Hilbert’s First Problem and the New Progress of Infinity Theory

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Hilbert's First Problem and the New Progress of Infinity Theory

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Abstract: In the 19th century, Cantor created the infinite cardinal number theory based on the "1-1 correspondence" principle. The continuum hypothesis is proposed under this theoretical framework. In 1900, Hilbert made it the first problem in his famous speech on mathematical problems, which shows the importance of this question. We know that the infinitesimal problem triggered the second mathematical crisis in the 17-18th centuries. The Infinity problem is no less important than the infinitesimal problem. In the 21st century, Sergeyev introduced the Grossone method from the principle of "whole is greater than part", and created another ruler for measuring infinite sets. The discussion in this paper shows that, compared with the cardinal number method, the Grossone method enables infinity calculation to achieve a leap from qualitative calculation to quantitative calculation. According to Grossone theory, there is neither the largest infinity and infinitesimal, nor the smallest infinity and infinitesimal. Hilbert's first problem is caused by the incompleteness of the infinity theory.

Key words: Hilbert's first problem; Cardinal numbers method; Grossone method; Continuum paradox; Infinity theory

1. Introduction

In 1874, Cantor introduced the concept of cardinal numbers based on the "1-1 correspondence" principle. Cantor proved that the cardinal number of the continuum， C, is equal to the cardinal number of the power set of the natural number set, $2^{\aleph_0}$, where $\aleph_0$ is the cardinal number of the natural number set. Cantor arranges the cardinal number of infinities from small to large as $\aleph_0$, $\aleph_1$, …, $\aleph_a$, …. Among them, a is an arbitrary ordinal number, which means that the cardinal number of the natural number set， $\aleph_0$， is the smallest infinity cardinal number. Cantor conjectured: $2^{\aleph_0}=\aleph_1$. This is the famous Continuum hypothesis (CH). For any ordinal a, $2^{\aleph_a}=\aleph_{a+1}$ holds, it is called the Generalized continuum hypothesis (GCH) [1].

In 1938 Gödel proved that the CH is not contradictory to the ZFC axiom system. In 1963, Cohen proved that the CH and the ZFC axiom system are independent of each other. Therefore, the CH cannot be proved in the ZFC axiom system [2]-[3].

However, people always have doubts about infinity theory. For example, in the study of Cosmic Continuum, the existing infinity theory shows great limitations[4]-[14].

In the 21st century, Sergeyev started from "the whole is greater than the part" and introduced a new method of counting infinity and infinitesimals, called the Grossone method. The introduced methodology (that is not related to non-standard analysis) gives the possibility to use the same numeral system for measuring infinite sets, working with divergent series, probability, fractals, optimization problems, numerical differentiation, ODEs, etc.[15]-[43]

The Grossone method introduced by Sergeyev takes the number of elements in the natural number set as a total number, marked as ①, as the basic numeral symbol for expressing infinity and infinitesimal, in order to more accurately describe infinity and infinitesimal.

The Grossone method was originally proposed as a Computational Mathematics, but its significance has far exceeded the category of Computational Mathematics. In particular, the Grossone method provides a new mathematical tool for the Cosmic Continuum Theory. A new infinity theory is about to emerge. But the mathematical community has not paid enough attention to this new development.

2. The traditional infinity paradox and the fourth mathematics crisis

In the history of mathematics, there have been three mathematics crises, each of which involves the foundation of mathematics. The first time was the discovery of irrational numbers, the second time was the infinitesimal problem, and the third time was the set theory paradox[44]-[45]. However, no one dare to say that the building of the mathematical theory system has been completed, and maybe the fourth mathematical crisis will appear someday.

In fact, the fourth mathematics crisis is already on the way. This is the infinity problem. In 1900, Hilbert put the Cantor continuum hypothesis as the first question in his famous lecture on 23 mathematics problems [46]. This will never be an impromptu work by an almighty mathematician.

The infinitesimal question unfold around whether the infinitesimal is zero or not. From the 1920s to the 1970s, this problem has been initially solved through the efforts of generations of mathematicians. However, there are still different opinions about the second mathematics crisis. I believe that the infinitesimal problem has not been completely solved, otherwise there would be no infinity problem. Because the infinity problem and the infinitesimal problem are actually two aspects of the same problem.

Let us first look at what is problem with infinity.

The first is the expression of infinity. Now, there are two ways to express the infinity, one is to express with infinity symbol ∞, and the other is to express with infinity cardinal number. However, neither the infinity symbol ∞ nor the infinity cardinal number can effectively express infinity and infinitesimal.

For example: when expressed in the infinity symbol ∞, we cannot distinguish the size of the natural number set and the real number set, nor can we distinguish the size of the natural number set and the integer set, they are all ∞. When expressed in infinity cardinal number, we can distinguish the size of the natural number set and the real number set, because the cardinal number of the natural number set is $\aleph_0$, and the cardinal number of the real number set is $C = 2^{\aleph_0}$; but it is still impossible to distinguish the size of the natural number set and the integer set, they are both $\aleph_0$.

The second is the calculation of infinity. Whether it is the infinity symbol A or the infinity cardinal number, it cannot play a mathematically precise role in calculations. E.g：

$$\infty + 1 = \infty,\quad \infty - 1 = \infty,\quad \infty \times \infty = \infty,\quad \infty^{\infty} = \infty.$$

And $\frac{\infty}{\infty}$, $\infty - \infty$, etc. have no meaning at all.

Relative to infinity symbol ∞, Cantor's infinite cardinal number is an improvement, but the cardinal number method of infinity can only be calculated qualitatively. The theory of infinity cardinal number is based on the principle of "1-1 correspondence". Although according to the principle of "power set is greater than the original set", infinite cardinal number can be compared in size, but it is only the size of classes of infinity , not the size of infinity individuals.

For example, according to the continuum hypothesis, the following equation holds:

$$\aleph_0 + 1 = \aleph_0,\quad \aleph_0 + \aleph_0 = \aleph_0,\quad \aleph_0 + 2^{\aleph_0} = 2^{\aleph_0},\quad 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}.$$

This obviously violates the calculation rules of finite numbers and does not meet the uniformity requirements of mathematical theory.

The reason for the infinity paradox in mathematical expressions and mathematical calculations is that the existing infinity theory does not need to follow the principle of "the whole is greater than the part", and this principle needs to be followed in the finite number theory. In this way, there is a problem of using different calculation rules in the same calculation formula.

Since there is an infinite problem, how can there be no infinitesimal problem?

For example: because the infinity and the infinitesimals are reciprocal of each other (when the infinitesimal is not zero), the following equation holds:

$$\frac{1}{\infty+1} = \frac{1}{\infty},\quad \frac{1}{\infty-1} = \frac{1}{\infty},\quad \frac{1}{\infty\times\infty} = \frac{1}{\infty},\quad \frac{1}{\infty^{\infty}} = \frac{1}{\infty};$$

$$\frac{1}{\aleph_0+1} = \frac{1}{\aleph_0},\quad \frac{1}{\aleph_0+\aleph_0} = \frac{1}{\aleph_0},\quad \frac{1}{\aleph_0+2^{\aleph_0}} = \frac{1}{2^{\aleph_0}},\quad \frac{1}{2^{\aleph_0}+2^{\aleph_0}} = \frac{1}{2^{\aleph_0}}.$$

Obviously, in these equations, although mathematical calculations can also be performed, the mathematical accuracy is lost. At the same time, treating zero as a special infinitesimal is inconsistent with the concept of infinitesimals. Because in modern mathematics, the infinitesimal is not a number but a variable, and zero is a specific number, which is inconsistent with the definition of infinitesimal.

It can be seen that the problem of infinity involves many basic mathematics problems, and the mathematics crisis caused by it is no less than the previous three mathematics crises. No wonder Hilbert listed the continuum problem as the top of the 23 mathematical problems.

3. Grossone method and quantitative calculation of infinity

Sergeyev used Grossone $\textcircled{1}$ to represent the number of elements in set of natural numbers, which is similar to Kantor's cardinal number method. Kantor's cardinal number and Sergeyev's Grossone $\textcircled{1}$ are superficially the same thing. Both represent the size of the set of natural numbers, but they are two completely different concepts.

The cardinal number represents the size of a type of set that satisfies the principle of "1-1 correspondence". For a finite set, the cardinal number is the "number" of elements, but for an infinite set, the cardinal number is not the "number" of elements. Is the size of a class of infinite sets that are equivalent to each other. And Grossone $\textcircled{1}$ represents the "number" of elements in a natural number set, just like any finite set. Using this as a ruler, you can measure every infinity and infinitesimal.

In Grossone theory, infinity and infinitesimal are not variables, but definite quantities. Infinity and infinitesimal are the reciprocal of each other. For example, the number of elements $\textcircled{1}$ of the natural number set is an infinity, and its reciprocal $\frac{1}{\textcircled{1}}$ is an infinitesimal. Obviously, zero is not an infinitesimal.

Let us see how numbers are expressed. The decimal numeral we generally use now are: 1,2,3,4,5,6,7,8,9,0. Among these 10 numeral, the largest numeral is 9, but we can use them to express all finite numbers, whether it is ten thousand digits, billion digits, or larger numbers.

As the number of elements in the natural number set, Grossone, together with 1,2,3,4,5,6,7,8,9,0, can express any finite number and infinity.

For example, according to the principle of "whole is greater than part", we can get:

$$\textcircled{1}+1 > \textcircled{1},\quad \textcircled{1}+\textcircled{1}=2\textcircled{1},\quad \textcircled{1}+2^{\textcircled{1}} > 2^{\textcircled{1}},\quad 2^{\textcircled{1}}+2^{\textcircled{1}}=2\times 2^{\textcircled{1}}$$

The Grossone method can not only accurately express infinity, but also accurately express infinitesimal. E.g:

$$\frac{1}{2\textcircled{1}},\qquad \frac{2}{3\textcircled{1}^2},\qquad \frac{3}{2^{\textcircled{1}}}$$

For example, infinity can be operated like a finite number:

$$0\cdot\textcircled{1}=\textcircled{1}\cdot 0=0,\quad \textcircled{1}-\textcircled{1}=0,\quad \frac{\textcircled{1}}{\textcircled{1}}=1,\quad \textcircled{1}^0=1,\quad 1^{\textcircled{1}}=1,\quad 0^{\textcircled{1}}=0$$

$$\lim_{x\to\textcircled{1}}\frac{1}{x}=\frac{1}{\textcircled{1}}\ , \qquad \lim_{x\to 2^{\textcircled{1}}}\frac{1}{x}=\frac{1}{2^{\textcircled{1}}}\ , \qquad \lim_{x\to\frac{1}{\textcircled{1}}}x^{3}=\frac{1}{\textcircled{1}^{3}}$$

$$\int_{0}^{\textcircled{1}}x^{2}dx=\frac{1}{3}\textcircled{1}^{3}\ , \quad \int_{\textcircled{1}}^{\textcircled{1}^{2}}x^{2}dx=\frac{1}{3}(\textcircled{1}^{6}-\textcircled{1}^{3})\ , \quad \int_{0}^{2^{\textcircled{1}}}x^{2}dx=\frac{1}{3}\cdot 2^{3\textcircled{1}}$$

More importantly, the Grossone method solves the calculation problems of $\frac{\infty}{\infty}$, $\infty-\infty$, etc. that cannot be performed in the infinity theory.

For example, the following calculations are possible:

$$\frac{\textcircled{1}}{2\textcircled{1}}=\frac{1}{2}, \qquad \frac{2\textcircled{1}}{3\textcircled{1}^{3}}=\frac{2}{3\textcircled{1}^{2}}, \qquad 3\textcircled{1}-\textcircled{1}=2\textcircled{1}$$

It can be seen that the Grossone method meets the requirements of the unity of mathematical theory. From the above discussion, we can see that the cardinal method uses the "1-1 correspondence" principle but violates the "whole is greater than the part" principle, while the Grossone method uses the "whole is greater than the part" principle, but does not violate the "1-1 correspondence" principle.

Therefore, the new infinity theory can integrate the infinity cardinal number method with the Grossone method. But when using the infinity cardinal number theory to calculate, we should not use the "=" symbol, but can use "$\equiv$" to indicate that it is equivalent under the "1-1 correspondence" principle. E.g:

$$\aleph_0+1\equiv\aleph_0, \quad \aleph_0+\aleph_0\equiv\aleph_0, \quad \aleph_0+2^{\aleph_0}\equiv 2^{\aleph_0}, \quad 2^{\aleph_0}+2^{\aleph_0}\equiv 2^{\aleph_0};$$

$$\frac{1}{\aleph_0+1}\equiv\frac{1}{\aleph_0}, \quad \frac{1}{\aleph_0+\aleph_0}\equiv\frac{1}{\aleph_0}, \quad \frac{1}{\aleph_0+2^{\aleph_0}}\equiv\frac{1}{2^{\aleph_0}}, \quad \frac{1}{2^{\aleph_0}+2^{\aleph_0}}\equiv\frac{1}{2^{\aleph_0}};$$

$$\textcircled{1}+1\equiv\aleph_0, \quad \textcircled{1}+\textcircled{1}\equiv\aleph_0\ , \quad \textcircled{1}+2^{\textcircled{1}}\equiv 2^{\aleph_0}, \quad 2^{\textcircled{1}}+2^{\textcircled{1}}\equiv 2^{\aleph_0};$$

$$\frac{1}{\textcircled{1}+1}\equiv\frac{1}{\aleph_0}, \quad \frac{1}{\textcircled{1}+\textcircled{1}}\equiv\frac{1}{\aleph_0}, \quad \frac{1}{\textcircled{1}+2^{\textcircled{1}}}\equiv\frac{1}{2^{\aleph_0}}, \quad \frac{1}{2^{\textcircled{1}}+2^{\textcircled{1}}}\equiv\frac{1}{2^{\aleph_0}}.$$

However, things are not so simple. Sergeyev also encountered a mathematical problem, which is the "maximal number paradox." Just imagine, if $\textcircled{1}$ represents the number of elements in a set of natural numbers, is $\textcircled{1}+1$ a natural number? If $\textcircled{1}+1$ is a natural number, because of $\textcircled{1}+1>\textcircled{1}$, then the number of elements in the natural number set is not $\textcircled{1}$.

Sergeyev thought $\textcircled{1}+1\notin N$, and the number greater than $\textcircled{1}$ is called an extended number [40]. But this is hard to make sense, because $\textcircled{1}+1$ fully conforms to the definition of natural numbers, and the extended natural numbers are still natural numbers. We will discuss this issue later.

4. Grossone is a number-like symbol used for calculations

In Cantor's infinite cardinal theory, the cardinal number of the natural number set, $\aleph_0$, is the smallest infinite cardinal number. Using Grossone method, the set of natural numbers can also be decomposed into smaller sets of infinity. For example: the natural numbers set N can be divided into two infinite sets, the odd set and the even set. Let O be the odd set and E be the even set. Then there are:

$$O = \{1,3,5,\ldots,\textcircled{1}-3,\textcircled{1}-1\},\ E = \{2,4,6,\ldots,\textcircled{1}-2,\textcircled{1}\}$$

$$N = O \cup E = \{1,2,3,\ldots,\textcircled{1}-3,\textcircled{1}-2,\textcircled{1}-1,\textcircled{1}\}$$

Obviously, the number of elements in the odd number set and the even number set is $\frac{\textcircled{1}}{2}$, which is less than the number of elements $\textcircled{1}$ in the natural number set.

Sergeyev also created a method of constructing an infinite subset of the natural number set [40]. He uses $N_{k,n}$ ($1 \le k \le n$, $n \in N$, n is a finite number) to indicate a set that the first number is k, and equal difference is n, and the size of the set is $\frac{\textcircled{1}}{n}$.

$$N_{k,n} = \{k, k+n, k+2n, k+3n, \ldots\}$$

$$N = \bigcup_{k=1}^{n} N_{k,n}$$

For example:

$$N_{1,2} = \{1,3,5,\ldots\} = O, \qquad N_{2,2} = \{2,4,6,\ldots\} = E$$

$$N = N_{1,2} \cup N_{2,2} = O \cup E$$

Or:

$$N_{1,3} = \{1,4,7,\ldots\}, \qquad N_{2,3} = \{2,5,8,\ldots\}, \qquad N_{3,3} = \{3,6,9,\ldots\}$$

$$N = N_{1,3} \cup N_{2,3} \cup N_{3,3}$$

Grossone $\textcircled{1}$ is a numeral symbol that represents the number of elements in natural numbers set. However, the set of integers and real numbers are larger than the set of natural numbers. According to the principle of "the whole is greater than the part", does it mean that there are integers and real numbers greater than $\textcircled{1}$?

Below we use Grossone method to examine the integer set Z and real number set R.

$$Z = \{-\textcircled{1}, -\textcircled{1}+1, \ldots, 2, 1, 0, 1, 2, \ldots, \textcircled{1}-1, \textcircled{1}\}$$

$$R = [-\textcircled{1}, -\textcircled{1}+1) \cup \ldots \cup [1,0) \cup \{0\} \cup (0,1] \cup \ldots \cup (\textcircled{1}-1, \textcircled{1}]$$

It is easy to see that there are no integers and real numbers exceeding $\textcircled{1}$ in both the integer set and the real number set.

The number of elements in the integer set is $2\textcircled{1}+1$; because the number of elements in (0,1] is $10^{\textcircled{1}}$, the number of elements in the real number set is $C = 2\textcircled{1} \cdot 10^{\textcircled{1}} + 1$. It can be seen that the set of real numbers is not the power set of the set of natural numbers. Obviously, Integer set and real number set the number of elements in are all greater than $\textcircled{1}$.

The integer set and real number set are larger than the natural number set, which refers to the number of elements, rather than the existence of numbers exceeding $\textcircled{1}$ in the integer set and real number set. In fact, $\textcircled{1}$ is not a number, but infinity. No number can exceed infinity, and $\textcircled{1}$ is a symbol for infinity.

Looking back at the problem of the "maximum number paradox" now, it is not difficult to solve it.

The problem lies in the qualitative aspect of A. In fact, A is just a number-like symbol used for infinity calculations, and is a ruler used to measure all infinity sets.

Take $\textcircled{1}+1$ as an example. First, $\textcircled{1}+1$, like $\textcircled{1}$, is infinity, not a numeral. Second, $\textcircled{1}+1 > \textcircled{1}$, indicating that this infinite set exceeds a single Grossone $\textcircled{1}$. Exceeding does not mean that it cannot be expressed. It is like measuring an object with a ruler. It does not matter if the object exceeds the ruler. You can measure a few more times. $\textcircled{1}$ is the ruler for measuring the infinite set. An infinite set is 1 more than this ruler. You can measure it more. After the measurement is accurate, mark it as $\textcircled{1}+1$.

Let A be an infinite set of $\textcircled{1}+1$ elements, then A can be written as:

$$A = N \cup \{1\} = \{1, 2, \ldots, \textcircled{1}-1, \textcircled{1}, 1\}$$

Or:

$$A = N \cup \{\textcircled{1}+1\} = \{1,2,\ldots,\textcircled{1}-1,\textcircled{1},\textcircled{1}+1\}$$

Or:

$$A = \{a_1, a_2, \ldots, a_{\textcircled{1}-1}, a_{\textcircled{1}}, a_{\textcircled{1}+1}\}$$

It can be seen that the so-called "maximum number paradox" does not exist for Grossone method.

5. "Continuum paradox" and relative continuum theory

The continuum originally refers to the real numbers set. Since the real number corresponds to the point 1-1 on the straight line, the straight line is intuitively composed of continuous and unbroken points, so the real number set is called the continuum. In the number sequence, the set that satisfies the "1-1 correspondence" relationship with the interval (0, 1) is called the continuum.

Traditional mathematics has an axiom: a point has no size. Taking the interval $(0,1]$ on the number line as an example, since there are infinitely many points on the interval $(0,1]$, the size s of the point in the interval $(0,1]$ is: $s = \lim_{x\to\infty}\frac{1}{x} = 0$. This proof uses the potential infinity thoughts. In mathematics, potential infinity and actual Infinity are two different views on infinity. Potential infinityists believe that infinity is not completed, but infinity in terms of its development, and infinity is only potential. Actual infinityists believe that infinity is a real, completed, existing whole. The theory of calculus adopts the concept of potential infinity, while Cantor's cardinality theory and Sergeyev's Grossone $\textcircled{1}$ theory adopt the concept of actual infinity.

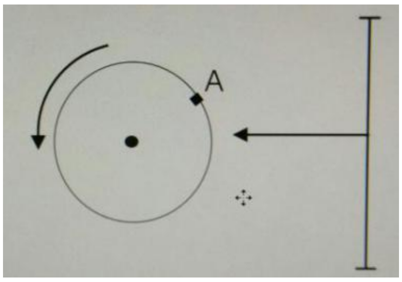
If the idea of actual infinity is adopted, by cardinal number method, the calculation method of the size of the point should be: because the interval $(0,1]$ is a continuum, its cardinal number is C, and the continuum is a linear ordered set of "dense and no holes", that is, the distance between two adjacent points is 0, so the size of the point in the interval $(0,1]$ is: $s = \frac{1}{\mathrm{C}}$. According to the cardinal number method, the cardinal number of the continuum is $C = 2^{\aleph_0} > \aleph_0$, so $\frac{1}{\mathrm{C}} < \frac{1}{\aleph_0}$, which indicates that the reciprocal of the cardinal number of the infinity is infinitesimal rather than zero, otherwise $\frac{1}{\mathrm{C}} = \frac{1}{\aleph_0}$, contradicts $\frac{1}{\mathrm{C}} < \frac{1}{\aleph_0}$. Therefore $s = \frac{1}{\mathrm{C}} > 0$.

However, according to the Grossone method, because the number of elements in $(0,1]$ is $10^{\textcircled{1}}$, the size of the point in the interval $(0,1]$ is: $s = \frac{1}{10^{\textcircled{1}}} > 0$.

Not only does the dot have a size, but the size of the dot is related to the decimal or binary system of the number on the number axis. For example, when using binary system, the number of elements in $(0,1]$ is $2^{①}$, and the size of the point in the interval $(0,1)$ is: $s = \frac{1}{2^{①}}$.

Imagine that one-dimensional straight lines, two-dimensional planes, three-dimensional and multi-dimensional spaces, etc. are all composed of points. If the size of a point is zero, how to form a straight line, plane and space with size? The Grossone method solves this infinitesimal puzzle.

We use a probability problem exemplified by Sergeyev to illustrate [40].



As shown in the figure above, suppose the radius of the disc in the figure is r, and the disc is rotating. We want to ask a probabilistic event E: What is the probability that point A on the disk stops just in front of the fixed arrow on the right? According to the traditional calculation method, point A has no size, so the probability of occurrence of E is:

$$P(E) = \lim_{h \to 0} \frac{h}{2\pi r} = 0$$

This is obviously contrary to experience and common sense. And if the size of the point is solved, such as $s = \frac{1}{10^{①}}$, then you can get:

$$P(E) = \frac{1}{2\pi r \cdot 10^{①}}$$

This is the logical result. This result can also be explained from the traditional mathematical axiom that "a point has no size", that is, the distance between two adjacent points in the continuum is not 0, but the continuum is not "dense and no holes". This forms a "Continuum paradox": either violate "a point has no size", or violate "the continuum is dense and no holes".

The concept of relative continuity proposed by Sergeyev in Grossone ① theory solves this problem well[40].

Sergeyev established the relative continuity on the function $f(x)$. The point that stipulates the range of the independent variable $[a,b]_S$ of $f(x)$ can be a finite number or an infinity, but the set $[a,b]_S$ is always discrete, where S represents a certain numeral system. In this way, for any point $x \in [a,b]_S$, its nearest left and right neighbors can always be determined:

$$x^{+} = \min\{z : z \in [a,b]_s, z > x\}$$

$$x^{-} = \max\{z : z \in [a,b]_s, z < x\}$$

Suppose a set $X = [a,b]_s = \{x_0, x_1, \ldots, x_{n-1}, x_n\}_s$, where $a = x_0$, $b = x_n$, and the numeral system S allow a certain unit of measure μ to be used to calculate the coordinates of the elements in the set. If for any $x \in (a,b)_S$, $x^{+} - x$ and $x - x^{-}$ are infinitesimal, then the set X is said to be continuous in the unit of measure μ. Otherwise, set X is said to be discrete in the unit of measure μ.

For example, if the unit of measure μ is used to calculate that the position difference between adjacent elements of set X is equal to ①^{-1}, then set X is continuous in the unit of measure μ; but if the unit of measure $\upsilon = \mu \cdot \text{①}^{-3}$ is used instead, calculate that the position difference between adjacent elements of the set X is equal to ①^{2}, then the set X is discrete in the unit of measure υ. Therefore, whether the set X is continuous or discrete depends on the size of the unit of measure μ.

Function $f(x)$ is continuous in the unit of measure at some point $x \in (a,b)_S$ in $[a,b]_S$, if $f(x^{+}) - f(x)$ and $f(x) - f(x^{-})$ are both infinitesimal. If only one is infinitesimal, it can be called left continuous or right continuous. If function $f(x)$ is continuous in the unit of measure μ at each point of $[a,b]_S$, then $f(x)$ is said to be continuous in the unit of measure μ on set $X = [a,b]_s$.

In layman's terms, relative continuity is the continuity associated with a unit of measure. Assuming that the distance between any adjacent elements in a set is infinitesimal under a certain unit of measurement, then the set is continuous for that unit of measurement, and discrete otherwise. By this definition, the same set that is continuous for one unit of measure may be discrete for another. The theory of relative continuity realizes the unity of continuity and discreteness. In the theory of relative continuity, the traditional mathematical axiom "a point has no size" still holds, but the distance between two adjacent points is not 0. In order to distinguish it from the existing continuum theories, I refer to the traditional continuum as the absolute continuum, and the relative continuity set as the relative continuum. It can be seen from the above discussion that the absolute continuum is only a special case of the relative continuum.

6. Discussion and conclusion

Actual infinity and potential infinity are two different views of infinity in the history of mathematics. Cardinal number theory and Grossone theory are actual infinite theory, while calculus theory is potential infinite theory, which shows that both actual infinite and potential infinite are reasonable. The question is, are these two views of infinity really incompatible? No!

The essence of mathematics is always contained in the essence of the universe. In other words, any mathematical theory is a reflection of some universal truth. The same is true for actual infinity and potential infinity. They reflect two mathematical truths in the infinite field, and they are compatible mathematical ideas.

The cognition of human logarithm has gone through the process from natural number to integer, from rational number to irrational number, from real number to complex number, and from potential infinity to actual infinity. And every breakthrough in the concept will lead to a mathematical revolution.

Before the calculus theory, people formed a philosophical understanding of actual infinity and potential infinity. Calculus theory makes potential infinity enter the mathematical kingdom with limit thought; Set theory makes actual infinity enter the realm of mathematics with cardinal number thought. However, the infinite theory has not been completely cracked so far. The discovery of Grosson's theory is a new development of actual infinite theory. Grossone theory not only adds new members to the mathematical kingdom, but also makes people have a further understanding of the concept of logarithm.

Now let's put actual infinity and potential infinity together into the family of numbers.

Limit theory: The number of elements in the natural number set and the number of elements in the real number set are both ∞.

Cardinality theory: The cardinality of the set of natural numbers is $\aleph_0$, and the cardinality of the set of real numbers is C, $C = 2^{\aleph_0}$.

Grossone theory: The number of elements in the set of natural numbers is ①, and the number of elements in the set of real numbers is $C = 2① \cdot 10^{①} + 1$.

It is not difficult to see that the above three infinity theories have constantly deepened their understanding of infinity, and the three infinity theories have different application fields.

The discussion in this article shows that:

(1) Cantor used the cardinal number method to solve the problem of comparing infinity; Sergeyev used Grossone method to solve the problem of unifying the calculation rules of infinity and finite numbers.

(2) The continuum in traditional mathematics refers to a collection of "dense and no holes", the relative continuum is a continuum that changes with the change of measurement units.

(3) Grossone method is a scientific infinity theory like the cardinal number method; in the new infinity theory, infinity and infinity can be mathematically calculated like finite numbers.

(4) Mathematics and the basic theories of physics have always been intertwined and developed, such as classical mechanics and calculus, relativity and non-Euclidean geometry, etc., which are all good stories in the history of science. The relative continuum theory provides a new path for the study of the cosmic continuum.

(5) Grossone theory makes Hilbert's first problem self-explanatory. According to the principles of "power set is greater than original set" and "whole is greater than part", there is neither the largest infinity and infinitesimal, nor the smallest infinity and infinitesimal.

Competing interests statement: I declare that there is no competing Interest.

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