists a unique common fixed point x^ of S and T in X , and for any $x_0 \in X$, the sequence $\{S^n T^n x_0\}_{n=0}^\infty$ converges in τ to x^* .*

Proof. Letting $h = \max\{m, m', n, n'\}$ and taking $x = y$ in (37), we have

$$\inf_{p, q \in O_{S,T}(S^h T^h x; 0, \infty)} M(p, q, t) \geq \inf_{p, q \in O_{S,T}(x; 0, \infty)} M(p, q, \Phi(t))$$

for all $x \in X$ and $t \geq 0$. By Theorem 11, we know that, for each $x_0 \in X$, the sequence $\{S^n T^n x_0\}_{n=0}^\infty$ converges in τ to some common fixed point $x^* \in X$ of S and T .

Now we prove that x^* is the unique common fixed point of S and T in X . Suppose that this is not the case. Then there exists another common fixed point y^* of S and T in X . From (37), we have

$$\begin{aligned} M(x^*, y^*, t) &= \inf_{\substack{p, q \in O_{S,T}(S^m T^{m'} x^*; 0, \infty) \\ \cup O_{S,T}(S^n T^{n'} y^*; 0, \infty)}} M(p, q, t) \\ &\geq \inf_{\substack{p, q \in O_{S,T}(x^*; 0, \infty) \\ \cup O_{S,T}(y^*; 0, \infty)}} M(p, q, \Phi(t)) \\ &= M(x^*, y^*, \Phi(t)), \quad t > 0. \end{aligned}$$

Repeating the procedure, for all $t \geq 0$ and $n = 1, 2, \dots$, we can prove

$$M(x^*, y^*, t) \geq M(x^*, y^*, \Phi^n(t)).$$

Letting $n \rightarrow \infty$ and using the condition (Φ) , we have

$$M(x^*, y^*, t) = H(t), \quad t \geq 0, \quad \text{i.e.,} \quad x^* = y^*.$$

This completes the proof. ■

As a consequence of Theorem 12 with $S = I_X$, we have

Corollary 4. *Let (X, M, Δ) , and Φ be the same in Theorem 12. Suppose that T is a τ -continuous self-mapping on (X, M, Δ) and that for each $x \in X$ the set $O_T(x; 0, \infty) = \{T^n x\}_{n=0}^\infty$ is fuzzy bounded. If there exist $m, n \in \mathbb{Z}^+$ such that for all $x, y \in X$ and $t \geq 0$,*

$$\inf_{\substack{p, q \in O_T(T^m x; 0, \infty) \\ \cup O_T(T^n y; 0, \infty)}} M(p, q, t) \geq \inf_{\substack{p, q \in O_T(x; 0, \infty) \\ \cup O_T(y; 0, \infty)}} M(p, q, \Phi(t)).$$

Then for any $x_0 \in X$, the sequence $\{T^n x_0\}_{n=0}^\infty$ τ -converges to some fixed point of T in X .

As an application, in the sequel we use some result stated above to show the existence and uniqueness of the solutions for nonlinear Volterra integral equations on a kind of particular fuzzy metric space.

In what follows, let $[0, a]$ be a fixed real interval ($0 < a < \infty$) and $(X, \|\cdot\|_X)$ a real Banach space. We denote by $C([0, a]; X)$ the Banach space of all X -valued continuous functions defined on $[0, a]$ with norm defined by

$$(38) \quad \|x\|_C = \sup_{0 \leq t \leq a} \|x(t)\|_X, \quad x(t) \in C([0, a]; X).$$

As well as the norm $\|\cdot\|_C$, the space $C([0, a]; X)$ can be endowed with another norm $\|\cdot\|_*$ which is defined as follows:

$$(39) \quad \|x\|_* = \sup_{0 \leq t \leq a} (e^{-Lt} \|x(t)\|), \quad x(t) \in C([0, a]; X),$$

where L is any positive number. It is clear that the norm $\|\cdot\|_*$ is equivalent to the norm $\|\cdot\|_C$.

We also denote by $(C([0, a]; X), M, \min)$ the induced non-archimedian FM -space, where M is the mapping from $C([0, a]; X) \times C([0, a]; X)$ into $[0, \infty)$ defined by

$$M(x, y, t) = \begin{cases} 0, & \text{if } t < \|x - y\|_*; \\ 1, & \text{if } t \geq \|x - y\|_*. \end{cases} \quad x(s), y(s) \in C([0, a]; X), t \in \mathbb{R}.$$

Now we study the existence and uniqueness of the solutions of the following kind of nonlinear Volterra integral equations:

$$(40) \quad x(t) = y(t) + \int_0^t K(t, s, x(s)) ds, \quad 0 \leq t \leq a,$$

where $y(t) \in C([0, a], X)$ is any given function.

Theorem 13. *Let $(X, \|\cdot\|_X)$, $C([0, a]; X)$ and $(C([0, a]; X), M, \min)$ be the same stated above. Suppose that the following conditions are satisfied:*

(a) $K(t, s, x(s)) \in C([0, a] \times [0, a] \times C([0, a]; X); X)$ and

$$\|K\|_C = \sup_{t, s \in [0, a], x \in X} \|K(t, s, x)\|_X < \infty,$$

(b) there exist $m \in \mathbb{Z}^+$ and a constant $\beta \in (0, 1)$ such that

$$M(T^m x, T^m y, t) = \min_{p, q \in \{x, y, T^m x, T^m y\}} M\left(p, q, \frac{t}{\beta}\right)$$

for all $x, y \in C([0, a]; X)$ and $t \in \mathbb{R}^+$, where the mappings T and T^m are defined as follows:

$$\begin{aligned} (Tx)(t) &= y(t) + \int_0^t K(t, s, x(s)) ds, \\ (T^m x)(t) &= y(t) + \int_0^t K(t, s, T^{m-1} x(s)) ds, \end{aligned}$$

(c) for any $x(t) \in C([0, a]; X)$, the set $O_T(x; 0, \infty) = \{T^n x(t)\}_{n=0}^\infty$ is bounded.

Then for any $x_0(t) \in C([0, a]; X)$, the sequence $\{T^n x_0(t)\}_{n=0}^\infty$ converges in the norm $\|\cdot\|_C$ to a solution $x^*(t) \in C([0, a]; X)$ of the equation (40).

Proof. The conclusion follows immediately from Corollary 2. ■

5 Conclusion

In this article we introduced a new fuzzy contraction mapping and prove that such mappings have fixed point in τ -complete fuzzy metric spaces. As an application, we utilized the results obtained to show the existence and uniqueness of random solution for the following random linear random operator equation. Also, we showed that the existence and uniqueness of the solutions for nonlinear Volterra integral equations on a kind of particular fuzzy metric space.

Compliance with Ethical Standards

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