New Type of Solitary Wave Solution With the Coexisting Peak and Valley for a Perturbed Wave Equation

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New type of solitary wave solution with the coexisting peak and valley for a perturbed wave equation

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Abstract

The perturbed mK(3,1) equation is restudied to further explore the dynamics of solitary wave solutions by combining the geometric singular perturbation theorem and bifurcation analysis in this paper. Besides the solitary waves presented in literature [1–3], we show that this equation possesses a family of solitary waves which decay to some constants determined by their wave speeds and a parameter. It is shown that a portion of the solitary wave solutions to the mK(3,1) equation will persist under small perturbations and the wave speed selection principle is presented as well. In addition to the solitary waves, each of which has only one peak or valley and approximates to a solitary wave of the unperturbed equation as the perturbation parameter tends to zero, we theoretically prove the existence of a new type of solitary waves with the coexisting peak and valley. The numerical simulations are carried out, and the results are in complete agreement with our theoretical analysis.

Key words: perturbed nonlinear wave equation, solitary wave solution, geometric singular perturbation theory, Melnikov's function.

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1 Introduction

Traveling wave solutions, as is well-known, are characterized as solutions invariant with respect to translation in space for nonlinear partial differential equations (NPDEs) which are usually used to describe the process of transmission, such as shallow water wave motions, traffic flows, ion acoustic waves and so on. In particular, a large number of NPDEs have been proposed to model the shallow water wave motions in fluid dynamics. The prototypical equation for soliton is the well-known KdV equation

$$u_t + (u^2)_x + u_{xxx} = 0 \quad (1)$$

which is named after Korteweg and de Vries [4] who proposed this model equation in 1895. To approximate the surface water waves in a uniform channel, Benjamin et al [5] derived a regularized version of the KdV equation for shallow water waves, which is known as the Benjamin-Bona-Mahony (BBM) equation

$$u_t + (u^2)_x - u_{xxt} = 0. \quad (2)$$

To investigate the role of nonlinear dispersion in the formation of patterns in liquid drops, Rosenau and Hyman [6] proposed and examined a family of fully nonlinear KdV equations

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad (3)$$

which is usually named as K(m,n) equation. The K(m,n) equation with generalized evolution term, that is the so-called mK(m,n) equation given by

$$(u^l)_t + (u^m)_x + (u^n)_{xxx} = 0, \quad (4)$$

is firstly explored by Biswas [7]. Clearly, by setting $l = 1, m = 2$ and $n = 1$ equation (4) becomes KdV equation (1), hence it is a generalized form of the KdV equation. Here we point out that the following equation

$$(u^l)_t + \alpha(u^m)_x + \beta(u^n)_{xxx} = 0 \quad (5)$$

studied in [8,9] can be regularized to equation (4) by rescaling if parameters α and β have same signs.

The traveling wave solutions, especially the solitary wave solutions of these equations have attracted extensive attentions and some effective techniques or methods have been proposed, among which the dynamical system method [10–14] are well applied to investigate the traveling wave solutions of various nonlinear wave equations. In particularly, it

has been shown that equation (4) (or(5)) with $n > 1$ admits a kind of solitary waves with compact support and therefore are named as compactons [6, 8, 9]. More recently, the perturbations of the backward diffusion u_{xx} and dissipation u_{xxxx} are taken into account for several model equations in some literature. Derks and Giles [15] examined the uniqueness of traveling wave solutions for the perturbed KdV equation

$$u_t + uu_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0, (0 < \epsilon \ll 1) \quad (6)$$

by using geometric singular perturbation theory. Later on Ogawa [16] showed firstly that solitary waves and periodic waves with certain chosen speed for the KdV equation will persist after small perturbation.

The persistence of solitary waves or periodic waves of the perturbed mK(m,1) equation given by

$$(u^l)_t + (u^m)_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0. \quad (7)$$

are considered to some extent by some researchers recently. Clearly, the perturbed KdV equation (6) is a special case $l = 1$ and $m = 2$ of equation (7), so it is a generalization of the perturbed KdV equation. Guo and Zhao [17] have examined the existence of periodic waves for equation (7) with $l = 3$ and $m = 5$; The particular case that (7) with $l = 1$ and arbitrary $m \in \mathbb{Z}^+$, also named as singularly perturbed higher-order KdV equation, has been investigated in [12, 18] via geometric singular perturbation theory.

The particular case of equation (7) with $l = 1$ and $m = 3$ which is the so-called perturbed mKdV equation is investigated detailedly in [13, 14]. It has been shown that some solitary waves and periodic waves of the unperturbed mKdV equation persist with certain wave speeds under small perturbation. In particular, Zhang et al. [13] observed a new type of solitary wave with both peak and valley for the perturbed mKdV equation by combining the geometric singular perturbation theorem and Melnikov method.

More recently, Chen et al. [1] investigated the persistence of solitary waves which decay to zero and periodic waves of the equation given by

$$(u^2)_t + (u^3)_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0, \quad (8)$$

which is the particular case of equation (7) that $l = 2$ and $m = 3$. Lately, the persistence of two solitary waves with particular nonzero limits respectively were investigated in [2]. However, for equation (8) with $\epsilon = 0$, Yuan [3] presented the exact solitary wave solutions which decay to constants cz_0 determined by the wave speed c and the parameter $z_0 \in [0, \frac{2}{3}]$. Unfortunately, Yuan did not give a fully investigation on the natural question whether all or a portion of these solitary wave solutions persist under the small singular perturbation and what the wave speed selection principle is.

The geometric singular perturbation theorem firstly established by Fenichel [19] has been well applied to explore the solitary waves, periodic waves, and even the wave fronts (also named as kink or anti-kink) of various nonlinear wave equations. Tang et al. [20] investigated the persistence of the solitary wave solution for singularly perturbed Gardner equation. Xu et al. [21] established the existence of wave fronts for a generalized Burgers-KdV equation with convolution kernel. Du et al. [22, 23] studied the existence of solitary wave solutions for delayed Camassa-Holm equation and also considered the existence of wave fronts for nonlinear Belousov-Zhabotinskii system with delay. Zhao [24, 25] dealt with the existence of solitary waves for generalized KdV equation with distributed delays and Korteweg-deVries equation with small delay. Mansour [26, 27] concerned with the singularly perturbed Burgers-KdV equation and constructed its traveling waves.

In this paper, we combine the geometric singular perturbation theorem and homoclinic bifurcation analysis, to be exact, the Melnikov's method, to further explore the solitary wave solutions of equation (8). We mainly focus on the following three parts:

- (1) Solving the solitary wave solutions for the unperturbed $mK(3, 1)$ equation with a quadratic evolution term.
- (2) Examining the persistence of these solitary waves under small singular perturbation.
- (3) Observing new solitary waves of the perturbed $mKdV$ equation.

It results that we obtain the explicit solitary wave solutions to the unperturbed $mK(3, 1)$ equation and present the wave speed selection principle for the solitary waves of the singularly perturbed equation (8). And, more importantly, we observe that singularly perturbed equation (8) has a family of new solitary waves which possess coexisting peak and valley.

The rest of this paper is arranged as follows. In Section 2, we firstly present the corresponding traveling wave equation of the singularly perturbed equation (8) which is reduced to a two-dimensional near-Hamiltonian system via geometric singular perturbation theorem. In Section 3, by studying the bifurcation and homoclinic orbits of the traveling wave system, we examine the explicit solitary wave solutions of the unperturbed $mK(3, 1)$ equation which are parameterized by a parameter y_0 and arbitrary wave speed c . In Section 4, the persistence of the solitary wave solutions of the singularly perturbed $mK(3, 1)$ equation is investigated by using the Melnikov's method, and the selection principle for wave speed is given. Numerical simulation results are presented in Section 5 and a short conclusion is made in Section 6. Finally in Appendix, the detailed calculations and proof are given.

2 Traveling wave equation and dimension reduction

By introducing $\xi = x - ct$ and letting $\varphi(\xi) = u(x, t)$, equation (8) becomes the following ordinary differential equation:

$$-2c\varphi\varphi' + 3\varphi^2\varphi' + \varphi''' + \epsilon(\varphi'' + \varphi''') = 0, \quad (9)$$

where prime means the derivative with respect to ξ and c is the wave speed. Integrating equation (9) once with respect to ξ yields

$$-c\varphi^2 + \varphi^3 + \varphi'' + \epsilon(\varphi' + \varphi''') = g, \quad (10)$$

where g is a constant of integration. By introducing new variables $\tau = c\xi$ and $y = \frac{\varphi}{c}$ and letting $C = \frac{g}{c^3}$, equation (10) transforms to

$$-y^2 + y^3 + \frac{d^2y}{d\tau^2} + \frac{\epsilon}{c} \frac{dy}{d\tau} + \epsilon c \frac{d^3y}{d\tau^3} = C, \quad (11)$$

which is equivalent to the following three-dimensional dynamical system

$$\frac{dy}{d\tau} = z, \quad \frac{dz}{d\tau} = v, \quad \epsilon c \frac{dv}{d\tau} = y^2 - y^3 - v - \frac{\epsilon}{c} z + C. \quad (12)$$

Taking ϵ as a small parameter, system (12) is a singularly perturbed system with a normally hyperbolic 2-dimensional critical manifold

$$M_0 = \{(y, z, v) \in \mathbb{R}^3 | v = y^2 - y^3 + C\}.$$

It follows from Fenichel's theorem [19] that, for $\epsilon > 0$ sufficiently small, there exists a two-dimensional submanifold $M_\epsilon \subset \mathbb{R}^3$ within the Hausdorff distance ϵ of M_0 which is invariant under the flow of system (12). The invariant submanifold M_ϵ is expressed as

$$M_\epsilon = \{(y, z, v) \in \mathbb{R}^3 : v = y^2 - y^3 + C + \epsilon cz(3y^2 - 2y - \frac{1}{c^2}) + O(\epsilon^2)\}. \quad (13)$$

Being confined on the slow invariant manifold M_ϵ and ignoring the term $O(\epsilon^2)$, system (12) reduces to

$$\frac{dy}{d\tau} = z, \quad \frac{dz}{d\tau} = y^2 - y^3 + C + \epsilon cz(3y^2 - 2y - \frac{1}{c^2}). \quad (14)$$

Therefore, one can examine the homoclinic orbits of system (12) by studying the homoclinic orbits of system (14). For more details, one can refer to [1–3]. Obviously, system (14)

is a near-Hamiltonian system. By bifurcation theorem for near-Hamiltonian system [29], one sees that the homoclinic orbits of the unperturbed system $(14)|_{\epsilon=0}$ will persist under small perturbation for suitable values of c .

Clearly, if $y = y(\tau)$ is a solution of equation (11) with $\lim_{\tau \rightarrow \infty} y(\tau) = y_0$, then

$$-y_0^2 + y_0^3 = C$$

and the set $\{(y(\tau), z(\tau), v(\tau)) \in \mathbb{R}^3 : \tau \in \mathbb{R}\}$ constructs a homoclinic orbit of system (12) to the equilibrium point $(y_0, 0, 0)$ and therefore it corresponds to a homoclinic orbit to a saddle $(y_0, 0)$ of system (14), and vice versa. In addition, the solution $y = y(\tau)$ of equation (11) with $\lim_{\tau \rightarrow \infty} y(\tau) = y_0$ corresponds to a solitary wave solution $u(x, t) = cy(c(x - ct))$ of equation (8) which decays to cy_0 as time approaches infinity. Therefore, one can explore the solitary wave solution of equation (8) which decays to cy_0 by studying the homoclinic orbits of system (14) to $(y_0, 0)$.

Remind that we aim to investigate the solitary wave solutions of equation (8) which correspond to the homoclinic orbits of system (12), and therefore we only need to focus on the homoclinic orbits of system (14). It is easy to check that the unperturbed system, namely system (14) with $\epsilon = 0$, has homoclinic orbit if and only if $-\frac{4}{27} \leq C \leq 0$. Chen et al. [1] considered a particular case when $C = 0$, and Zhu et al. [2] considered another two special cases for $C = -\frac{2}{27}$ and $C = -\frac{4}{27}$. We will consider the general case to further explore the solitary wave solutions of equation (8).

If we assume that system (14) has a homoclinic orbit to an equilibrium point $(y_0, 0)$, then it is a saddle and $0 \leq y_0 \leq \frac{2}{3}$. It seems to be more convenient and suitable to take y_0 as the new parameter for system (12) since the corresponding solitary waves of equation (8) are parameterized by y_0 (see main result). To this end, we rewrite system (14) as

$$\frac{dy}{d\tau} = z, \quad \frac{dz}{d\tau} = y^2 - y^3 - (y_0^2 - y_0^3) + \epsilon cz(3y^2 - 2y - \frac{1}{c^2}), \quad (0 \leq y_0 \leq \frac{2}{3}). \quad (15)$$

3 Solitary wave solutions of the mK(3, 1) equation

It follows from the discussion in last section that the homoclinic orbits of system $(15)|_{\epsilon=0}$, namely

$$\frac{dy}{d\tau} = z, \quad \frac{dz}{d\tau} = y^2 - y^3 - (y_0^2 - y_0^3) \quad (16)$$

determine the traveling wave solutions of the mK(3, 1) equation, namely equation (8)| $_{\epsilon=0}$. To be exact, $u(x, t) = cy(\tau) = cy(c(x - ct))$ is a solitary wave of the mK(3, 1) equation with wave speed c provided that $y = y(\tau)$ and $z = y'(\tau)$ satisfy system (16) and $\lim_{\tau \rightarrow \infty} y(\tau) = y_0$ for $0 \leq y_0 \leq \frac{2}{3}$. It implies that one homoclinic orbit $(y(\tau), y'(\tau))$ of system (16) for a given y_0 corresponds to a family of solitary wave solutions $u(x, t) = cy(\tau) = cy(c(x - ct))$ with arbitrary wave speed c . To derive the solitary waves of the mK(3, 1) equation, we examine the solutions determined by the homoclinic orbits of system (16) with $0 \leq y_0 \leq \frac{2}{3}$.

Obviously, system (16) is a Hamiltonian system with Hamiltonian

$$H(y, z) = \frac{1}{2}z^2 + \frac{1}{4}y^4 - \frac{1}{3}y^3 + (y_0^2 - y_0^3)y.$$

The bifurcation and phase portraits of (16) have been analyzed in [2, 3]. Here we recall some of their results. Note that we apply the new parameter y_0 since it is closely related to the solitary waves we seek in this paper. Refer to Fig. 1 for the homoclinic orbits for $0 \leq y_0 \leq \frac{2}{3}$.

Fig. 1 Homoclinic orbits of system (16) for $0 \leq y_0 \leq \frac{2}{3}$

The homoclinic orbits $L_0^\pm(y_0)$ (to the saddle $(y_0, 0)$) are determined by

$$H(y, z) = \frac{1}{2}z^2 + \frac{1}{4}y^4 - \frac{1}{3}y^3 + (y_0^2 - y_0^3)y = \frac{2}{3}y_0^3 - \frac{3}{4}y_0^4 \quad (17)$$

on the phase plane. Solving equation (17) for z yields

$$\begin{aligned} z &= \pm \sqrt{-\frac{1}{2}y^4 + \frac{2}{3}y^3 - 2(y_0^2 - y_0^3)y + \frac{4}{3}y_0^3 - \frac{3}{2}y_0^4} \\ &= \pm \sqrt{\frac{1}{2}(y - y_0)^2(m_+ - y)(y - m_-)}, \end{aligned} \quad (18)$$

where

$$m_{\pm} = \frac{2}{3} - y_0 \pm \sqrt{-2y_0^2 + \frac{4}{3}y_0 + \frac{4}{9}}. \quad (19)$$

It is easy to see that $m_- \leq y_0 \leq m_+$ for $0 \leq y_0 \leq \frac{2}{3}$.

Inserting (18) in the first equation of (16), one gets

$$\frac{dy}{d\tau} = \pm \sqrt{\frac{1}{2}(y - y_0)^2(m_+ - y)(y - m_-)}. \quad (20)$$

Integrating equation (20) along the homoclinic orbits $L_0^{\pm}(y_0)$ yields $y = y_{\pm}(\tau)$ which can be expressed as

$$y_+(\tau) = y_0 - \frac{(6y_0^2 - 4y_0)e^{\tau\sqrt{2y_0-3y_0^2}}}{\frac{4}{9} + \frac{4}{3}y_0 - 2y_0^2 - (\frac{2}{3} - 2y_0)e^{\tau\sqrt{2y_0-3y_0^2}} + \frac{1}{4}e^{2\tau\sqrt{2y_0-3y_0^2}}} \quad (21)$$

and

$$y_-(\tau) = y_0 + \frac{(6y_0^2 - 4y_0)e^{\tau\sqrt{2y_0-3y_0^2}}}{\frac{4}{9} + \frac{4}{3}y_0 - 2y_0^2 + (\frac{2}{3} - 2y_0)e^{\tau\sqrt{2y_0-3y_0^2}} + \frac{1}{4}e^{2\tau\sqrt{2y_0-3y_0^2}}} \quad (22)$$

for $0 < y_0 < \frac{2}{3}$;

$$y(\tau) = \frac{12}{9 + 2\tau^2} \quad (23)$$

for $y_0 = 0$ and

$$y(\tau) = \frac{2}{3} - \frac{12}{9 + 2\tau^2} \quad (24)$$

for $y_0 = \frac{2}{3}$. Therefore, we obtain the solitary waves of the mK(3, 1) equation.

Theorem 1. For arbitrary $0 < y_0 < \frac{2}{3}$ and wave speed c ,

$$u_+(x, t; c, y_0) = cy_0 - \frac{c(6y_0^2 - 4y_0)e^{c(x-ct)}\sqrt{2y_0-3y_0^2}}{\frac{4}{9} + \frac{4}{3}y_0 - 2y_0^2 - (\frac{2}{3} - 2y_0)e^{c(x-ct)}\sqrt{2y_0-3y_0^2} + \frac{1}{4}e^{2c(x-ct)}\sqrt{2y_0-3y_0^2}}; \quad (25)$$

$$u_-(x, t; c, y_0) = cy_0 + \frac{c(6y_0^2 - 4y_0)e^{c(x-ct)}\sqrt{2y_0-3y_0^2}}{\frac{4}{9} + \frac{4}{3}y_0 - 2y_0^2 + (\frac{2}{3} - 2y_0)e^{c(x-ct)}\sqrt{2y_0-3y_0^2} + \frac{1}{4}e^{2c(x-ct)}\sqrt{2y_0-3y_0^2}}; \quad (26)$$

$$u_1(x, t) = \frac{12c}{9 + 2c^2(x - ct)^2}; \quad (27)$$

and

$$u_2(x, t) = \frac{2c}{3} - \frac{12c}{9 + 2c^2(x - ct)^2} \quad (28)$$

are solitary wave solutions for the $mK(3, 1)$ equation, namely equation (8) with $\epsilon = 0$.

Fig. 2 Solitary waves of equation (8) with $\epsilon = 0$ given by (25) with $y_0 = \frac{2}{9}$ and different wave speeds c

We have derived all exact solitary wave solutions with arbitrary wave speed c for the unperturbed mK(3, 1) equation $(8)|_{\epsilon=0}$. For given $y_0 \in (0, \frac{2}{3})$, the solitary waves $u_{\pm}(x, t; c, y_0)$ with different wave speeds c correspond to the same homoclinic orbit $L_0^{\pm}(y_0)$ of system (16). We will show in the following section that for given $y_0 \in (0, \frac{2}{3})$, only the solitary wave $u_{\pm}(x, t; c, y_0)$ with a particular value of wave speed will persist under small perturbation. For instance, for the eight solitary waves of the unperturbed equation $(8)|_{\epsilon=0}$ with $y_0 = \frac{2}{9}$ and different wave speeds shown in Fig. 2 and Fig. 3, the ones shown in (a) and (c) will persist while those shown in (b) and (d) will vanish under small perturbation, which will be proven in the following section.

Fig. 3 Solitary waves of equation (8) with $\epsilon = 0$ given by (26) with $y_0 = \frac{2}{9}$ and different wave speeds c

4 Persistence of the solitary wave solutions under small perturbation and selection principle of wave speed

We now firstly study the homoclinic bifurcation of system (15) by using the Melnikov method to examine the persistence and the selection principle for the wave speed of the solitary wave solutions under small perturbation.

4.1 Homoclinic bifurcations of system (15)

We introduce $\delta = c^2$ for the convenience of further discussion. Then by the homoclinic bifurcation theorem [28, 29], the Melnikov functions for the homoclinic orbits $L_0^\pm(y_0)$ of the near-Hamiltonian system (15) are defined as

$$M_\pm(\delta, y_0) = \oint_{L_0^\pm(y_0)} \left(3y^2 - 2y - \frac{1}{\delta}\right) z^2 d\tau. \quad (29)$$

Remind that the homoclinic orbits $L_0^+(y_0) \cup L_0^-(y_0)$ of the Hamiltonian system (15)| $_\epsilon = 0$ are determined by the algebraic curve

$$\frac{1}{2}z^2 + \frac{1}{4}y^4 - \frac{1}{3}y^3 + (y_0^2 - y_0^3)y = \frac{2}{3}y_0^3 - \frac{3}{4}y_0^4$$

on the phase plane and oriented by the dynamical system (16). Furthermore, the right homoclinic orbit $L_0^+(y_0)$ locates in the range $y_0 < y \leq m_+$ and the left one $L_0^-(y_0)$ locates in the range $m_- \leq y < y_0$. To investigate bifurcation of the big homoclinic orbit $L_0^+(y_0) \cup L_0^-(y_0)$, we define

$$\begin{aligned} M(\delta, y_0) &= M_+(\delta, y_0) + M_-(\delta, y_0) \\ &= \oint_{L_0^+(y_0) \cup L_0^-(y_0)} \left(3y^2 - 2y - \frac{1}{\delta}\right) z^2 d\tau. \end{aligned} \quad (30)$$

Let

$$I_\pm(y_0) = \oint_{L_0^\pm(y_0)} z^2 d\tau \quad (31)$$

and

$$J_\pm(y_0) = \oint_{L_0^\pm(y_0)} (3y^2 - 2y) z^2 d\tau, \quad (32)$$

then $M_{\pm}(\delta, y_0)$ is represented as

$$M_{\pm}(\delta, y_0) = J_{\pm}(y_0) - \frac{1}{\delta} I_{\pm}(y_0). \quad (33)$$

Lemma 1. *For arbitrary $0 < y_0 < \frac{2}{3}$, $M_{\pm}(\delta, y_0) = 0$ has a positive root $\delta = \delta_{\pm}(y_0)$. Furthermore, $\frac{d}{d\delta} M_{\pm}(\delta, y_0)|_{\delta=\delta_{\pm}(y_0)} \neq 0$.*

Proof. It follows directly from equation (31) that $I_{\pm}(y_0) > 0$. Note that the homoclinic orbits $L_0^{\pm}(y_0)$ are defined by planar dynamical system (16), and thus

$$J_{\pm}(y_0) = \oint_{L_0^{\pm}(y_0)} (3y^2 - 2y) z^2 d\tau = \oint_{L_0^{\pm}(y_0)} z d(-z').$$

By integration of parts, one has

$$J_{\pm}(y_0) = \oint_{L_0^{\pm}(y_0)} z d(-z') = \int_{\mathbb{R}} z'^2 d\tau,$$

which is positive obviously.

It follows from equation (33) that

$$\delta = \delta_{\pm}(y_0) = \frac{I_{\pm}(y_0)}{J_{\pm}(y_0)} \quad (34)$$

is a positive root of $M_{\pm}(\delta, y_0) = 0$. Clearly, $\frac{d}{d\delta} M_{\pm}(\delta, y_0)|_{\delta=\delta_{\pm}(y_0)} = \frac{(J_{\pm}(y_0))^2}{I_{\pm}(y_0)} > 0$. This completes the proof. $\square$

Remark 1. *In fact, the formulas for $I_{\pm}(y_0)$ and $J_{\pm}(y_0)$ can be derived by direct integration (see Appendix for details), and thus $\delta_{\pm}(y_0)$ can be presented explicitly by y_0 .*

Proposition 1. *For arbitrary $0 < y_0 < \frac{2}{3}$ and $0 < \epsilon \ll 1$, there exist two functions $c_{\pm}^{(+)}(y_0, \epsilon) = \pm\sqrt{\delta_{+}(y_0)} + O(\epsilon)$ such that system (15) with $c = c_{\pm}^{(+)}(y_0, \epsilon)$ has a homoclinic orbit near $L_0^{+}(y_0)$. There exist two functions $c_{\pm}^{(-)}(y_0, \epsilon) = \pm\sqrt{\delta_{-}(y_0)} + O(\epsilon)$ such that system (15) with $c = c_{\pm}^{(-)}(y_0, \epsilon)$ has a homoclinic orbit near $L_0^{-}(y_0)$.*

Proof. It follows directly from Lemma 1 and the homoclinic bifurcation theorem (Refer to Proposition 1 in [13] or see [28, 29] for more details) that system (15) with $\delta = \delta_{+}(y_0) + O(\epsilon)$, that is $c = \pm\sqrt{\delta_{+}(y_0)} + O(\epsilon)$, has a homoclinic orbit near $L_0^{+}(y_0)$. Similarly, one can prove the existence of $c_{\pm}^{(-)}(y_0, \epsilon) = \pm\sqrt{\delta_{-}(y_0)} + O(\epsilon)$ such that system (15) with $c = c_{\pm}^{(-)}(y_0, \epsilon)$ has a homoclinic orbit near $L_0^{-}(y_0)$. It completes the proof. $\square$

Lemma 2. *For arbitrary $y_0 \in (0, \frac{2}{3})$, $M(\delta, y_0) = 0$ has a positive root $\delta = \delta(y_0)$ and $\frac{d}{d\delta}M(\delta, y_0)|_{\delta=\delta(y_0)} \neq 0$. Furthermore, $M_+(\delta(y_0), y_0) \neq 0$ for $y_0 \in (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3})$.*

Proof. Substitute (33) in the first equation of (30) and re-organize the terms to have

$$M(\delta, y_0) = (J_+ + J_-) - \frac{1}{\delta}(I_+ + I_-). \quad (35)$$

Clearly,

$$\delta = \delta(y_0) = \frac{I_+ + I_-}{J_+ + J_-} \quad (36)$$

is a positive root of $M(\delta, y_0) = 0$ and $\frac{d}{d\delta}M(\delta, y_0)|_{\delta=\delta(y_0)} = \frac{(J_+ + J_-)^2}{I_+ + I_-} > 0$.

We now show that $M_+(\delta(y_0), y_0) \neq 0$ for $y_0 \in (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3})$ by contradiction. If we assume that $M_+(\delta(y_0), y_0) = 0$, then from (36) we have

$$J_+ - \frac{1}{\delta(y_0)}I_+ = J_+ - \frac{J_+ + J_-}{I_+ + I_-}I_+ = 0,$$

which is equivalent to

$$I_+J_- - I_-J_+ = 0.$$

We have reached a contradiction (see Appendix for more details). It completes the proof. $\square$

Proposition 2. *For arbitrary $y_0 \in (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3})$ and $0 < \epsilon \ll 1$, there exist two functions $c(y_0, \epsilon) = \pm\sqrt{\delta(y_0)} + O(\epsilon)$, such that equation (15) with $c = \pm\sqrt{\delta(y_0)} + O(\epsilon)$ has a big homoclinic loop near $L_0^+(y_0) \cup L_0^-(y_0)$.*

Proof. It follows directly from Lemma 2 and the homoclinic bifurcation theorem (Refer to Proposition 2 in [13] or see [28, 29] for more details) that system (15) with $\delta = \delta(y_0) + O(\epsilon)$, that is $c = \pm\sqrt{\delta(y_0)} + O(\epsilon)$, has a big homoclinic orbit near $L_0^+(y_0) \cup L_0^-(y_0)$. $\square$

4.2 The selection principle of wave speed for solitary wave solutions

Recall from Section 2 that a homoclinic orbit $\{(y(\tau), z(\tau), v(\tau)) \in \mathbb{R}^3 : \tau \in \mathbb{R}\}$ to $(y_0, 0, 0)$ of system (12) gives a solitary wave solution $u(x, t) = cy(c(x - ct))$ of equation (8) which

decays to cy_0 as time approaches infinity. However, it has been shown that the homoclinic orbits of system (12) corresponds to those of the perturbed Hamiltonian system (15). Therefore, the solitary wave solutions of the perturbed equation (8) are totally determined by the homoclinic orbits of the near-Hamiltonian system (15). The following two theorems follow directly from Proposition 1 and 2 and Theorem 1 directly.

Theorem 2. (Persistence of the solitary waves and selection principle of wave speed)

For arbitrary $0 < y_0 < \frac{2}{3}$ and ϵ sufficiently small, the perturbed $mK(3,1)$ equation (8) has

(1) two families of solitary wave solutions $u = u_+(x, t, c, y_0, \epsilon)$ with wave speed $c_{\pm}^{(+)}(y_0, \epsilon) = \pm\sqrt{\delta_+(y_0)} + O(\epsilon)$ which decay to $c_{\pm}^{(+)}(y_0, \epsilon)y_0$ as time approaches infinity. When ϵ approaches 0, they converge to the solitary waves (25) with $c = \pm\sqrt{\delta_+(y_0)}$ of the unperturbed equation, respectively.

(2) two families of solitary wave solutions $u = u_-(x, t, c, y_0, \epsilon)$ with wave speed $c_{\pm}^{(-)}(y_0, \epsilon) = \pm\sqrt{\delta_-(y_0)} + O(\epsilon)$ which decay to $c_{\pm}^{(-)}(y_0, \epsilon)y_0$ as time approaches infinity. When ϵ approaches 0, they converge to the solitary waves (26) with $c = \pm\sqrt{\delta_-(y_0)}$ of the unperturbed equation, respectively.

Remark 2. *Theorem 2 implies that the solitary wave solutions given by (25) (resp. (26)) with two particularly selected wave speed $c = \pm\sqrt{\delta_+(y_0)}$ (resp. $c = \pm\sqrt{\delta_-(y_0)}$) persist under small singular perturbation (see (a) and (c) in Fig. 2 and Fig. 3).*

Theorem 3. (New solitary waves with coexisting peak and valley)

For arbitrary $y_0 \in (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3})$ and ϵ sufficiently small, the perturbed nonlinear equation (8) has a solitary wave solution $u(x, t, c, y_0, \epsilon)$ with wave speed $c = \pm\sqrt{\delta(y_0)} + O(\epsilon)$ which possesses the coexisting peak and valley.

5 Numerical simulation

In this section, numerical simulations are carried out to illustrate the results derived theoretically in previous sections. Set $y_0 = \frac{2}{9}$, then we have $m_{\pm}(\frac{2}{9}) = \frac{4 \pm 2\sqrt{13}}{9}$. By using equations (34), (36) and (39)-(42)(see Appendix), we have $\delta_+(\frac{2}{9}) \approx 1.731612570$, $\delta_-(\frac{2}{9}) \approx 3.079092826$ and $\delta(\frac{2}{9}) \approx 1.903972672$.

Let $\epsilon = 0.01$ and $c = \pm\sqrt{\delta(\frac{2}{9})} = \pm 1.315907508$. Then with the help of Maple, we get the trajectories of system (15) passing through $(m_+(\frac{2}{9}), 0)$ and $(m_-(\frac{2}{9}), 0)$, respectively (see (b) and (c) of Fig. 4). It is clearly shown that the homoclinic orbits of the unperturbed system $(15)|_{\epsilon=0}$ is broken” due to the small perturbation.

Fig. 4 For $y_0 = \frac{2}{9}$, the trajectories of system (15) passing through $(m_+(\frac{2}{9}), 0)$ and $(m_-(\frac{2}{9}), 0)$

To capture the bifurcation values $\delta = c^2 = \delta_+(\frac{2}{9}, \epsilon)$, we set $(y(0), z(0)) = (m_+(\frac{2}{9}) + 0.000005, 0)$, $\delta = \delta_+(\frac{2}{9}) - 0.001$ and $\delta = \delta_+(\frac{2}{9})$, respectively. Then we obtain the graphs in Fig. 5. We observe that there exists $\delta_+(\frac{2}{9}, \epsilon) \in (\delta_+(\frac{2}{9}) - 0.001, \delta_+(\frac{2}{9}))$ such that system (15) with $c = \sqrt{\delta_+(\frac{2}{9}, \epsilon)}$ has a homoclinic orbit which corresponds to a solitary wave solution of equation (8).

(a) $\delta = \delta_+(\frac{2}{9}) - 0.001$

(b) $\delta = \delta_+(\frac{2}{9}) - 0.001$

(c) $\delta = \delta_+(\frac{2}{9})$

(d) $\delta = \delta_+(\frac{2}{9})$

Fig. 5 Trajectories and solutions $y(\tau)$ of (15) with $y_0 = \frac{2}{9}$, $\epsilon = 0.01$, and initial value $(y(0), z(0)) = (m_+(\frac{2}{9}) + 0.000005, 0)$

Set the initial value $(y(0), z(0)) = (m_-(\frac{2}{9}) - 0.0000015, 0)$ and let $\delta = \delta_-(\frac{2}{9}) - 0.002$ and $\delta = \delta_-(\frac{2}{9})$ respectively, then we get the graphs shown in Fig. 6. Consequently, we can claim that there exists $\delta_-(\frac{2}{9}, \epsilon) \in (\delta_-(\frac{2}{9}) - 0.002, \delta_-(\frac{2}{9}))$ such that system (15) with $c = \sqrt{\delta_-(\frac{2}{9}, \epsilon)}$ has a homoclinic orbit which corresponds to a solitary wave solution of the perturbed mK(3,1) equation (8).

(a) $\delta = \delta_-(\frac{2}{9}) - 0.002$

(b) $\delta = \delta_-(\frac{2}{9}) - 0.002$

(c) $\delta = \delta_-(\frac{2}{9})$

(d) $\delta = \delta_-(\frac{2}{9})$

Fig. 6 Trajectories and solutions $y(\tau)$ of (15) with $y_0 = \frac{2}{9}$, $\epsilon = 0.01$, and initial value $(y(0), z(0)) = (m_-(\frac{2}{9}) - 0.0000015, 0)$

In order to capture the bifurcation values of $\delta(\frac{2}{9}, \epsilon)$, we set the initial value $(y(0), z(0)) = (0, 10^{-7})$, $\delta = \delta(\frac{2}{9}) + 0.0007$ and $\delta = \delta(\frac{2}{9}) + 0.1$, respectively. Then we obtain Fig. 7 from which one observes that $\delta(\frac{2}{9}, \epsilon) \in (\delta(\frac{2}{9}) + 0.0007, \delta(\frac{2}{9}) + 0.1)$. Therefore, there exist a solitary wave solution with wave speed $c = \sqrt{\delta(\frac{2}{9}, \epsilon)}$ which possesses coexisting peak and valley.

(a) $\delta = \delta(\frac{2}{9}) + 0.0007$

(b) $\delta = \delta(\frac{2}{9}) + 0.0007$

(c) $\delta = \delta(\frac{2}{9}) + 0.1$

(d) $\delta = \delta(\frac{2}{9}) + 0.1$

Fig. 7 Trajectories and solutions $y(\tau)$ of (15) with $y_0 = \frac{2}{9}$, $\epsilon = 0.01$, and the initial value $(y(0), z(0)) = (0, 10^{-7})$

6 Conclusion

In this paper, we focus on the existence of solitary wave solutions of a singularly perturbed mKdV equation with a quadratic evolution term by combining the geometric singular perturbation theorem and homoclinic bifurcation analysis. The results show that not all solitary wave solutions of the unperturbed equation can persist, but only the ones with particularly chosen wave speeds persist under small singular perturbation. More importantly, we have discovered that this equation possesses a new type of solitary waves that possess coexisting peak and valley, which refresh our understanding on the solitary wave solutions to the singularly perturbed nonlinear wave equations.

7 Appendix: calculation of integrals

Recall that

$$I_{\pm}(y_0) = \oint_{L_0^{\pm}(y_0)} z^2 d\tau, \quad J_{\pm}(y_0) = \oint_{L_0^{\pm}(y_0)} (3y^2 - 2y) z^2 d\tau,$$

and the homoclinic orbits $L_0^+(y_0) \cup L_0^-(y_0)$ are determined by

$$\frac{1}{2}z^2 + \frac{1}{4}y^4 - \frac{1}{3}y^3 + (y_0^2 - y_0^3)y = \frac{2}{3}y_0^3 - \frac{3}{4}y_0^4,$$

that is

$$z_{\pm} = \pm \sqrt{-\frac{1}{2}y^4 + \frac{2}{3}y^3 - 2(y_0^2 - y_0^3)y + \frac{4}{3}y_0^3 - \frac{3}{2}y_0^4} \quad (37)$$

with $y_0 < y \leq m_+$ for the right homoclinic loop $L_0^+(y_0)$ and with $m_- \leq y < y_0$ for the left one $L_0^-(y_0)$. They are both oriented by system (16).

By noticing that $\frac{dy}{d\tau} = z$ and the orbits $L_0^{\pm}(y_0)$ are symmetry with respect to the y -axis, one has

$$I_+(y_0) = 2 \int_{y_0}^{m_+} \sqrt{-\frac{1}{2}y^4 + \frac{2}{3}y^3 - 2(y_0^2 - y_0^3)y + \frac{4}{3}y_0^3 - \frac{3}{2}y_0^4} dy. \quad (38)$$

Direct calculation of integral yields

$$I_+(y_0) = 2\sqrt{2} \left(y_0^3 - y_0^2 + \frac{2}{27} \right) \left(\frac{\pi}{2} - \arcsin \frac{\sqrt{2}(3y_0 - 1)}{\sqrt{-9y_0^2 + 6y_0 + 2}} \right) + \frac{4}{9} \sqrt{y_0(2 - 3y_0)}. \quad (39)$$

Similarly, we have

$$\begin{aligned} J_+(y_0) &= 2 \int_{y_0}^{m_+} (3y^2 - 2y) \sqrt{-\frac{1}{2}y^4 + \frac{2}{3}y^3 - 2(y_0^2 - y_0^3)y + \frac{4}{3}y_0^3 - \frac{3}{2}y_0^4} dy \\ &= \frac{4\sqrt{2}}{3} \left(y_0^3 - y_0^2 + \frac{2}{27} \right) \left(\frac{\pi}{2} - \arcsin \frac{\sqrt{2}(3y_0 - 1)}{\sqrt{-9y_0^2 + 6y_0 + 2}} \right) \\ &\quad + 2 \left(\frac{24}{5}y_0^3 - \frac{18}{5}y_0^4 - \frac{8}{5}y_0^2 + \frac{4}{27} \right) \sqrt{y_0(2 - 3y_0)}, \end{aligned} \quad (40)$$

$$\begin{aligned}
I_-(y_0) &= 2 \int_{m_-}^{y_0} \sqrt{-\frac{1}{2}y^4 + \frac{2}{3}y^3 - 2(y_0^2 - y_0^3)y + \frac{4}{3}y_0^3 - \frac{3}{2}y_0^4} dy \\
&= -2\sqrt{2} \left(y_0^3 - y_0^2 + \frac{2}{27} \right) \left(\frac{\pi}{2} + \arcsin \frac{\sqrt{2}(3y_0 - 1)}{\sqrt{-9y_0^2 + 6y_0 + 2}} \right) + \frac{4}{9} \sqrt{y_0(2 - 3y_0)} ,
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
J_-(y_0) &= 2 \int_{m_-}^{y_0} (3y^2 - 2y) \sqrt{-\frac{1}{2}y^4 + \frac{2}{3}y^3 - 2(y_0^2 - y_0^3)y + \frac{4}{3}y_0^3 - \frac{3}{2}y_0^4} dy \\
&= -\frac{4\sqrt{2}}{3} \left(y_0^3 - y_0^2 + \frac{2}{27} \right) \left(\frac{\pi}{2} + \arcsin \frac{\sqrt{2}(3y_0 - 1)}{\sqrt{-9y_0^2 + 6y_0 + 2}} \right) \\
&\quad + 2 \left(\frac{24}{5}y_0^3 - \frac{18}{5}y_0^4 - \frac{8}{5}y_0^2 + \frac{4}{27} \right) \sqrt{y_0(2 - 3y_0)} .
\end{aligned} \tag{42}$$

It follows that

$$I_+J_- - I_-J_+ = \frac{8\sqrt{2}}{135} \pi (3y_0 - 1)(9y_0^2 - 6y_0 - 2)y_0^{\frac{5}{2}}(2 - 3y_0)^{\frac{5}{2}} \tag{43}$$

Clearly, for $y_0 \in (0, \frac{2}{3})$, $I_+J_- - I_-J_+ = 0$ if and only if $y_0 = \frac{1}{3}$.

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Conflicts of interest

All authors have declared that no conflict of interest exists.

Date Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study

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Figures

Figure 1

"Please see the Manuscript PDF file for the complete figure caption".

(a) $c = 1.315907508$

(b) $c = 2$

(c) $c = -1.315907508$

(d) $c = -2$

Figure 2

"Please see the Manuscript PDF file for the complete figure caption".

(a) $c = 1.754734403$

(b) $c = 1$

(c) $c = -1.754734403$

(d) $c = -1$

Figure 3

"Please see the Manuscript PDF file for the complete figure caption".

(a) $\epsilon = 0$

(b) $\epsilon = 0.01$ and $c = \sqrt{\delta(\frac{2}{9})}$

(c) $\epsilon = 0.01$ and $c = -\sqrt{\delta(\frac{2}{9})}$

Figure 4

"Please see the Manuscript PDF file for the complete figure caption".

(a) $\delta = \delta_+(\frac{2}{9}) - 0.001$

(b) $\delta = \delta_+(\frac{2}{9}) - 0.001$

(c) $\delta = \delta_+(\frac{2}{9})$

(d) $\delta = \delta_+(\frac{2}{9})$

Figure 5

"Please see the Manuscript PDF file for the complete figure caption".

(a) $\delta = \delta_-(\frac{2}{9}) - 0.002$

(b) $\delta = \delta_-(\frac{2}{9}) - 0.002$

(c) $\delta = \delta_-(\frac{2}{9})$

(d) $\delta = \delta_-(\frac{2}{9})$

Figure 6

"Please see the Manuscript PDF file for the complete figure caption".

(a) $\delta = \delta(\frac{2}{9}) + 0.0007$

(b) $\delta = \delta(\frac{2}{9}) + 0.0007$

(c) $\delta = \delta(\frac{2}{9}) + 0.1$

(d) $\delta = \delta(\frac{2}{9}) + 0.1$

Figure 7

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