On the order of the non-primes of type $6n+5$ and $6n+1$

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Abstract

I introduce three natural number sequences which together cover all non-primes of type $6n + 5$ and $6n + 1$.

- $\langle a_n \rangle = 35, 65, 77, 95, 119, 125, 143, \dots$
- $\langle a_n \rangle = 49, 91, 133, 169, 175, 217, 247, \dots$ This sequence is not known in OEIS until today.
- $\langle a_n \rangle = 25, 55, 85, 115, 121, 145, 175, \dots$

These three natural number sequences are significant because all prime numbers > 3 are contained in the number sequences $6n + 5$ and $6n + 1$. Therefore, this paper shows that prime numbers are not randomly distributed. Furthermore, I show a deterministic primality test using these three natural number sequences.

1 Introduction

Let $\mathbb{N}$ be the set of natural numbers including zero. Then the number sequences $6n + 5$ for all $n \in \mathbb{N} \geq 0$ and $6n + 1$ for all $n \in \mathbb{N} > 0$ cover all odd natural numbers > 3 that are not divisible by 3.

Because all prime numbers > 3 are odd natural numbers and not divisible by 3, they must be of type $6n + 5$ or $6n + 1$. Let us now consider, which natural numbers of type $6n + 5$ and $6n + 1$ are not prime.

2 Under what conditions are numbers of type $6n + 5$ composite and what regularity do these follow?

Let $\mathbb{N}_{even}$ be the set of even natural numbers, $\mathbb{N}_{odd}$ the set of odd natural numbers and “ $\cdot$ ” the symbol for common multiplication. $6n + 5$ is not a prime number if:

Proof.

$$\frac{6n + 5}{a} = b \quad \leftrightarrow \quad a \cdot b = 6n + 5 \quad \leftrightarrow \quad \frac{a \cdot b - 5}{6} = n \quad a, b, n \in \mathbb{N} \quad \text{and} \quad a, b > 1$$

Now let z be $(a \cdot b - 5)$.

- $\frac{z}{6} \in \mathbb{N} \leftrightarrow \frac{z}{2} \in \mathbb{N} \wedge \frac{z}{3} \in \mathbb{N}$ for all $z \in \mathbb{N}$
- $2 \mid (a \cdot b - 5) \leftrightarrow (a \wedge b) \in \mathbb{N}_{odd}$ because $(\mathbb{N}_{odd} \cdot \mathbb{N}_{odd}) \in \mathbb{N}_{odd}$ and $(\mathbb{N}_{odd} \geq 7) - 5 \in \mathbb{N}_{even}$
- $3 \mid (a \cdot b - 5) \leftrightarrow a \cdot b$ has the structure $(3x + 1)(3y - 1)$ for all $x, y \in \mathbb{N} > 0$

because

$$\frac{(3x+1)(3y-1)-5}{3} = \frac{9xy-3x+3y-1-5}{3} = \frac{9xy-3x+3y-6}{3} = 3xy-x+y-2$$

Therefore, for a non-prime number of type $6n+5$, the following must hold for an index n :

$$\frac{(3x+1)(3y-1)-5}{6} = n \quad (1)$$

for all $n \in \mathbb{N}$ and $x, y \in N_{\text{even}}$. $\square$

Let us now insert values for x and y in (1) as described in the following table. The n calculated in this way must be substituted into $6n+5$ to obtain the natural number sequences of the non-primes of type $6n+5$.

x	y	$6n+5$	$\langle a_j \rangle$
$x_0 = 2$	$y = 2, 4, 6, \dots$	$35, 77, 119, \dots$	$\langle a_0 \rangle = 35 + 42m$
$x_1 = 4$	$y = 2, 4, 6, \dots$	$65, 143, 221, \dots$	$\langle a_1 \rangle = 65 + 78m \Rightarrow \langle a_1 \rangle = (35+30) + (42+36)m$
$x_2 = 6$	$y = 2, 4, 6, \dots$	$95, 209, 323, \dots$	$\langle a_2 \rangle = 95 + 114m \Rightarrow \langle a_2 \rangle = (35+2 \cdot 30) + (42+2 \cdot 36)m$
$x_3 = 8$	$y = 2, 4, 6, \dots$	$125, 275, 425, \dots$	$\langle a_3 \rangle = 125 + 150m \Rightarrow \langle a_3 \rangle = (35+3 \cdot 30) + (42+3 \cdot 36)m$
$\dots$	$\dots$	$\dots$	$\langle a_k \rangle = (35+k \cdot 30) + (42+k \cdot 36)m$

All non-primes of type $6n+5$ must therefore be part of the number sequences $\langle a \rangle_{k,m} = (35+30k) + (42+36k)m$ for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$.

All non-primes of type $6n+5$ can thus be represented as zeros of the following parametric sine function:

$$f_1(x) = \sin\left(\left(\frac{\pi}{42+36t}\right) \cdot (x - 35 - 30t)\right) \text{ for all } x \in [35+30t; \infty] \text{ and } t \in \mathbb{N}$$

If x and y are interchanged, the same numbers are produced, just in a different order. Therefore, the interchange of the variables x and y can be ignored in this context.

3 Under what conditions are numbers of type $6n+1$ composite and what regularity do these follow?

$6n+1$ is not a prime number if:

Proof.

$$\frac{6n+1}{a} = b \quad \leftrightarrow \quad a \cdot b = 6n+1 \quad \leftrightarrow \quad \frac{a \cdot b - 1}{6} = n \quad a, b, n \in \mathbb{N} \quad \text{and} \quad a, b > 1$$

Now let z be $(a \cdot b - 1)$.

- $\frac{z}{6} \in \mathbb{N} \leftrightarrow \frac{z}{2} \in \mathbb{N} \wedge \frac{z}{3} \in \mathbb{N}$ for all $z \in \mathbb{N}$
- $2 \mid (a \cdot b - 1) \leftrightarrow (a \wedge b) \in \mathbb{N}_{\text{odd}}$ because $(\mathbb{N}_{\text{odd}} \cdot \mathbb{N}_{\text{odd}}) \in \mathbb{N}_{\text{odd}}$ and $(\mathbb{N}_{\text{odd}} \geq 7) - 5 \in \mathbb{N}_{\text{even}}$
- $3 \mid (a \cdot b - 1) \leftrightarrow a \cdot b$ has either the structure $(3x+1)(3y+1)$ or $(3x-1)(3y-1)$ for all $x, y \in \mathbb{N} > 0$

because:

$$\frac{(3x+1)(3y+1)-1}{3} = \frac{9xy+3x+3y+1-1}{3} = \frac{9xy+3x+3y}{3} = 3xy+x+y$$

and

$$\frac{(3x-1)(3y-1)-1}{3} = \frac{9xy-3x-3y+1-1}{3} = \frac{9xy-3x-3y}{3} = 3xy-x-y$$

for all $x, y \in \mathbb{N}$. Therefore, for a non-prime number of type $6n + 1$, the following must hold for an index n : either

$$\frac{(3x + 1)(3y + 1) - 1}{6} = n \quad (2)$$

or

$$\frac{(3x - 1)(3y - 1) - 1}{6} = n \quad (3)$$

for all $n \in \mathbb{N}$ and $x, y \in \mathbb{N}_{\text{even}}$. $\square$

All non-primes of type $6n + 1$ therefore are contained in two different number sequences. Let us call them “number sequence A of non-primes of type $6n + 1$ ” and “number sequence B of non-primes of type $6n + 1$ ”. Let us first consider number sequence A of non-primes of type $6n + 1$ by using (2).

3.1 Number sequence A of non-primes of type $6n + 1$

x	y	$6n + 1$	$\langle a_j \rangle$
$x_0 = 2$	$y = 2, 4, 6, \dots$	$49, 91, 133, \dots$	$\langle a_0 \rangle = 49 + 42m$
$x_1 = 4$	$y = 2, 4, 6, \dots$	$91, 169, 247, \dots$	$\langle a_1 \rangle = 91 + 78m \Rightarrow \langle a_1 \rangle = (49 + 42) + (42 + 36)m$
$x_2 = 6$	$y = 2, 4, 6, \dots$	$133, 247, 361, \dots$	$\langle a_2 \rangle = 133 + 114m \Rightarrow \langle a_2 \rangle = (49 + 2 \cdot 42) + (42 + 2 \cdot 36)m$
$x_3 = 8$	$y = 2, 4, 6, \dots$	$175, 325, 475, \dots$	$\langle a_3 \rangle = 175 + 150m \Rightarrow \langle a_3 \rangle = (49 + 3 \cdot 42) + (42 + 3 \cdot 36)m$
$\dots$	$\dots$	$\dots$	$\langle a_k \rangle = (49 + k \cdot 42) + (42 + k \cdot 36)m$

All non-primes of type $6n + 1$ of number sequence A must therefore be part of the number sequences $\langle a \rangle_{k,m} = (49 + 42k) + (42 + 36k)m$ for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$.

All non-primes of type $6n + 1$ of number sequence A can thus be represented as zeros of the following parametric sine function:

$$f_2(x) = \sin\left(\left(\frac{\pi}{42+36t}\right) \cdot (x - 49 - 42t)\right) \text{ for all } x \in [49 + 42t; \infty] \text{ and } t \in \mathbb{N}$$

Let us now consider number sequence B of non-primes of type $6n + 1$ by using (3).

3.2 Number sequence B of non-primes of type $6n + 1$

x	y	$6n + 1$	$\langle a_j \rangle$
$x_0 = 2$	$y = 2, 4, 6, \dots$	$25, 55, 85, \dots$	$\langle a_0 \rangle = 25 + 30m$
$x_1 = 4$	$y = 2, 4, 6, \dots$	$55, 121, 187, \dots$	$\langle a_1 \rangle = 55 + 66m \Rightarrow \langle a_1 \rangle = (25 + 30) + (30 + 36)m$
$x_2 = 6$	$y = 2, 4, 6, \dots$	$85, 187, 289, \dots$	$\langle a_2 \rangle = 85 + 102m \Rightarrow \langle a_2 \rangle = (25 + 2 \cdot 30) + (30 + 2 \cdot 36)m$
$x_3 = 8$	$y = 2, 4, 6, \dots$	$115, 253, 391, \dots$	$\langle a_3 \rangle = 115 + 138m \Rightarrow \langle a_3 \rangle = (25 + 3 \cdot 30) + (30 + 3 \cdot 36)m$
$\dots$	$\dots$	$\dots$	$\langle a_k \rangle = (25 + k \cdot 30) + (30 + k \cdot 36)m$

All non-primes of type $6n + 1$ of number sequence B must therefore be part of the number sequences $\langle a \rangle_{k,m} = (25 + 30k) + (30 + 36k)m$ for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$.

All non-primes of type $6n + 1$ of number sequence B can thus be represented as zeros of the following parametric sine function:

$$f_3(x) = \sin\left(\left(\frac{\pi}{30+36t}\right) \cdot (x - 25 - 30t)\right) \text{ for all } x \in [25 + 30t; \infty] \text{ and } t \in \mathbb{N}$$

4 A deterministic primality test

Prove whether a *candidate* is of type $6n + 5$ or $6n + 1$.

- If $\frac{\text{candidate}-5}{6} = n$ for all $n \in \mathbb{N}$, the *candidate* is of type $6n + 5$.
- If $\frac{\text{candidate}-1}{6} = n$ for all $n \in \mathbb{N}$, the *candidate* is of type $6n + 1$.

4.1 Testing numbers of type $6n + 5$

Prove, if

$$f_1(x) = \sin\left(\left(\frac{\pi}{42 + 36t}\right) \cdot (x - 35 - 30t)\right) \quad \text{for all} \quad x \in [35 + 30t; \infty] \quad \text{and} \quad t \in \mathbb{N}$$

has a zero at this candidate:

$$\frac{\text{candidate} - 35 - 30t}{42 + 36t} = a \quad t, a \in \mathbb{N} \quad (4)$$

If there is at least one integer natural number solution for t and a existing, the *candidate* is a composite. If there is no integer natural solution existing for t and a , the *candidate* is prime.

4.2 Testing numbers of type $6n + 1$

Because the non-primes of type $6n + 1$ follow two different number sequences it is necessary to prove both sequences:

4.2.1 Using number sequence A

Prove, if

$$f_2(x) = \sin\left(\left(\frac{\pi}{42 + 36t}\right) \cdot (x - 49 - 42t)\right) \quad \text{for all} \quad x \in [49 + 42t; \infty] \quad \text{and} \quad t \in \mathbb{N}$$

has a zero at this *candidate*:

$$\frac{\text{candidate} - 49 - 42t}{42 + 36t} = a \quad t, a \in \mathbb{N} \quad (5)$$

4.2.2 Using number sequence B

Prove, if

$$f_3(x) = \sin\left(\left(\frac{\pi}{30 + 36t}\right) \cdot (x - 25 - 30t)\right) \quad \text{for all} \quad x \in [25 + 30t; \infty] \quad \text{and} \quad t \in \mathbb{N}$$

has a zero at this *candidate*:

$$\frac{\text{candidate} - 25 - 30t}{30 + 36t} = a \quad t, a \in \mathbb{N} \quad (6)$$

If either (5) or (6) yields any integer natural solution for a and t , the *candidate* is composite. If neither shows an integer natural solution for a and t , the *candidate* is prime.

5 Use of the primality test in practice

The following calculations have been done on <https://wolframalpha.com/> over the integers.

5.1 Candidate 999746703869

999746703869 is a pseudoprime of type $6n + 5$. (<https://oeis.org/A001567/b001567.txt>). To test 999746703869 with wolframalpha the following must be written into the input line:

$$\text{solve}(999746703869 - 35 - 30t)/(42 + 36t) - a = 0, \text{over the integers}$$

This leads to the following result:

$$\frac{999746703869 - 35 - 30t}{42 + 36t} - a = 0$$

Integer solutions:

$$\begin{array}{ll}
a = -408197 & t = -68034 \\
a = -1 & t = -166624450646 \\
a = 68032 & t = 408195 \\
a = 166624450644 & t = -1
\end{array}$$

The natural number solution $a = 68032$ and $t = 408195$ indicates 999746703869 is composite.

5.2 Candidate 999746703713

999746703713 is a prime number of type $6n + 5$.

$$\frac{999746703713 - 35 - 30t}{42 + 36t} - a = 0$$

Integer solutions:

$$\begin{array}{ll}
a = -1 & t = -166624450620 \\
a = 166624450618 & t = -1
\end{array}$$

Because there is no integer natural solution existing for a and t the number 999746703713 is prime.

5.3 Candidate 299945917

299945917 is a strong pseudoprime of type $6n + 1$. (<https://oeis.org/A020243/b020243.txt>)

5.3.1 Testing with number sequence A

$$\frac{299945917 - 49 - 42t}{42 + 36t} - a = 0$$

Integer solutions:

$$\begin{array}{ll}
a = -1 & t = 49990985 \\
a = 26 & t = 306692 \\
a = 134 & t = 61640 \\
a = 377 & t = 22031 \\
a = 22031 & t = 377
\end{array}$$

5.3.2 Testing with number sequence B

$$\frac{299945917 - 25 - 30t}{30 + 36t} - a = 0$$

Integer solutions:

$$\begin{array}{ll}
a = -49990987 & t = -1 \\
a = -306694 & t = -28 \\
a = -61642 & t = -136 \\
a = -22033 & t = -379 \\
a = -379 & t = -22033
\end{array}$$

The natural number solutions seen over a and t indicate 299945917 is composite.

5.4 Candidate 999986341177

999986341177 is a prime number of type $6n + 1$.

5.4.1 Testing with number sequence A

$$\frac{999986341177 - 49 - 42t}{42 + 36t} - a = 0 \quad (7)$$

Integer solutions:

$$\begin{array}{ll} a = -1 & t = 166664390195 \\ a = 166664390195 & t = -1 \end{array}$$

5.4.2 Testing with number sequence B

$$\frac{999986341177 - 25 - 30t}{30 + 36t} - a = 0 \quad (8)$$

Integer solutions:

$$\begin{array}{ll} a = -166664390197 & t = -1 \\ a = -1 & t = -166664390197 \end{array}$$

Because neither equation, [(7) or (8)], has shown at least one integer natural solution for a and t the number 999986341177 is prime.

Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- [OrdnungNichtprimzahlen.xlsx](#)
- [PrimalitytestExamples.pdf](#)
- [Countingprimenumbers.pdf](#)