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Focusing properties and focal shift of vortex Hermite-cosh- Gaussian beams

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Abstract

In this work, we investigate the focusing properties of a vortex Hermite-cosh-Gaussian beam (vHChGB) passing through a converging lens system. The analytical propagation equation as well as the beam width expression of a focused vHChGB is derived based on the Huygens-Fresnel diffraction principle. From the obtained formulae, the effects of the Gaussian Fresnel number N_F and the beam parameters on the structure of the light intensity distribution and the beam spot size in the focal region are analyzed numerically. It is shown that the focal shift, which is determined from the minimum beam spot width criterion, is affected strongly by the change of the Fresnel number, the parameter b , and it is slightly altered by the vortex charge M and nearly insensitive to the beam order n . For a fixed value of N_F , the focal shift is smaller when the parameter b or the vortex charge M is larger. The focal shift decreases monotonously with the increase of Fresnel number until it vanishes asymptotically for large N_F . The obtained results may be useful for the applications of the vHChGBs in beam shaping and beam focusing.

Keywords: *Vortex Hermite-cosh-Gaussian beam; Focusing properties; Focal shift; Fresnel number.*

1. Introduction

In last few years, the beam focusing has drawn the attention of laser researchers because of its theoretical and practical interests. It is known that the point of maximum intensity in the propagated field for focused laser beams does not coincide with the geometric focus but is shifted toward the focusing lens. This effect, referred to as focal shift, is crucial due to the need to accurately determine the actual focal plane in practical applications. The focal shift has been firstly discovered and analyzed for the fundamental Gaussian beam by Li and Wolf [1-2], after that, the phenomenon has been investigated for various types of laser beams as Laguerre-Gaussian beams [3], Bessel-Gaussian beams [4], flat-topped beams [5],

partially coherent conical Bessel-Gauss beams [6], hyperbolic-cosine-Gaussian beams [7], the axisymmetric Bessel-modulated Gaussian beam [8], Hermite-Gaussian beams [9], Hermite-Cosh-Gaussian beams [10] and vortex cosine-hyperbolic Gaussian beams [11] and so on. Recently, a generalized form of the hollow vortex Gaussian beam, which is called as vortex Hermite-cosh-Gaussian beam has been investigated by our research group [12]. The beam can be generated from a standard Hermite-cosh-Gaussian beam (HChGB) passing through a spiral phase plate [13-14]. The profile of the beam has zero intensity at the center and a spirally wave-front phase which is associated to the orbital angular momentum of light [15-16]. The vHChGB possesses three key parameters, namely the beam orders, the decentered parameter b and the vortex charge number M . By choosing the adequate beam parameters conditions, the beam can be reduced to well-known hollow beams, e.g., the vortex-Gaussian beam [18], vortex Hermite-Gaussian beam [19-20], and vortex-cosh-Gaussian beam [21]. As practical applications, the beam can be used as optical traps for micro-particles or a beam spanner by use of the orbital angular momentum [17]. The propagation properties of the vHChGB in free space and through turbulent atmosphere have been examined [22].

In the present paper, we will investigate the focusing properties including the focal shift of a vHChGB passing through a converging lens system. In the rest of the paper, the intensity distribution of a vHChGB at the initial plane is illustrated as function of the beam parameters in Section 2. Then, the analytical formulas for the vHChGB passing through a thin lens system and the beam spot width of the focused vHChGB are derived in detail in Sections 3 and 4. The characteristics of the intensity distribution in the focal region and the behavior of the focal shift are discussed with numerical examples as function of the Fresnel Number and the beam parameters in Section 5. Finally, the main results are outlined in the conclusion.

2. Field distribution of a vHChGB at the initial plane

In the Cartesian coordinates system, the z -axis is taken to be the propagation direction. The electric field of a vHChGB with $(x-y)$ symmetry at the source plane $z=0$ takes the form as [12]

$$E_n(x_0, y_0, z=0) = (x_0 + iy_0)^M E_n(x_0, z=0) E_n(y_0, z=0), \quad n=1, 2, \dots \quad (1a)$$

where

$$E_n(u, z=0) = H_n\left(\frac{\sqrt{2}u}{\omega_0}\right) \cosh\left(\frac{b}{\omega_0}u\right) \exp\left(-\frac{u^2}{\omega_0^2}\right), \quad (1b)$$

with $u = x_0$ or y_0 . (x_0, y_0) are the Cartesian coordinates at the source plane. n is the mode index associated with the Hermite polynomial $H_n(\cdot)$, and is called the beam order. $\cosh(\cdot)$ is the hyperbolic-cosine function, ω_0 is the waist size of the Gaussian part, b is the decentered parameter associated with the cosh part, and the integer parameter M denotes the topological charge of the vortex.

If $n = 0$, Eq. (1a) reduces to be the vortex-cosh-Gaussian beam (vChGB) [21] while when $b = 0$ or $M = 0$, one obtains the vortex Hermite-Gaussian beam (vHGB) [19-20] and HChGB [23-27], respectively. Furthermore, the case $n = b = 0$ will leads to the hollow Gaussian vortex beam [18].

Eq. (1a.) can also be written in the form

$$E_n(x_0, y_0, z = 0) = \frac{1}{4} \exp\left(\frac{b^2}{4}\right) H_n\left(\frac{\sqrt{2}}{\omega_0} x_0\right) H_n\left(\frac{\sqrt{2}}{\omega_0} y_0\right) \times \left\{ \exp\left[-\left(\frac{x_0}{\omega_0} - \frac{b}{2}\right)^2\right] + \exp\left[-\left(\frac{x_0}{\omega_0} + \frac{b}{2}\right)^2\right] \right\} \times \left\{ \exp\left[-\left(\frac{y_0}{\omega_0} - \frac{b}{2}\right)^2\right] + \exp\left[-\left(\frac{y_0}{\omega_0} + \frac{b}{2}\right)^2\right] \right\} (x_0 + iy_0)^M. \quad (2)$$

The above equation means that a vHChGB can be obtained by a superposition of four Hermite-decentered Gaussian modes of order n with the embedded vortex charge M .

The typical intensity distribution patterns of a vHChGB at the source plane are illustrated in Figs. 1 and 2 for different values of the parameters n , b and M . For convenience, in the numerical calculations the width of the waist is set to be $\omega_0 = 1 \text{ mm}$. The plots of Figs. (1-2) show that the vHChGB is hollow dark-like, with a central dark region surrounded by an array spot structure. The beam profile is mirror symmetric, and the intensity distribution can be either with multi-spot or four spot patterns depending on the value of the parameter b . Indeed, for small b ($b < 1$), the beam exhibits $4n$ lobes with the four main ones are located at the square vertices of the beam spot (see Fig. 1). While for large b , the beam is four-petal like whatever the value of the beam order n . In addition, one can note also the elongation of the main lobes due to the increasing of vortex charge M (see Fig. 2).

Figure 1: The intensity distribution of a vHChGB at the source plane with $\omega_0=1$ mm and $M=1$ for different values of n and b .

Figure 2: The intensity distribution of vHChGB at the initial plane with $\omega_0 = 1$ mm and $n=3$ for $M=1$ (top row), $M=2$ (middle row), 4 and $M=6$ (bottom row) and (a.1-4) $b=0$, (b.1-4) $b=1.5$ and (c.1-4) $b=4$.

3. Focusing of a vHChGB by a thin lens system

Let us consider an incident vHChGB passing through an aplanatic lens of focal length f , as schematized in Fig. 3. Assuming that the incident beam is placed at the lens plane ($z=0$), the transfer matrix for the optical system reads as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-s & fs \\ -1/f & 1 \end{pmatrix}, \quad (3)$$

where $s = z/f$ is the normalized (with respect to the focal length) propagation distance with z is the distance from the lens plane to the output plane. A, B, C, D are the matrix elements associated with the lens system.

Figure 3: Schematic illustration of the focusing lens system.

Within the paraxial approximation, the propagation of a vHChGB through the optical system obeys the Huygens Fresnel diffraction integral, which can be expressed as [28]

$$E(x, y, z) = \frac{iN_F}{\omega_0^2 s} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_n(x_0, y_0, 0) \exp \left\{ -\frac{i\pi N_F}{\omega_0^2 s} \left[(1-s)(x_0^2 + y_0^2) - 2(xx_0 + yy_0) + (x^2 + y^2) \right] \right\} dx_0 dy_0, \quad (4)$$

where $E(x, y, z)$ is the field at the receiver plane z , and $N_F = \omega_0^2 / \lambda f$ is the Gaussian Fresnel number. In Eq. (4), an unimportant term of propagation was omitted for convenience. Substituting from Eqs. (1a) into Eq. (4), and recalling the following binomial formula

$$(x_0 + iy_0)^M = \sum_{l=0}^M C_l^M x_0^l (iy_0)^{M-l}, \quad (5a)$$

where

$$C_l^M = \frac{M!}{l!(M-l)!}, \quad (5b)$$

then by making some algebraic operations, Eq. (4) can be expressed as

$$E(x, y, z) = \frac{iN_F}{\omega_0^2 s} \exp \left[-\frac{i\pi N_F}{\omega_0^2 s} (x^2 + y^2) \right] \sum_{l=0}^M C_l^M (i)^{M-l} I_l(x) I_{M-l}(y), \quad (6a)$$

where

$$I_q(t) = \int_{-\infty}^{+\infty} u_0^q H_n \left(\frac{\sqrt{2}}{\omega_0} u_0 \right) \cosh \left(b \frac{u_0}{\omega_0} \right) \exp \left\{ -\left(\frac{1}{\omega_0^2} + \frac{i\pi N_F (1-s)}{\omega_0^2 s} \right) u_0^2 + \frac{i2\pi N_F}{\omega_0^2 s} t u_0 \right\} du_0, \quad (6b)$$

with $t = x$ or y , $q = l$ or $M-l$.

Now, by using the definition of $\cosh(.)$ function, Eq. (6b) can be written as

$$I_q(t) = \frac{1}{2} [J_q^+(t) + J_q^-(t)], \quad (7a)$$

where

$$J_q^\pm(t) = \int_{-\infty}^{+\infty} u_0^q H_n \left(\frac{\sqrt{2}u_0}{\omega_0} \right) \exp \left(-\eta u_0^2 + \left(\frac{i2\pi N_F}{\omega_0^2 s} t \pm \frac{b}{\omega_0} \right) u_0 \right) du_0. \quad (7b)$$

with

$$\eta = \frac{1}{\omega_0^2} + \frac{i\pi N_F (1-s)}{\omega_0^2 s}. \quad (7c)$$

Recalling the following formulas [29-30]

$$H_n(x) = \sum_{p=0}^{[n/2]} \frac{(-1)^p n!}{p! (n-2p)!} (2x)^{n-2p}, \quad (8)$$

and

$$\int_{-\infty}^{+\infty} x^m \exp(-px^2 + 2qx) dx = \sqrt{\frac{\pi}{p}} \exp \left(\frac{q^2}{p} \right) \left(\frac{1}{2i\sqrt{p}} \right)^m H_m \left(\frac{iq}{\sqrt{p}} \right), \quad \text{Re}(p) > 0, \quad (9)$$

then after carrying on the tedious calculations, Eq. (7b) turns out to be

$$J_q^\pm(t) = \sqrt{\frac{\pi}{\eta}} \left(\frac{1}{2i\sqrt{\eta}} \right)^q \sum_{p=0}^{[n/2]} \frac{(-1)^p n!}{p! (n-2p)!} \left(\frac{\sqrt{2}}{\omega_0 i \sqrt{\eta}} \right)^{n-2p} f^\pm(t) H_{q+n-2p} \left(\frac{i}{\sqrt{\eta}} \left(\frac{i\pi N_F}{\omega_0^2 s} t \pm \frac{b}{2\omega_0} \right) \right), \quad (10a)$$

where

$$f^\pm(t) = \exp \left\{ \frac{1}{4\eta} \left(\frac{i2\pi N_F}{\omega_0^2 s} t \pm \frac{b}{\omega_0} \right)^2 \right\}. \quad (10b)$$

Substituting from Eqs. (10a) and (7a) into (6a), one obtains

$$E(u, v, s) = \frac{(-1)^{-n} i \pi N_F}{2^{M-n+2} \beta} \left(\frac{s}{\beta} \right)^{\frac{M}{2}+n} \exp \left\{ \frac{s b^2}{2 \beta} - \frac{i \pi N_F (1 - i \pi N_F)}{\beta} (u^2 + v^2) \right\} \sum_{l=0}^M C_l^M i^{-l} [F_{n,l}^+(u) + F_{n,l}^-(u)] [F_{n,M-l}^+(v) + F_{n,M-l}^-(v)], \quad (11a)$$

where

$$F_{n,l}^{\pm}(u) = \sum_{p=0}^{\left[\frac{n}{2}\right]} \frac{n!}{p!(n-2p)!} \left(\frac{\beta}{2s} \right)^p \exp \left(\frac{\pm i \pi b N_F}{\beta} u \right) H_{l+n-2p} \left(\frac{1}{2\sqrt{s}\beta} (\pm i b s - 2\pi N_F u) \right), \quad (11b)$$

and

$$\beta = s + i \pi N_F (1 - s), \quad (11c)$$

with (u, v) denote the normalized transverse coordinates, $u = x/\omega_0$ and $v = y/\omega_0$.

Eq. (11a) is the analytical expression of the vHChGB passing through a thin lens system, which is a convenient way to study the propagation characteristics of the focused beam.

From the general formula Eq. (11a) one can distinguish some special cases, which are described as following:

i) When $n=0$, Eq. (11a) will give the field expression of a focused vChGB, which can be expressed as

$$E(u, v, s) = \frac{i \pi N_F}{2^{M+2} \beta} \left(\frac{s}{\beta} \right)^{\frac{M}{2}} \exp \left\{ \frac{s b^2}{2 \beta} - \frac{i \pi N_F (1 - i \pi N_F)}{\beta} (u^2 + v^2) \right\} \sum_{l=0}^M C_l^M i^{-l} [F_l^+(u) + F_l^-(u)] [F_{M-l}^+(v) + F_{M-l}^-(v)], \quad (12a)$$

where

$$F_l^{\pm}(u) = \exp \left(\frac{\pm i \pi b N_F}{\beta} u \right) H_l \left(\frac{1}{2\sqrt{s}\beta} (\pm i b s - 2\pi N_F u) \right), \quad (12b)$$

and

$$F_{M-l}^{\pm}(v) = \exp \left(\frac{\pm i \pi b N_F}{\beta} v \right) H_{M-l} \left(\frac{1}{2\sqrt{s}\beta} (\pm i b s - 2\pi N_F v) \right). \quad (12c)$$

Eq. (12a) is consistent with the result obtained in Ref. [11].

ii) In the limiting case $M=0$, i.e., in the absence of the vortex, Eq. (11a) reduces to

$$E(u, v, s) = \frac{i\pi N_F}{\beta} \exp \left\{ \frac{s}{2\beta} b^2 + Q(u^2 + v^2) \right\} \cosh(Su) \cosh(Sv), \quad (13a)$$

where

$$S = \frac{i\pi b N_F}{\beta}, \quad (13b)$$

Eq. (13a) is the field expression of the focused ChGB. This result is consistent with Eq. (7a) of Ref. [7].

ii) If one takes $M=n=0$, Eq. (11) will be simplified to

$$E(u, v, s) = \frac{i\pi N_F}{\beta} \exp \left\{ Q(u^2 + v^2) \right\}, \quad (14)$$

which is the propagation formula of the focused Gaussian beam [1-2].

The irradiance $I(u, v, s)$ of the focused vHChGB is defined as the squared modulus of the field,

$$I(u, v, s) = |E(u, v, s)|^2. \quad (15)$$

By substituting from Eq. (11a) into Eq. (15), and after some algebraic operations we obtain the irradiance of the focused vHChGB, which can be expressed as

$$I(u, v, s) = I_a(s) \exp \left\{ \frac{-2 \left(\frac{\pi N_F}{s} \right)^2 (u^2 + v^2)}{\left[1 + \left(\frac{\pi N_F}{s} (s-1) \right)^2 \right]} \right\} \times \sum_{l=0}^M C_l^M (i)^{-l} \times \sum_{\kappa=0}^M C_{\kappa}^M (-i)^{-\kappa} \times \left[G_{n,(l,\kappa)}^-(u) + G_{n,(l,\kappa)}^+(u) \right] \left[G_{n,(M-l,M-\kappa)}^-(v) + G_{n,(M-l,M-\kappa)}^+(v) \right], \quad (16a)$$

where

$$G_{n,(l,\kappa)}^m(u) = \sum_{p=0}^{\left[\frac{n}{2} \right]} \frac{n!}{p!(n-2p)!} \sum_{\tau=0}^{\left[\frac{n}{2} \right]} \frac{n!}{\tau!(n-2\tau)!} \left(\frac{\beta}{2s} \right)^{\tau+p} \times H_{l+n-2p}(\delta(u m \alpha s)) \left\{ \begin{array}{l} \exp(\pm 2S_1 u) H_{\kappa+n-2\tau}(\delta^*(u \pm \alpha s)) \\ + \exp(\pm 2S_2 u) H_{\kappa+n-2\tau}(\delta^*(u m \alpha s)) \end{array} \right\}, \quad (16b)$$

where

$$I_a(s) = \frac{(\pi/4)^2 (s)^{M+2n} (2)^{2n-2M}}{\left[\left(\frac{s}{N_F} \right)^2 + (\pi(s-1))^2 \right] \left[(s)^2 + (\pi N_F (s-1))^2 \right]^{\frac{M}{2}+n}} \times \exp \left\{ \frac{b^2}{\left[1 + \left(\frac{\pi N_F (s-1)}{s} \right)^2 \right]} \right\}, \quad (16c)$$

with S_1, S_2, δ , and α are given, respectively by

$$S_1 = \frac{\pi^2 b (1-s)}{\left[\left(\frac{s}{N_F} \right)^2 + (\pi(s-1))^2 \right]}, \quad (16d)$$

$$S_2 = \frac{i\pi b N_F s}{\left[s^2 + (\pi N_F (s-1))^2 \right]}, \quad (16e)$$

$$\delta = \frac{\pi N_F}{\sqrt{s(s + i\pi N_F (1-s))}}, \quad (16f)$$

and

$$\alpha = \frac{ib}{2\pi N_F}. \quad (16g)$$

The on-axis intensity of the focused beam is obtained by substituting $(u, v) = (0, 0)$ into Eq. (16a), so one obtains

$$I(s) = I_a(s) \times \sum_{l=0}^M C_l^M (i)^{-l} \times \sum_{\kappa=0}^M C_{\kappa}^M (-i)^{-\kappa} \times \left[G_{n,(l,\kappa)}^-(s) + G_{n,(l,\kappa)}^+(s) \right] \left[G_{n,(M-l,M-\kappa)}^-(s) + G_{n,(M-l,M-\kappa)}^+(s) \right], \quad (17a)$$

Where

$$G_{n,(l,\kappa)}^m(s) = \sum_{p=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{p!(n-2p)!} \sum_{\tau=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{\tau!(n-2\tau)!} \left(\frac{\beta}{2s} \right)^{\tau+p} \times H_{l+n-2p}(\delta(m\alpha s)) \left[H_{\kappa+n-2\tau}(\delta^*(\pm\alpha s)) + H_{\kappa+n-2\tau}(\delta^*(m\alpha s)) \right], \quad (17b)$$

It is readily seen from Eq. (17a) that in general the on-axis intensity of the focused vHChG beam is not always zero, i.e., there may exist some special beam parameters conditions (i.e., for particular values for n, b and M) for which the focused beam presents a central bright intensity. From the numerical results presented in Section 4, it is noticed that when $M=2$, and $n=1, 3$, the focused vHChGB will present central peak intensity.

4. Beam with of the focused vHChGB

To further analyze the propagation properties of the focused vHChGB, it is useful to investigate the beam spot width (root mean square width or rms for short) within the focus region.

As it is known, the rms width of a general laser beam can be defined from the second-order moment intensity as [31-32]

$$W_{\sigma} = \left(\frac{4}{P_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma^2 |E(x, y, z)|^2 dx dy \right)^{1/2}, \quad (18a)$$

where σ denotes either x or y transverse coordinate, and P_0 is the total power of the beam and is given by

$$P_0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |E(x_0, y_0, z=0)|^2 dx_0 dy_0. \quad (18b)$$

Because of the symmetry of the vHChGB, the second-order moment in x - and y - directions are identical. In the following, therefore, only the calculation steps for x -coordinate will be presented.

The total power P_0 of a vHChGB can be expressed as

$$P_0 = \sum_{l=0}^M C_l^M I_l(b) I_{M-l}(b), \quad (19a)$$

where the typical integral expression $I_R(b)$ is defined as

$$I_R(b) = \int_{-\infty}^{+\infty} u_0^{2R} H_n^2 \left(\frac{\sqrt{2}u_0}{\omega_0} \right) \cosh^2 \left(b \frac{u_0}{\omega_0} \right) \exp \left(-\frac{2u_0^2}{\omega_0^2} \right) du_0. \quad (19b)$$

By applying the integral formula of Eq. (9), and after straightforward algebraic calculations,

$I_R(b)$ can be expressed as

$$I_R(b) = \frac{1}{2} \sqrt{\pi} \left(\frac{1}{2i} \right)^{2R} \left(\frac{w_0}{\sqrt{2}} \right)^{2R+1} \sum_{p=0}^{\left[\frac{n}{2} \right]} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\left[\frac{n}{2} \right]} \frac{n!(-1)^n}{q!(n-2q)!} \times \left[\exp \left(\frac{b^2}{2} \right) H_{2R+2n-2p-2q} \left(\frac{ib}{\sqrt{2}} \right) + H_{2R+2n-2p-2q}(0) \right] \quad (19c)$$

It is too complicated to perform the beam width directly from Eq. (11a), fortunately, one can calculate it by means of the transformation law of the second-order moments, which is given as [32]

$$\langle x^2 \rangle = A^2 \langle x_0^2 \rangle + 2AB \langle x_0 \theta_{x0} \rangle + B^2 \langle \theta_{x0}^2 \rangle, \quad (20)$$

where x and x_0 are the transverse coordinates in the output plane z and input plane $z=0$, respectively. At the input plane $z=0$, the second-order intensity moments are given by [32]

$$\langle x_0^2 \rangle = \frac{1}{P_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_0^2 |E(x_0, y_0, z=0)|^2 dx_0 dy_0, \quad (21a)$$

$$\langle x_0 \theta_0 \rangle = \frac{1}{2ikP_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_0 E(x_0, y_0, z=0) \frac{\partial E^*(x_0, y_0, z=0)}{\partial x_0} dx_0 dy_0 + c.c., \quad (21b)$$

and

$$\langle \theta_0^2 \rangle = \frac{-1}{P_0 k^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x_0, y_0, z=0) \frac{\partial^2 E^*(x_0, y_0, z=0)}{\partial x_0^2} dx_0 dy_0 + c.c. \quad (21c)$$

The application of Eq. (21b) for the incident vHChGB gives the value zero, $\langle x_0 \theta_{x0} \rangle = 0$.

Thus, for our lens system, Eq. (20) leads to

$$\langle x^2 \rangle = (s-1)^2 \langle x_0^2 \rangle + \frac{w_0^4}{\lambda^2 N_F^2} s^2 \langle \theta_0^2 \rangle. \quad (22)$$

Substituting from Eq. (1) into Eq. (21a), and after some algebraic manipulations, one obtains

$$\langle x_0^2 \rangle = \frac{1}{P_0} \sum_{l=0}^M C_l^M I_{l+1}(b) I_{M-l}(b). \quad (23)$$

Substituting from Eq. (1) into Eq. (21c) and using the same calculation technique as above, then after algebraic operations, the result can be expressed as

$$\langle \theta_{0x}^2 \rangle = \frac{-1}{P_0 k^2} [Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + Q_{10}], \quad (24)$$

where

$$Q_1 = M(M-1) \sum_{l=0}^{M-2} C_l^{M-2} [I_{l+1}(b) I_{M-l-2}(b) - I_l(b) I_{M-l-1}(b)], \quad (25a)$$

$$Q_2 = \left(\frac{2\sqrt{2}}{\omega_0} \right)^2 n(n-1) \sum_{l=0}^M C_l^M Y_l(b) I_{M-l}(b), \quad (25b)$$

$$Q_3 = \left(\frac{b}{w_0} \right)^2 P_0, \quad (25c)$$

$$Q_4 = \frac{2P_0}{w_0^2} \left[\frac{2}{w_0^2} \langle x_0^2 \rangle - 1 \right], \quad (25d)$$

$$Q_5 = \left(\frac{4\sqrt{2}}{\omega_0} \right) nM \sum_{l=0}^{M-1} C_l^{M-1} B_l(b) I_{M-l-1}(b), \quad (25e)$$

$$Q_6 = \left(\frac{2Mb}{w_0} \right) \sum_{l=0}^{M-1} C_l^{M-1} J_l(b) I_{M-l-1}(b), \quad (25f)$$

$$Q_7 = \left(\frac{-4M}{w_0^2} \right) \sum_{l=0}^{M-1} C_l^{M-1} I_{l+1}(b) I_{M-l-1}(b), \quad (25g)$$

$$Q_8 = \left(\frac{-2}{\omega_0^2} \right) \left(\frac{4\sqrt{2}}{\omega_0} \right) n \sum_{l=0}^M C_l^M B_l(b) I_{M-l}(b), \quad (25h)$$

$$Q_9 = \left(\frac{-4}{w_0^2} \right) \left(\frac{b}{w_0} \right) \sum_{l=0}^M C_l^M J_l(b) I_{M-l}(b), \quad (25i)$$

and

$$Q_{10} = \left(\frac{b}{\omega_0} \right) \left(\frac{4\sqrt{2}}{\omega_0} \right) n \sum_{l=0}^M C_l^M G_l(b) I_{M-l}(b), \quad (25j)$$

where $I_R(b)$ is given by Eq. (19e) and $J_R(b)$, $Y_R(b)$, $B_R(b)$ and $G_R(b)$ are indicated by

$$J_R(b) = \frac{1}{2} \sqrt{\pi} \left(\frac{1}{2i} \right)^{2R+1} \left(\frac{w_0}{\sqrt{2}} \right)^{2R+2} \sum_{p=0}^{\left[\frac{n}{2} \right]} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\left[\frac{n}{2} \right]} \frac{n!(-1)^n}{q!(n-2q)!} \exp\left(\frac{b^2}{2} \right) H_{2R+2n-2p-2q+1} \left(\frac{ib}{\sqrt{2}} \right), \quad (25k)$$

$$Y_R(b) = \frac{1}{2} \sqrt{\pi} \left(\frac{1}{2i} \right)^{2R} \left(\frac{w_0}{\sqrt{2}} \right)^{2R+1} \sum_{p=0}^{\left[\frac{n}{2} \right]} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\left[\frac{n-2}{2} \right]} \frac{(-1)^{n-1} (n-2)!}{q!(n-2q-2)!} \left[\exp\left(\frac{b^2}{2} \right) H_{2R+2n-2p-2q-2} \left(\frac{ib}{\sqrt{2}} \right) + H_{2R+2n-2p-2q-2}(0) \right], \quad (25l)$$

$$B_R(b) = \frac{1}{4} \sqrt{\pi} \left(\frac{1}{2i} \right)^{2R} \left(\frac{w_0}{\sqrt{2}} \right)^{2R+2} \sum_{p=0}^{\left[\frac{n}{2} \right]} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\left[\frac{n-1}{2} \right]} \frac{(-1)^n (n-1)!}{q!(n-2q-1)!} \left[\exp\left(\frac{b^2}{2} \right) H_{2R+2n-2p-2q} \left(\frac{ib}{\sqrt{2}} \right) + H_{2R+2n-2p-2q}(0) \right], \quad (25m)$$

and

$$G_R(b) = \frac{1}{4} \sqrt{\pi} \left(\frac{1}{2i} \right)^{2R-1} \left(\frac{w_0}{\sqrt{2}} \right)^{2R+1} \sum_{p=0}^{\left[\frac{n}{2} \right]} \frac{n!}{p!(n-2p)!} \sum_{q=0}^{\left[\frac{n-1}{2} \right]} \frac{(n-1)!(-1)^n}{q!(n-2q-1)!} \exp\left(\frac{b^2}{2} \right) H_{2R+2n-2p-2q-1} \left(\frac{ib}{\sqrt{2}} \right). \quad (25n)$$

The analytical formula for the beam width of the focused vHChGB is deduced directly from from Eq. (22) to Eq. (25). The obtained expression, which is dependent explicitly on the beam parameters and the Fresnel number, is convenient to investigate the evolution of beam within the focus region.

5. Numerical calculations and analysis

Based on the analytical formulas obtained above, several numerical calculations are performed to illustrate the focusing properties and focal shift of the vHChGB in the focus region under different beam parameters conditions. The calculation parameters are set to be $\lambda=632.8 \text{ nm}$ and $\omega_0=1 \text{ mm}$, and $f=50 \text{ mm}$. Since the incident vHChGB can have one of two configurations depending on the value of b (see Fig. 1 in Sec. 2), thus, in the following the two beam configurations are examined separately.

Fig. 4 presents the irradiance distribution (contour lines) of the focused vHChGB at different propagation distances. The plots show an asymmetry in the evolution of the irradiance with respect to the geometric focus plane ($z=f$) and the beam energy is focused slightly just before the lens focal plane. The complete symmetry of the irradiance evolution is realized in front of and behind the actual focus plane. Furthermore, one can clearly see that nearby the focus the irradiance pattern of the vHChGB with $M=1$ is fan blades like pattern; the beam profile consists of $4n$ main blade spots and a hole-intensity at the beam center. The sense of rotation of the light blades changes from anticlockwise to clockwise as the beam passes beyond the actual focal plane.

Figure 4: Contour lines of the intensity distribution of the focused vHChGB with $M=1$ in the focal region for different b and n , with $\omega_0 = 1\text{ mm}$ and $N_F = 1$.

From the top to bottom rows : (a) $n=1$ (rows 1-2) (b) $n=2$ (rows 3-4) and (c) $n=3$ (rows 5-6).

The calculation parameters used in Fig. 5 are the same as those in Fig. 4 except that the vortex charge is $M=2$. It can be seen from the plots of Fig. 5 that when $n=2$, the beam profile is quite similar to that obtained for $M=1$ in Fig. 4. While for small b and odd valued n , $n=1, 3$, the output field presents a central bright lobe (see Fig. 6). One can notice also that in far-field the beam with large values of b will have four main lobes surrounding many faint secondary lobes.

Figure 5: the parameters are the same as Fig. 4, except $M=2$.

Figure 6: Contour lines of irradiance distribution of vHChGB with $M=2$ at the real focal plane z_m with $w_0 = 1\text{mm}$, $M=2$, $N_F = 1$ for different values of b and n .

The beam spot width of the focused vHChGB is calculated from Eq. (18a) and its variation against the propagation distance under different parameters conditions is shown in Fig. 7, from which it can be seen that the beam spot size reaches a minimum at certain plane z_m just before the lens focus. Also, one can note that when the parameter b is large the output beam at the real focus has nearly the same width even if n varies (see the bottom row in Fig. 7), and in this case the focal shift is slightly affected by the change of n . When $M=1, 2$ and 4 , the plots show that the real focus plane is more shifted when M is smaller.

The focal shift variation under different beam parameters conditions is illustrated in Fig. 8, from which it is seen that it always negative and decreases monotonously with the increase of N_F until it conceals out asymptotically for large N_F . Furthermore, one can easily see that for a fixed N_F , the focal shift is smaller when the parameter b is larger, and it is nearly intensive to the beam order value, which confirms the result obtained in Fig. 7.

Figure 7: Variations of the beam with $W_x(z)$ against relative axial distance z/f with $w_0 = 1\text{ mm}$ and $N_F = 1$ for different values of M and n .

Figure 8: The relative focal shift Δz versus N_F for different values of b and M : (a) $n=1$, (b) $n=2$ and (c) $n=3$.

6. Conclusion

Based on the Huygens-Fresnel diffraction integral, we have derived the analytical expression for a vHChGB focused by a thin lens system. Also, the closed-form formula for the beam spot width (rms) is obtained by using the second-moment intensity method. It is shown from the numerical examples that the intensity distribution of the beam in the focal region depends on the Fresnel number N_F and the beam parameters b , n and M . The focal shift which is determined from the minimum beam spot width criterion, is affected strongly by the change of N_F , the parameter b , and slightly altered by the vortex charge M and nearly insensitive to the beam order n . For a fixed value of N_F , the focal shift is smaller when the parameter b or M is larger. The focal shift decreases monotonously with the increase of N_F until it vanishes asymptotically for large N_F . The obtained results may be useful for the applications of the vHChGB in beam shaping and beam focusing.

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Figures

Figure 1

The intensity distribution of a vHChGB at the source plane with $\omega_0=1$ mm and $M=1$ for different values of n and b

Figure 2

The intensity distribution of vHChGB at the initial plane with $\omega_0 = 1$ and $n=3$ for $M=1$ (top row), $M=2$ (middle row), 4 and $M=6$ (bottom row) and (a.1-4) $b=0$, (b.1-4) $b=1.5$ and (c.1-4) $b=4$.

Incident vHChGB

Figure 3

Schematic illustration of the focusing lens system.

Figure 4

Contour lines of the intensity distribution of the focused vHChGB with $M=1$ in the focal region for different b and n , with $\omega_0=1$ and $NF=1$. From the top to bottom rows : (a) $n=1$ (rows 1-2) (b) $n=2$ (rows 3-4) and (c) $n=3$ (rows 5-6).

Figure 5

The parameters are the same as Fig. 4, except $M=2$.

Figure 6

Contour lines of irradiance distribution of vHChGB with $M=2$ at the real focal plane Z_m with $W_0 = 1\text{mm}$, $M=2$, $NF=1$ for different values of b and n .

Figure 7

Variations of the beam with $W(z)$ against relative axial distance Z/F with and $W_0 = 1\text{mm}$ and $N_F = 1$ for different values of M and n .

Figure 8

The relative focal shift Δz versus NF for different values of b and M : (a) $n=1$, (b) $n=2$ and (c) $n=3$.