A New Stabilization Algorithm for Mechanical Systems with Underactuation Degree One based on Controlled Lagrangians

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A new stabilization algorithm for mechanical systems with underactuation degree one based on controlled Lagrangians

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Abstract A new stabilization control algorithm based on controlled Lagrangian method for a class of mechanical systems with underactuation degree one is presented in this paper. Firstly, a desired controlled system with the Lagrangian structure and desired properties is constructed. By equating the underactuated system with the desired system, the matching condition and controller structure are determined. A sufficient condition for the matching condition to be held is derived, and from the sufficient condition desired kinetic energy, potential energy, gyroscopic forces and dissipative forces of the desired system can be solved explicitly. Compared with the existing matching conditions, to solve proposed sufficient condition at most two partial differential equations needs to be solved, and the rests are all algebraic equations, which is easier to solve. An algorithm to solve this sufficient condition is given in detail, and a nonlinear smooth feedback control law can be obtained to stabilize the underactuated systems. Finally, the novel control algorithm is applied to achieve almost global stability for a vertical takeoff and landing aircraft and to locally stabilize a Pendubot with two degrees of freedom at the highest point. Simulation results demonstrate effectiveness of the proposed method.

Keywords Underactuated system · Stabilization · Controlled Lagrangians · Matching condition

1 Introduction

The controlled Lagrangian (CL) method is a nonlinear controller design technique from an energy perspective. Because the CL method has great advantages and theoretical value in the control problem of underactuated mechanical systems such as inverted pendulum, acrobot, aircraft, satellite and so on, the method has been widely studied and developed by many scholars and researchers in theory and application since it was first proposed in [1]. For instance, Bloch, Leonard, and Marsden in [2] designed a controller through kinetic energy shaping to stabilize a class of underactuated system whose driving variables are symmetric group variables, and they extended the method in [2] by introducing potential energy shaping in [3], which enables the full state of mechanical system asymptotically stable. Further, they presented a constructive approach to the determination of stabilizing control laws for a class of symmetry Lagrangian systems described by the Euler-Poincaré equations in [4] and realized the stabilization of a class of discrete systems by extending the CL method in [33]. The author in [5]-[7] analyzed the feasibility of the CL method, and the author in [8]-[9] managed to increase degrees of freedom of the system to achieve the purpose of broadening application fields of the CL method and improving control performances. In order to reduce the complexity of solving matching conditions, Hamberg, Auckly, and Kapitanski in [10][11] proposed the λ -method that makes the matching conditions simplified and innovates the construction of controlled kinetic energy through Riemann measurement. Therefore, after that, the research object was no longer limited to the system with kinetic energy symmetry. Since the λ -method was improved, a simpler matching condition v-equation was obtained in [12]. Dong introduced gyroscopic forces in his doctoral thesis [13], which further enhanced λ -method and obtained more general matching condition-

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s, and Craig Woolsey described the design steps of the CL method after the introduction of artificial gyroscopic forces and pointed out in [14] that the artificial gyroscopic forces can increase the degree of freedom for expanding stable area of the balance point and adjusting performances of the closed-loop system. In [15], the criteria for matching and stabilizing of several types of mechanical systems were given and the constraints of matching conditions were relaxed by adding force shaping.

For the asymptotic stabilization problem of general n degrees of freedom underactuated mechanical systems with input coupling, many scholars have done special research on the energy shaping method based on Lagrangian framework and Hamilton framework. Shaft in [19] used the symplectic geometry theory to study the energy shaping of mechanical systems based on the Hamilton framework and then demonstrated the role of geometric mechanics in control. The authors in [20] proposed passivity-based interconnection and damping allocation control. A set of explicit solutions of partial differential equations in matching conditions under some conditions were obtained. In the same year, the author in [21] assumed that the controlled kinetic energy has the same cyclic variable as the original kinetic energy, which makes the kinetic energy equation become a set of ordinary differential equations. Similar techniques have been described for general controlled Hamiltonian and Lagrangian systems with underactuation degree one in [22], [23] and [24], respectively. Chang D. in [25] and Blankenstein G et al. in [26] explored equivalence between CL method and IDA-PBC method and found that they are equivalent for simple mechanical systems. In order to simplify the design of controller, in [18], Li and Huo obtained a sufficient condition satisfying the matching condition by extending the results of [17] which comprised the new K -equation and one P -equation cascaded, the regular condition, and the explicit gyroscopic forces. Further, Viola, Ortega, and Banavar proved that the inhomogeneous terms in K -equations can be eliminated or simplified by modifying the target dynamics and introducing a change of coordinates in the original system in [16]. In summary, so far the main existing matching conditions consist of two sets of cascaded partial differential equations. The controller design process based on the CL method is still complicated. Therefore, it is quite necessary to develop a simpler stabilization control algorithm for underactuated mechanical systems.

In this paper, we are encouraged by [17] [18] [31] [34] to further develop the CL method for a class of mechanical systems with underactuation degree one. Main contributions of the paper are summarized as follows:

1. A feasible sufficient condition satisfying the matching condition is derived. By introducing a vector $w(q)$ into the sufficient condition, the processes of solving the desired kinetic energy matrix and gyroscopic matrix are

completely decoupled. After $w(q)$ is first solved from a partial differential equation, the desired kinetic energy and the gyroscopic force can be obtained sequentially. In addition, more free parameters provided by introducing $w(q)$ are helpful for solving the desired potential energy with desired properties.

2. According to whether the inertia matrix of the underactuated system is a constant matrix, two different approaches to solve the sufficient conditions are provided respectively.
3. The gyroscopic matrix is configured as a specific structure, which has fewer unknown gyroscopic force parameters so that the computational complexity of solving the gyroscopic matrix is greatly reduced. Compared with the result in [18], the number of gyroscopic force parameters to be solved is reduced from $nC_n^2 = \frac{n^2(n-1)}{2}$ to $C_{n+1}^2 - 1 = \frac{n(n+1)}{2} - 1$.
4. Last but not least, Compared with the methods in [2] [3] [4] [7] needed using the knowledge of differential geometry such as group theory, the method proposed in this paper only use matrix theory in the whole derivation process, which is more conducive to understanding and application by engineers and technicians.

In order to verify the effectiveness of the proposed approach, the control algorithm is applied to an almost globally stabilizing scheme for a vertical takeoff and landing aircraft with strong input coupling, and obtain a controller for a two degrees of freedom Pendubot that can be asymptotically stable at the highest point within a given range, respectively. In both cases above, we all obtain a nonlinear feedback control law to give the system desired performance and rigorously prove the asymptotic stability of the closed-loop system. The simulation experiments are carried out on these two examples to confirm the correctness of the control algorithm.

The remainder of this paper is organized as follows. In Section 2, the underactuated system model and control objective are presented. A desired controlled system is designed in Section 3. In Section 4, the controller structure and matching condition are determined and a feasible sufficient condition satisfying the matching condition is derived. We solve this sufficient condition to obtain the control law in Section 5. In Section 6, the algorithm proposed in this paper is applied to a vertical take-off and landing aircraft and a two degrees of freedom Pendubot, and the effectiveness of the controller is verified by simulation experiments. Section 7 concludes this paper and proposes possible research directions in the future.

2 System model and control objective

Consider a n degrees of freedom (DOF) mechanical system with underactuation degree one, where $n \geq 2$. Let $q =$

$[q_1, \dots, q_n]^T$ and $\dot{q} = [\dot{q}_1, \dots, \dot{q}_n]^T$ represent its generalized coordinates and generalized velocities, respectively. Denote $u = B(q)v$ as the non-conservative generalized force applied to the system, where v is independent control input and $B(q)$ is input coupling matrix with rank $B(q) = n - 1$. The kinetic energy and potential energy of the system are expressed as $E_k(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q}$, where $M(q)$ is inertia matrix satisfying $M(q) = M^T(q) > 0$, and $E_p(q) : \mathbb{R}^n \rightarrow \mathbb{R}$, respectively. Then, the Lagrangian of the system can be written as

$$L(q, \dot{q}) = E_k(q, \dot{q}) - E_p(q) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - E_p(q). \quad (1)$$

By the use of Euler-Lagrange equation, the system dynamical equation can be obtained as follows:

$$\begin{aligned} & \frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} - \frac{\partial L(q, \dot{q})}{\partial q} \\ &= \frac{dM(q)\dot{q}}{dt} - \frac{\partial L(q, \dot{q})}{\partial q} \\ &= M(q)\ddot{q} + \frac{\partial M(q)\dot{q}}{\partial q^T} \dot{q} - \frac{1}{2} \frac{\partial \dot{q}^T M(q) \dot{q}}{\partial q} + \frac{\partial E_p(q)}{\partial q} \\ &= u = B(q)v. \end{aligned} \quad (2)$$

Objective of this paper is to design control v such that the underactuated system (2) can be stabilized to the desired equilibrium point $(q, \dot{q}) = (0, 0)$ from any initial state $(q(0), \dot{q}(0))$.

3 Desired controlled system design

In order to design a controller based on the CL method, we first construct a desired controlled system with the desired controlled kinetic energy $\bar{E}_k(q, \dot{q}) = \frac{1}{2} \dot{q}^T \bar{M}(q) \dot{q}$, where $\bar{M}(q) = \bar{M}^T(q) > 0$ is a positive definite matrix, and the desired controlled potential energy $\bar{E}_p(q) \geq 0$, which possesses the minimum value zero only at $q = 0$. Then, the Lagrangian of the desired controlled system can be written as

$$\bar{L}(q, \dot{q}) = \bar{E}_k(q, \dot{q}) - \bar{E}_p(q) = \frac{1}{2} \dot{q}^T \bar{M}(q) \dot{q} - \bar{E}_p(q). \quad (3)$$

Denoting $\bar{u}$ as the non-conservative generalized force applied to the desired controlled system and using the Euler-Lagrange equation, we can obtain the desired controlled system as follows:

$$\begin{aligned} & \frac{d}{dt} \frac{\partial \bar{L}(q, \dot{q})}{\partial \dot{q}} - \frac{\partial \bar{L}(q, \dot{q})}{\partial q} \\ &= \frac{d\bar{M}(q)\dot{q}}{dt} - \frac{\partial \bar{L}(q, \dot{q})}{\partial q} \\ &= \bar{M}(q)\ddot{q} + \frac{\partial \bar{M}(q)\dot{q}}{\partial q^T} \dot{q} - \frac{1}{2} \frac{\partial \dot{q}^T \bar{M}(q) \dot{q}}{\partial q} + \frac{\partial \bar{E}_p(q)}{\partial q} = \bar{u}. \end{aligned} \quad (4)$$

Design desired generalized force

$$\bar{u} = \frac{1}{2} \left[\frac{\partial \bar{M}(q)\dot{q}}{\partial q^T} - \left(\frac{\partial \bar{M}(q)\dot{q}}{\partial q^T} \right)^T \right] \dot{q} + \bar{G}(q, \dot{q})\dot{q} - \bar{D}(q)\dot{q}, \quad (5)$$

where gyroscopic matrix $\bar{G}(q, \dot{q})$ satisfies $\bar{G}(q, \dot{q}) = -\bar{G}^T(q, \dot{q})$

$$\bar{G}(q, \dot{q}) \triangleq \begin{bmatrix} 0 & \dots & 0 & -\dot{q}^T \bar{g}_1(q) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -\dot{q}^T \bar{g}_{n-1}(q) \\ \dot{q}^T \bar{g}_1(q) & \dots & \dot{q}^T \bar{g}_{n-1}(q) & 0 \end{bmatrix},$$

$\bar{g}_i(q) \in \mathbb{R}^n, i = 1, \dots, n-1$,

and dissipative matrix $\bar{D}(q)$ is a semi-positive definite matrix satisfying $\bar{D}(q) = \bar{D}^T(q) \geq 0$. Substituting (5) into (4) gives the desired closed-loop system equation

$$\bar{M}(q)\ddot{q} + \frac{1}{2} \frac{\partial \bar{M}(q)\dot{q}}{\partial q^T} \dot{q} - \bar{G}(q, \dot{q})\dot{q} + \bar{D}(q)\dot{q} + \frac{\partial \bar{E}_p(q)}{\partial q} = 0_{n \times 1}. \quad (6)$$

Its a equilibrium point is $(q, \dot{q}) = (0, 0)$.

For the desired closed-loop system (6), its mechanical energy $\bar{E}(q, \dot{q}) = \bar{E}_k(q, \dot{q}) + \bar{E}_p(q) = \frac{1}{2} \dot{q}^T \bar{M}(q) \dot{q} + \bar{E}_p(q)$ is a positive definite function of q and $\dot{q}$, so it can be used as a Lyapunov function of system (6). The derivative of $\bar{E}(q, \dot{q})$ along trajectory the system (6) can be calculated as follows:

$$\begin{aligned} \dot{\bar{E}}(q, \dot{q}) &= \dot{q}^T \bar{M}(q)\ddot{q} + \frac{1}{2} \dot{q}^T \dot{\bar{M}}(q)\dot{q} + \dot{q}^T \frac{\partial \bar{E}_p(q)}{\partial q} \\ &= \dot{q}^T \bar{G}(q, \dot{q})\dot{q} - \dot{q}^T \bar{D}(q)\dot{q} = -\dot{q}^T \bar{D}(q)\dot{q} \leq 0. \end{aligned} \quad (7)$$

From the Lyapunov stability theorem, we know that the desired closed-loop system is stable at the equilibrium point $(q, \dot{q}) = (0, 0)$.

4 Controller structure and matching condition

4.1 Determination of controller structure and matching condition

When the underactuated system (2) is equivalent to the desired system (6), the acceleration $\ddot{q}$ in the two systems must be equal. From (2), we can get the acceleration of the underactuated system as

$$\begin{aligned} \ddot{q} &= M^{-1}(q)B(q)v - M^{-1}(q) \frac{\partial M(q)\dot{q}}{\partial q^T} \dot{q} \\ &\quad + \frac{1}{2} M^{-1}(q) \frac{\partial \dot{q}^T M(q) \dot{q}}{\partial q} - M^{-1}(q) \frac{\partial E_p(q)}{\partial q}. \end{aligned} \quad (8)$$

The acceleration $\ddot{q}$ of the desired closed-loop system (6) can be obtained as follows:

$$\ddot{q} = -\frac{1}{2}\overline{M}^{-1}(q)\frac{\partial\overline{M}(q)\dot{q}}{\partial q^T}\dot{q} + \overline{M}^{-1}(q)\overline{G}(q, \dot{q})\dot{q} - \overline{M}^{-1}(q)\overline{D}(q)\dot{q} - \overline{M}^{-1}(q)\frac{\partial\overline{E}_p(q)}{\partial q}. \quad (9)$$

Equating (8) with (9) gives

$$\begin{aligned} B(q)v &= \frac{\partial M(q)\dot{q}}{\partial q^T}\dot{q} - \frac{1}{2}\frac{\partial\dot{q}^T M(q)}{\partial q}\dot{q} \\ &\quad - \frac{1}{2}M(q)\overline{M}^{-1}(q)\frac{\partial\overline{M}(q)\dot{q}}{\partial q^T}\dot{q} \\ &\quad + M(q)\overline{M}^{-1}(q)\overline{G}(q, \dot{q})\dot{q} - M(q)\overline{M}^{-1}(q)\overline{D}(q)\dot{q} \\ &\quad + \frac{\partial E_p(q)}{\partial q} - M(q)\overline{M}^{-1}(q)\frac{\partial\overline{E}_p(q)}{\partial q}. \end{aligned} \quad (10)$$

Since $\text{rank} B(q) = n-1$, there exists an invertible matrix $P(q) \in \mathbb{R}^{n \times n}$ such that $P(q)B(q) = [I_{n-1}, \mathbf{0}_{(n-1) \times 1}]^T$. Premultiplying (8) by $P(q)$ and denoting $W(q) \triangleq P(q)M(q)\overline{M}^{-1}(q)$, we get

$$\begin{aligned} \begin{bmatrix} v \\ 0 \end{bmatrix} &= P(q)\frac{\partial M(q)\dot{q}}{\partial q^T}\dot{q} - \frac{1}{2}P(q)\frac{\partial\dot{q}^T M(q)}{\partial q}\dot{q} \\ &\quad - \frac{1}{2}W(q)\frac{\partial\overline{M}(q)\dot{q}}{\partial q^T}\dot{q} + W(q)\overline{G}(q, \dot{q})\dot{q} \\ &\quad - W(q)\overline{D}(q)\dot{q} + P(q)\frac{\partial E_p(q)}{\partial q} - W(q)\frac{\partial\overline{E}_p(q)}{\partial q}. \end{aligned} \quad (11)$$

The first $n-1$ lines of (11) show controller structure, which can be written as

$$\begin{aligned} v &= [I_{n-1}, \mathbf{0}_{(n-1) \times 1}] \left[P(q)\frac{\partial M(q)\dot{q}}{\partial q^T}\dot{q} - \frac{1}{2}P(q)\frac{\partial\dot{q}^T M(q)}{\partial q}\dot{q} \right. \\ &\quad - \frac{1}{2}W(q)\frac{\partial\overline{M}(q)\dot{q}}{\partial q^T}\dot{q} + W(q)\overline{G}(q, \dot{q})\dot{q} \\ &\quad \left. - W(q)\overline{D}(q)\dot{q} + P(q)\frac{\partial E_p(q)}{\partial q} - W(q)\frac{\partial\overline{E}_p(q)}{\partial q} \right], \end{aligned} \quad (12)$$

and the last line of (11) gives the matching condition to be satisfied by the desired system (6):

$$\begin{aligned} 0 &= e_n^T \left[P(q)\frac{\partial M(q)\dot{q}}{\partial q^T}\dot{q} - \frac{1}{2}P(q)\frac{\partial\dot{q}^T M(q)}{\partial q}\dot{q} \right. \\ &\quad - \frac{1}{2}W(q)\frac{\partial\overline{M}(q)\dot{q}}{\partial q^T}\dot{q} + W(q)\overline{G}(q, \dot{q})\dot{q} \\ &\quad \left. - W(q)\overline{D}(q)\dot{q} + P(q)\frac{\partial E_p(q)}{\partial q} - W(q)\frac{\partial\overline{E}_p(q)}{\partial q} \right], \end{aligned} \quad (13)$$

where $e_n^T = [0, \dots, 0, 1] \in \mathbb{R}^n$. Define vectors

$$p^T(q) = e_n^T P(q), \quad (14)$$

$$w^T(q) = [w_1, \dots, w_n] = e_n^T W(q) = p^T(q)M(q)\overline{M}^{-1}(q). \quad (15)$$

The matching condition (13) can be written as

$$\begin{aligned} 0 &= p^T(q)\frac{\partial M(q)\dot{q}}{\partial q^T}\dot{q} - \frac{1}{2}p^T(q)\frac{\partial\dot{q}^T M(q)}{\partial q}\dot{q} \\ &\quad - \frac{1}{2}w^T(q)\frac{\partial\overline{M}(q)\dot{q}}{\partial q^T}\dot{q} + w^T(q)\overline{G}(q, \dot{q})\dot{q} \\ &\quad - w^T(q)\overline{D}(q)\dot{q} + p^T(q)\frac{\partial E_p(q)}{\partial q} - w^T(q)\frac{\partial\overline{E}_p(q)}{\partial q}. \end{aligned} \quad (16)$$

If $\overline{M}(q)$, $\overline{E}_p(q)$, $\overline{G}(q, \dot{q})$ and $\overline{D}(q)$ can be solved from the matching condition (16), the desired controller can be obtained by substituting them into (12).

4.2 A sufficient condition for matching condition

In order to solve the matching condition (16) easily, a feasible sufficient condition satisfying the matching condition is derived firstly. It is easy to verify following results:

$$\begin{aligned} p^T(q)\frac{\partial M(q)\dot{q}}{\partial q^T}\dot{q} &= \left[p^T(q)\frac{\partial M(q)}{\partial q_1}\dot{q}, \dots, p^T(q)\frac{\partial M(q)}{\partial q_n}\dot{q} \right] \\ &= \left[\dot{q}^T \frac{\partial M(q)}{\partial q_1} p(q), \dots, \dot{q}^T \frac{\partial M(q)}{\partial q_n} p(q) \right] \\ &= \dot{q}^T \left(\frac{\partial M(q)p(q)}{\partial q^T} - M(q)\frac{\partial p(q)}{\partial q^T} \right), \end{aligned} \quad (17)$$

$$\begin{aligned} p^T(q)\frac{\partial\dot{q}^T M(q)}{\partial q}\dot{q} &= p^T(q) \begin{bmatrix} \dot{q}^T \frac{\partial M(q)}{\partial q_1} \\ \vdots \\ \dot{q}^T \frac{\partial M(q)}{\partial q_n} \end{bmatrix} \\ &= p^T(q) \left[\frac{\partial e_1^T M(q)}{\partial q}\dot{q}, \dots, \frac{\partial e_n^T M(q)}{\partial q}\dot{q} \right] \\ &= \dot{q}^T \left[\frac{\partial M(q)e_1}{\partial q^T} p(q), \dots, \frac{\partial M(q)e_n}{\partial q^T} p(q) \right] \\ &\triangleq \dot{q}^T \tilde{M}(q), \end{aligned} \quad (18)$$

where $e_i \in \mathbb{R}^n$ denotes the i -th column of identity matrix $I_n \in \mathbb{R}^{n \times n}$. Considering the definition of $w^T(q)$ and using the same method to derive (17) we have

$$\begin{aligned} w^T(q)\frac{\partial\overline{M}(q)\dot{q}}{\partial q^T}\dot{q} &= \dot{q}^T \left(\frac{\partial\overline{M}(q)w(q)}{\partial q^T} - \overline{M}(q)\frac{\partial w(q)}{\partial q^T} \right) \\ &= \dot{q}^T \left(\frac{\partial M(q)p(q)}{\partial q^T} - \overline{M}(q)\frac{\partial w(q)}{\partial q^T} \right). \end{aligned} \quad (19)$$

In addition, we can prove that, for any $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$,

$$x^T \overline{G}(q, \dot{q}) = \dot{q}^T \tilde{G}(q)T(x^T), \quad (20)$$

where

$$\tilde{\mathbf{G}}(\mathbf{q}) \triangleq [\bar{\mathbf{g}}_1(\mathbf{q}), \dots, \bar{\mathbf{g}}_{n-1}(\mathbf{q})],$$

$$\mathbf{T}(\mathbf{x}^T) = \begin{bmatrix} -x_n & 0 & \cdots & 0 & x_1 \\ 0 & -x_n & \cdots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -x_n & x_{n-1} \end{bmatrix}.$$

Substituting (17)-(19) into the matching condition (16) and using (20), we get

$$\begin{aligned} 0 = & \dot{\mathbf{q}}^T \left[\frac{1}{2} \frac{\partial \mathbf{M}(\mathbf{q}) \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} - \frac{1}{2} \tilde{\mathbf{M}}(\mathbf{q}) - \mathbf{M}(\mathbf{q}) \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} \right. \\ & \left. + \frac{1}{2} \bar{\mathbf{M}}(\mathbf{q}) \frac{\partial \mathbf{w}(\mathbf{q})}{\partial \mathbf{q}^T} + \tilde{\mathbf{G}}(\mathbf{q}) \mathbf{T}(\mathbf{w}^T) \right] \dot{\mathbf{q}} - \mathbf{w}^T(\mathbf{q}) \bar{\mathbf{D}}(\mathbf{q}) \dot{\mathbf{q}} \\ & + \mathbf{p}^T(\mathbf{q}) \frac{\partial E_p(\mathbf{q})}{\partial \mathbf{q}} - \mathbf{w}^T(\mathbf{q}) \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}}. \end{aligned} \quad (21)$$

Define

$$\begin{aligned} \mathbf{A}(\mathbf{q}) = & \frac{1}{2} \frac{\partial \mathbf{M}(\mathbf{q}) \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} - \frac{1}{2} \tilde{\mathbf{M}}(\mathbf{q}) - \mathbf{M}(\mathbf{q}) \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} \\ & + \frac{1}{2} \bar{\mathbf{M}}(\mathbf{q}) \frac{\partial \mathbf{w}(\mathbf{q})}{\partial \mathbf{q}^T} + \tilde{\mathbf{G}}(\mathbf{q}) \mathbf{T}(\mathbf{w}^T). \end{aligned}$$

the matching condition (21) can be converted to

$$\begin{aligned} 0 = & \dot{\mathbf{q}}^T \mathbf{A}(\mathbf{q}) \dot{\mathbf{q}} - \mathbf{w}^T(\mathbf{q}) \bar{\mathbf{D}}(\mathbf{q}) \dot{\mathbf{q}} \\ & + \mathbf{p}^T(\mathbf{q}) \frac{\partial E_p(\mathbf{q})}{\partial \mathbf{q}} - \mathbf{w}^T(\mathbf{q}) \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}} \\ = & \frac{1}{2} \dot{\mathbf{q}}^T [\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q})] \dot{\mathbf{q}} - \mathbf{w}^T(\mathbf{q}) \bar{\mathbf{D}}(\mathbf{q}) \dot{\mathbf{q}} \\ & + \mathbf{p}^T(\mathbf{q}) \frac{\partial E_p(\mathbf{q})}{\partial \mathbf{q}} - \mathbf{w}^T(\mathbf{q}) \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}}. \end{aligned} \quad (22)$$

A sufficient condition to satisfy the matching condition(22) is that the following three equations hold simultaneously:

$$\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q}) = \mathbf{0}_{n \times n}, \quad (23)$$

$$\mathbf{p}^T(\mathbf{q}) \frac{\partial E_p(\mathbf{q})}{\partial \mathbf{q}} - \mathbf{w}^T(\mathbf{q}) \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}} = 0, \quad (24)$$

$$\mathbf{w}^T(\mathbf{q}) \bar{\mathbf{D}}(\mathbf{q}) = \mathbf{0}_{1 \times n}. \quad (25)$$

Furthermore, in order to realize decoupling of $\bar{\mathbf{M}}(\mathbf{q})$, $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{w}(\mathbf{q})$ in (23), we can prove that if $w_n \neq 0$, (23) is equivalent to following equations:

$$\begin{aligned} \mathbf{w}^T(\mathbf{q}) \left[\frac{1}{2} \frac{\partial \mathbf{M}(\mathbf{q}) \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} - \frac{1}{2} \tilde{\mathbf{M}}(\mathbf{q}) - \mathbf{M}(\mathbf{q}) \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} \right] \mathbf{w}(\mathbf{q}) \\ + \frac{1}{2} \mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \frac{\partial \mathbf{w}(\mathbf{q})}{\partial \mathbf{q}^T} \mathbf{w}(\mathbf{q}) = 0, \end{aligned} \quad (26)$$

$$[\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q})]_{[-nn]} = \mathbf{0}_{n \times n}, \quad (27)$$

where (26) is a partial differential equation about $\mathbf{w}(\mathbf{q})$ only, and $[\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q})]_{[-nn]}$ indicates that the element at the

n -th row and the n -th column of matrix $[\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q})]$ is replaced by zero.

In fact, since (27) and (23) are the same except for the n -th row and n -th column element, it is only necessary to prove that (26) is equivalent to $a_{nn} = 0$ under the condition of $w_n \neq 0$, where $\mathbf{A}(\mathbf{q}) \triangleq [a_{ij}]$. Using the definition of $\mathbf{w}(\mathbf{q})$, $\mathbf{A}(\mathbf{q})$, the property of skew symmetric matrix $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$ and (27), from (26) we obtain

$$\begin{aligned} 0 = & \mathbf{w}^T(\mathbf{q}) \left[\frac{1}{2} \frac{\partial \mathbf{M}(\mathbf{q}) \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} - \frac{1}{2} \tilde{\mathbf{M}}(\mathbf{q}) - \mathbf{M}(\mathbf{q}) \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} \right. \\ & \left. + \frac{1}{2} \bar{\mathbf{M}}(\mathbf{q}) \frac{\partial \mathbf{w}(\mathbf{q})}{\partial \mathbf{q}^T} \right] \mathbf{w}(\mathbf{q}) \\ = & \mathbf{w}^T(\mathbf{q}) \mathbf{A}(\mathbf{q}) \mathbf{w}(\mathbf{q}) - \mathbf{w}^T(\mathbf{q}) \tilde{\mathbf{G}}(\mathbf{q}) \mathbf{T}(\mathbf{w}^T) \mathbf{w}(\mathbf{q}) \\ = & \frac{1}{2} \mathbf{w}^T(\mathbf{q}) [\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q})] \mathbf{w}(\mathbf{q}) - \mathbf{w}^T(\mathbf{q}) \bar{\mathbf{G}}(\mathbf{q}, \mathbf{w}) \mathbf{w}(\mathbf{q}) \\ = & \frac{1}{2} \mathbf{w}^T(\mathbf{q}) [\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q})] \mathbf{w}(\mathbf{q}) \\ = & \frac{1}{2} \mathbf{w}^T(\mathbf{q}) [\mathbf{A}(\mathbf{q}) + \mathbf{A}^T(\mathbf{q})]_{[-nn]} \mathbf{w}(\mathbf{q}) \\ & + \mathbf{w}^T(\mathbf{q}) (a_{nn} \mathbf{e}_n \mathbf{e}_n^T) \mathbf{w}(\mathbf{q}) \\ = & \mathbf{w}^T(\mathbf{q}) (a_{nn} \mathbf{e}_n \mathbf{e}_n^T) \mathbf{w}(\mathbf{q}) = w_n^2 a_{nn}. \end{aligned}$$

Obviously, $a_{nn} = 0$, if $w_n \neq 0$.

To sum up, we conclude that a sufficient condition for the matching condition (16) to be true is that $w_n \neq 0$ and (24)-(27) are satisfied simultaneously. The advantage of this sufficient condition is that it realizes decoupling of the matching condition, such that the desired $\bar{E}_p(\mathbf{q})$, $\bar{\mathbf{M}}(\mathbf{q})$, $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$ and $\bar{\mathbf{D}}(\mathbf{q})$ can be solved separately.

5 Solution of matching condition

Regarding the sufficient conditions (24)-(27), we find that (26) is a partial differential equation only with respect to $\mathbf{w}(\mathbf{q})$, so we first solve (26) to obtain $\mathbf{w}(\mathbf{q})$. Then, solving the potential energy equation (24) gives $\bar{E}_p(\mathbf{q})$. Finally, using $\mathbf{w}(\mathbf{q})$, we can design $\bar{\mathbf{M}}(\mathbf{q})$ and obtain $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$ and $\bar{\mathbf{D}}(\mathbf{q})$ by solving (27) and (25), respectively. The solution process is described as following algorithm in detail.

Step 1: According to whether $\mathbf{M}(\mathbf{q})$ is a constant matrix, we can solve $\mathbf{w}(\mathbf{q})$ by two different approaches from (26).

Case 1: When $\mathbf{M}(\mathbf{q})$ is a constant matrix, we have

$$\frac{\partial \mathbf{M} \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} = \mathbf{M} \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T}, \quad \tilde{\mathbf{M}} = \mathbf{0},$$

and (26) becomes

$$-\frac{1}{2} \left(\mathbf{w}^T(\mathbf{q}) \mathbf{M} \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} - \mathbf{p}^T(\mathbf{q}) \mathbf{M} \frac{\partial \mathbf{w}(\mathbf{q})}{\partial \mathbf{q}^T} \right) \mathbf{w}(\mathbf{q}) = 0. \quad (28)$$

Design

$$\mathbf{w}(\mathbf{q}) = \Gamma \mathbf{p}(\mathbf{q}), \quad (29)$$

where the constant matrix $\Gamma \in \mathbb{R}^{n \times n}$ is to be solved. Substituting (29) into (28) yields

$$\frac{1}{2} \mathbf{p}^T(\mathbf{q}) (\mathbf{M}\Gamma - \Gamma^T \mathbf{M}) \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} \Gamma \mathbf{p}(\mathbf{q}) = 0. \quad (30)$$

A sufficient condition to satisfy (30) is

$$\mathbf{M}\Gamma - \Gamma^T \mathbf{M} = \mathbf{0}_{n \times n}. \quad (31)$$

Design Γ as a symmetric matrix, (31) is equivalent to a linear equation as follows:

$$(\mathbf{M} \otimes \mathbf{I}_n - \mathbf{I}_n \otimes \mathbf{M}) \bar{\Gamma} = \mathbf{0}_{n^2 \times 1}, \quad (32)$$

where $\bar{\Gamma} = [\mathbf{e}_1^T \Gamma, \dots, \mathbf{e}_n^T \Gamma]^T \in \mathbb{R}^{n^2}$. In (32), there are $\frac{1}{2}n(n-1)$ independent equations and $\frac{1}{2}n(n+1)$ unknowns. Since $\frac{1}{2}n(n+1) - \frac{1}{2}n(n-1) = n > 0$, there must exist a non-zero solution Γ for (32), and it contains n free parameters at least. In order to ensure that $w_n \neq 0$ and $\bar{\mathbf{M}}(\mathbf{q})$ to be designed is a positive definite matrix, the free parameters in Γ must satisfy

$$\begin{cases} \mathbf{e}_n^T \Gamma \mathbf{p}(\mathbf{q}) = w_n \neq 0, \\ \mathbf{p}^T(\mathbf{q}) \mathbf{M} \Gamma \mathbf{p}(\mathbf{q}) = \mathbf{p}^T(\mathbf{q}) \mathbf{M} \mathbf{w}(\mathbf{q}) > 0. \end{cases} \quad (33)$$

Case 2: when $\mathbf{M}(\mathbf{q})$ is not a constant matrix, $\mathbf{w}(\mathbf{q})$ is solved from (26). Since (26) contains n unknowns, among them $n-1$ unknowns can be regarded as free parameters when solving $\mathbf{w}(\mathbf{q})$. Similar to Case 1, if we want to guarantee that $w_n \neq 0$ and $\bar{\mathbf{M}}(\mathbf{q})$ to be designed is a positive definite matrix, the $n-1$ free parameters in $\mathbf{w}(\mathbf{q})$ must satisfy

$$\begin{cases} w_n \neq 0, \\ \mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{w}(\mathbf{q}) > 0. \end{cases} \quad (34)$$

Remark 1: In fact, using the definition of $\mathbf{w}(\mathbf{q})$, we know that if the designed $\bar{\mathbf{M}}(\mathbf{q})$ is a positive definite matrix, it follows that

$$\begin{aligned} \mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{w}(\mathbf{q}) &= \mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \bar{\mathbf{M}}^{-1}(\mathbf{q}) \bar{\mathbf{M}}(\mathbf{q}) \mathbf{w}(\mathbf{q}) \\ &= \mathbf{w}^T(\mathbf{q}) \bar{\mathbf{M}}(\mathbf{q}) \mathbf{w}(\mathbf{q}) > 0. \end{aligned}$$

This implies that $\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{w}(\mathbf{q}) > 0$ is a necessary condition for designed $\bar{\mathbf{M}}(\mathbf{q})$ to be a positive definite matrix. Roughly speaking, if the free parameters in Γ or $\mathbf{w}(\mathbf{q})$ do not satisfy $\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{w}(\mathbf{q}) > 0$, we can never get a desired positive definite matrix $\bar{\mathbf{M}}(\mathbf{q})$ at all.

Remark 2: In order to facilitate solving the desired potential energy $\bar{E}_p(\mathbf{q})$ with an minimum value zero only at $\mathbf{q} = \mathbf{0}$, the free parameters in Γ or $\mathbf{w}(\mathbf{q})$ should be reserved as much as possible.

Step 2: Solve the desired potential energy $\bar{E}_p(\mathbf{q})$. Substituting the obtained $\mathbf{w}(\mathbf{q})$ into (24) gives

$$\mathbf{w}^T(\mathbf{q}) \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}} = \mathbf{p}^T(\mathbf{q}) \frac{\partial E_p(\mathbf{q})}{\partial \mathbf{q}}. \quad (35)$$

This is a quasilinear partial differential equation. Solve it to get the expression of $\bar{E}_p(\mathbf{q})$. By choosing the integral constants and the free parameters in $\mathbf{w}(\mathbf{q})$ such that $\bar{E}_p(\mathbf{q})$ satisfies

$$\frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}} \Big|_{\mathbf{q}=\mathbf{0}} = \mathbf{0}_{n \times 1}, \quad \frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}^T} \Big|_{\mathbf{q}=\mathbf{0}} > 0,$$

which ensure that $\bar{E}_p(\mathbf{q})$ has a unique minimum value zero at $\mathbf{q} = \mathbf{0}$. It is worth noting that so far, $\mathbf{w}(\mathbf{q})$ has been completely determined after $\bar{E}_p(\mathbf{q})$ is solved.

Step 3: Design the desired positive definite matrix $\bar{\mathbf{M}}(\mathbf{q})$. Since $\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{w}(\mathbf{q}) > 0$, we must be able to design parameters $\bar{m}_{kk}(\mathbf{q}) > 0, k = 1, \dots, n-1$, and satisfy

$$\sum_{k=1}^{n-1} \frac{(\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{e}_k)^2}{\bar{m}_{kk}} < \mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{w}(\mathbf{q}). \quad (36)$$

According to $\mathbf{w}^T(\mathbf{q}) = \mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \bar{\mathbf{M}}^{-1}(\mathbf{q})$, we design

$$\bar{\mathbf{M}}(\mathbf{q}) = \begin{bmatrix} \bar{m}_{11} & & \bar{m}_{1n} \\ & \ddots & \vdots \\ & & \bar{m}_{(n-1)(n-1)} & \bar{m}_{(n-1)n} \\ \bar{m}_{1n} & \dots & \bar{m}_{(n-1)n} & \bar{m}_{nn} \end{bmatrix}, \quad (37)$$

where

$$\begin{cases} \bar{m}_{kn} = \frac{\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{e}_k - w_k \bar{m}_{kk}}{w_n}, k = 1, \dots, n-1, \\ \bar{m}_{nn} = \frac{\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{e}_n - \sum_{k=1}^{n-1} w_k \bar{m}_{kn}}{w_n}, \end{cases} \quad (38)$$

and unshown elements in the matrix are all zero. At this time, it is easy to verify

$$\begin{aligned} |\bar{\mathbf{M}}(\mathbf{q})| &= \left(\prod_{k=1}^{n-1} \bar{m}_{kk} \right) \cdot \left(\bar{m}_{nn} - \sum_{k=1}^{n-1} \frac{\bar{m}_{kn}^2}{\bar{m}_{kk}} \right) \\ &= \frac{1}{w_n^2} \left(\prod_{k=1}^{n-1} \bar{m}_{kk} \right) \cdot \\ &\quad \left(\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{w}(\mathbf{q}) - \sum_{k=1}^{n-1} \frac{(\mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q}) \mathbf{e}_k)^2}{\bar{m}_{kk}} \right) > 0, \end{aligned}$$

which ensures that $\bar{\mathbf{M}}(\mathbf{q})$ is a positive definite matrix.

Step 4: Solve gyroscopic matrix $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$ from (27). Using the definition of $\mathbf{A}(\mathbf{q})$, we have

$$\begin{aligned} \mathbf{0}_{n \times n} &= \frac{1}{2} \left[\frac{\partial \mathbf{M}(\mathbf{q}) \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} + \frac{\partial \mathbf{p}^T(\mathbf{q}) \mathbf{M}(\mathbf{q})}{\partial \mathbf{q}} - \tilde{\mathbf{M}}(\mathbf{q}) - \tilde{\mathbf{M}}^T(\mathbf{q}) \right. \\ &\quad \left. - 2\mathbf{M}(\mathbf{q}) \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}^T} - 2 \frac{\partial \mathbf{p}^T(\mathbf{q})}{\partial \mathbf{q}} \mathbf{M}(\mathbf{q}) + \bar{\mathbf{M}}(\mathbf{q}) \frac{\partial \mathbf{w}(\mathbf{q})}{\partial \mathbf{q}^T} + \right. \\ &\quad \left. \frac{\partial \mathbf{w}^T(\mathbf{q})}{\partial \mathbf{q}} \bar{\mathbf{M}}(\mathbf{q}) \right]_{[-nm]} + [\tilde{\mathbf{G}}(\mathbf{q}) \mathbf{T}(\mathbf{w}^T) + \mathbf{T}^T(\mathbf{w}^T) \tilde{\mathbf{G}}^T(\mathbf{q})]_{[-nm]} \\ &\triangleq \mathbf{H}(\mathbf{q}) + [\tilde{\mathbf{G}}(\mathbf{q}) \mathbf{T}(\mathbf{w}^T) + \mathbf{T}^T(\mathbf{w}^T) \tilde{\mathbf{G}}^T(\mathbf{q})]_{[-nm]}. \end{aligned}$$

(39)

It is a linear algebraic equation with respect to gyroscopic force functions and can be solved as follows.

1) When $\mathbf{M}(\mathbf{q})$ is a constant matrix and $\mathbf{p}(\mathbf{q})$ is a constant vector, we have $\mathbf{H}(\mathbf{q}) \equiv \mathbf{0}_{n \times n}$. According to (39), we can choose $\tilde{\mathbf{G}}(\mathbf{q}) = \mathbf{0}_{n \times (n-1)}$, which implies that it is unnecessary to introduce gyroscopic forces into the desired closed-loop system (6).

2) In other cases, we have $\mathbf{H}(\mathbf{q}) \neq \mathbf{0}_{n \times n}$. Considering the symmetry of (39), we know that in (39) there are $\frac{1}{2}n(n+1) - 1$ independent equations corresponding to the left lower triangle matrix to be solved. Thus, design

$$\tilde{\mathbf{G}}(\mathbf{q}) = [\bar{g}_1(\mathbf{q}), \dots, \bar{g}_{n-1}(\mathbf{q})]$$

$$= \begin{bmatrix} \bar{g}_{1,1} & & & & \\ \bar{g}_{1,2} & \bar{g}_{2,2} & & & \\ \vdots & \vdots & \ddots & & \\ \bar{g}_{1,n-1} & \bar{g}_{2,n-1} & \dots & \bar{g}_{n-1,n-1} & \\ \bar{g}_{1,n} & \bar{g}_{2,n} & \dots & \bar{g}_{n-1,n} & \end{bmatrix}, \quad (40)$$

where the unshown elements are all zero. Then, substituting (40) into (39) yields

$$\mathbf{H}(\mathbf{q}) = -w_n \begin{bmatrix} 2\bar{g}_{1,1} & * & \dots & * & * & * \\ \bar{g}_{1,2} & 2\bar{g}_{2,2} & \dots & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \bar{g}_{1,n-1} & \bar{g}_{2,n-1} & \dots & \bar{g}_{n-2,n-1} & 2\bar{g}_{n-1,n-1} & * \\ \tilde{g}_1 & \tilde{g}_2 & \dots & \tilde{g}_{n-2} & \tilde{g}_{n-1} & * \end{bmatrix}, \quad (41)$$

where $\tilde{g}_j = \bar{g}_{j,n} - \sum_{k=1}^j \frac{w_k}{w_n} \bar{g}_{k,j}$, $j = 1, \dots, n-1$. By solving the equations at the left lower triangular matrix in (41), we get

$$\tilde{\mathbf{G}}(\mathbf{q}) = [\bar{g}_1(\mathbf{q}), \dots, \bar{g}_{n-1}(\mathbf{q})]$$

$$= -\frac{1}{w_n} \begin{bmatrix} \frac{1}{2}h_{11} & & & & \\ h_{21} & \frac{1}{2}h_{22} & & & \\ \vdots & \vdots & \ddots & & \\ h_{(n-2)1} & h_{(n-2)2} & \dots & \frac{1}{2}h_{(n-2)(n-2)} & \\ h_{(n-1)1} & h_{(n-1)2} & \dots & h_{(n-1)(n-2)} & \frac{1}{2}h_{(n-1)(n-1)} \\ \tilde{h}_1 & \tilde{h}_2 & \dots & \tilde{h}_{n-2} & \tilde{h}_{n-1} \end{bmatrix}, \quad (42)$$

where

$$\mathbf{H}(\mathbf{q}) = [h_{ij}];$$

$$\tilde{h}_j = h_{nj} + \frac{w_j}{2w_n} h_{jj} + \sum_{k=1}^{j-1} \frac{w_k}{w_n} h_{jk}, \quad j = 1, \dots, n-1.$$

Substituting (42) into the expression of $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$ yields

$$\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & \dots & 0 & -g_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -g_{n-1} \\ g_1 & \dots & g_{n-1} & 0 \end{bmatrix}, \quad (43)$$

where

$$g_j \triangleq \dot{\mathbf{q}}^T \bar{\mathbf{g}}_j(\mathbf{q})$$

$$= -\frac{1}{w_n} \left(\frac{1}{2} h_{jj} \dot{q}_j + \sum_{k=j+1}^{n-1} h_{kj} \dot{q}_k + \tilde{h}_j \dot{q}_n \right), \quad j = 1, \dots, n-1.$$

Step 5: Design desired dissipative $\bar{\mathbf{D}}$ from (35). Firstly, design $n-1$ non-zero linearly independent vectors $\alpha_1(\mathbf{q}), \dots, \alpha_{n-1}(\mathbf{q})$ such that

$$\mathbf{w}^T(\mathbf{q}) \alpha_i(\mathbf{q}) = 0, \quad i = 1, \dots, n-1.$$

Denoting matrix $\tilde{\mathbf{D}}(\mathbf{q}) = [\alpha_1(\mathbf{q}), \dots, \alpha_{n-1}(\mathbf{q})]$, the desired dissipative matrix $\bar{\mathbf{D}}$ can be designed as

$$\bar{\mathbf{D}}(\mathbf{q}) = \tilde{\mathbf{D}}(\mathbf{q}) \text{diag}(d_1, d_2, \dots, d_{n-1}) \tilde{\mathbf{D}}(\mathbf{q})^T, \quad (44)$$

where $d_1, \dots, d_{n-1}$ are positive numbers. It is easy to verify that the obtained $\bar{\mathbf{D}}(\mathbf{q})$ shown in (44) is a symmetric semi-positive definite matrix and satisfies (25).

So far, the desired $\bar{E}_p(\mathbf{q}), \bar{\mathbf{M}}(\mathbf{q}), \bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$ and $\bar{\mathbf{D}}(\mathbf{q})$ have been completely solved from the sufficient conditions (24)-(27). Substituting them into (12), we can get the controller expression. So above process to solve the matching condition is also an algorithm to design controller for the under-actuated system (2).

Besides, we have proved that the closed-loop system is stable in Section 3. For a specific system, we can prove that if there is no non-zero trajectory in the set of $\dot{\bar{E}}(\mathbf{q}, \dot{\mathbf{q}}) \equiv 0$, then the closed-loop system is asymptotically stable based on the LaSalle invariant principle.

6 Simulation Examples

6.1 Vertical Takeoff and Landing Aircraft

Our first example is a vertical takeoff and landing (VTOL) aircraft moving in the vertical $x-y$ plane as shown in Fig.1,

Fig. 1 VTOL aircraft.

its dynamical model is given by [27][28][34] as follows:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix} = \begin{bmatrix} -s_\theta & \varepsilon c_\theta \\ c_\theta & \varepsilon s_\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \triangleq \mathbf{B}\mathbf{v},$$

where (x, y) indicates its position, θ is roll angle, m is mass, J is moment of inertia, g is gravitational acceleration, $\varepsilon > 0$ is a parameter describing the relationship between rolling torque and lateral acceleration, $\mathbf{B}$ is input coupling matrix and $\mathbf{v}$ is control input, $s_{(\cdot)} \triangleq \sin(\cdot)$ and $c_{(\cdot)} \triangleq \cos(\cdot)$. Our objective is to design control law $\mathbf{v}$ with the method proposed in this paper so that the VTOL aircraft is asymptotically stabilized from any initial state to the desired equilibrium point $[x_d, y_d, 0, 0, 0, 0]^T$. Define $\mathbf{M} = \text{diag}\{m, m, J\}$, $\mathbf{g} = [0, mg, 0]^T$ and $\mathbf{q} = [x_e, y_e, \theta]^T = [x - x_d, y - y_d, \theta]^T$, the aircraft dynamical model can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{g} = \mathbf{B}\mathbf{v}. \quad (45)$$

Next, follow the steps described in Section 5 to design the controller $\mathbf{v}$ such that $\mathbf{q} \rightarrow [0, 0, 0]^T$.

Step 1: Solve $\mathbf{w}(\mathbf{q})$. First, choose a reversible matrix

$$\mathbf{P}(\mathbf{q}) = \begin{bmatrix} -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \\ c_\theta & s_\theta & -\varepsilon \end{bmatrix},$$

such that $\mathbf{P}(\mathbf{q})\mathbf{B} = [\mathbf{I}_{2 \times 2}, \mathbf{0}_{2 \times 1}]^T$. From the definition of $\mathbf{p}^T(\mathbf{q})$, we have

$$\mathbf{p}^T(\mathbf{q}) = \mathbf{e}_3^T \mathbf{P}(\mathbf{q}) = [c_\theta, s_\theta, -\varepsilon]. \quad (46)$$

Solving (32) gives

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{12} & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix}, \quad (47)$$

where $\gamma_{11}, \gamma_{12}, \gamma_{22}, \gamma_{33}$ are free parameters. Substituting the solved Γ into (29) yields

$$\begin{aligned} \mathbf{w}^T(\mathbf{q}) &= \mathbf{p}^T(\mathbf{q})\Gamma \\ &= [\gamma_{11}c_\theta + \gamma_{12}s_\theta, \gamma_{12}c_\theta + \gamma_{22}s_\theta, -\varepsilon\gamma_{33}]. \end{aligned} \quad (48)$$

From (33), we know that $\gamma_{11}, \gamma_{12}, \gamma_{22}, \gamma_{33}$ should satisfy

$$\begin{cases} \gamma_{33} \neq 0 \\ m(\gamma_{11}c_\theta^2 + \gamma_{22}s_\theta^2 + 2\gamma_{12}s_\theta c_\theta) + J\varepsilon^2\gamma_{33} > 0. \end{cases} \quad (49)$$

Step 2: Solve $\bar{E}_p(\mathbf{q})$. Substituting (46) and (48) into (35) gives

$$\begin{aligned} (\gamma_{11}c_\theta + \gamma_{12}s_\theta) \frac{\partial \bar{E}_p(\mathbf{q})}{\partial x_e} + (\gamma_{12}c_\theta + \gamma_{22}s_\theta) \frac{\partial \bar{E}_p(\mathbf{q})}{\partial y_e} \\ - \varepsilon\gamma_{33} \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \theta} = mgs_\theta. \end{aligned} \quad (50)$$

Choose $\gamma_{12} = 0$, a solution satisfying (50) is

$$\begin{aligned} \bar{E}_p(\mathbf{q}) &= \frac{mg(1-c_\theta)}{-\varepsilon\gamma_{33}} + \frac{1}{2}p_1(x_e + a_1s_\theta)^2 \\ &\quad + \frac{1}{2}p_2(y_e + a_2(1-c_\theta))^2, \end{aligned}$$

where $a_1 \triangleq \frac{\gamma_{11}}{\varepsilon\gamma_{33}}$, $a_2 \triangleq \frac{\gamma_{22}}{\varepsilon\gamma_{33}}$, p_1 and p_2 are positive constants. Let

$$\frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} p_1(x_e + a_1s_\theta) \\ p_2(y_e + a_2(1-c_\theta)) \\ \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \theta} \end{bmatrix} = 0,$$

where

$$\begin{aligned} \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \theta} &= \frac{mgs_\theta}{-\varepsilon\gamma_{33}} + p_1a_1c_\theta(x_e + a_1s_\theta) \\ &\quad + p_2a_2s_\theta(y_e + a_2(1-c_\theta)), \end{aligned}$$

we find that the possible extreme points of $\bar{E}_p(\mathbf{q})$ are $(0, 0, 0)$ and $(0, -2a_2, \pi)$ in the range of $(x_e, y_e) \in \mathbb{R}^2$, $\theta \in (-\pi, \pi]$. The Hessian matrix of $\bar{E}_p(\mathbf{q})$ is calculated as follows:

$$\frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}^T} = \begin{bmatrix} p_1 & 0 & p_1a_1c_\theta \\ 0 & p_2 & p_2a_2s_\theta \\ p_1a_1c_\theta & p_2a_2s_\theta & \frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \theta^2} \end{bmatrix},$$

where

$$\begin{aligned} \frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \theta^2} &= -\frac{mgc_\theta}{\varepsilon\gamma_{33}} + p_1a_1^2c_\theta^2 - p_1a_1s_\theta(x_e + a_1s_\theta) \\ &\quad + p_2a_2^2s_\theta^2 + p_2a_2c_\theta(y_e + a_2(1-c_\theta)). \end{aligned}$$

Since $p_1 > 0$, $p_2 > 0$, choosing $\gamma_{33} < 0$ we have

$$\begin{cases} \left| \frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}^T} \right|_{(0,0,0)} = \frac{p_1p_2mg}{-\varepsilon\gamma_{33}} > 0, \\ \left| \frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}^T} \right|_{(0,-2a_2,\pi)} = \frac{p_1p_2mg}{\varepsilon\gamma_{33}} < 0. \end{cases}$$

Thus, at this time, $\bar{E}_p(\mathbf{q})$ is positive for any $\mathbf{q}$ in the range $(x_e, y_e) \in \mathbb{R}^2$, $\theta \in (-\pi, \pi]$, and has a minimum value zero only at $\mathbf{q} = [0, 0, 0]^T$. From (49), $\gamma_{12} = 0$ and $\gamma_{33} < 0$, we know that the free parameters in Γ should satisfy

$$\gamma_{12} = 0, \gamma_{33} < 0, m(\gamma_{11}c_\theta^2 + \gamma_{22}s_\theta^2) + J\varepsilon^2\gamma_{33} > 0. \quad (51)$$

As a result, the free parameters can be chosen in the following order: $\gamma_{12} = 0$, $\gamma_{33} < 0$, $\gamma_{11} = \gamma_{22} > -\frac{J\varepsilon^2\gamma_{33}}{m}$.

Step 3: Design $\bar{\mathbf{M}}(\mathbf{q})$. According to (37), we design

$$\bar{\mathbf{M}}(\mathbf{q}) = \begin{bmatrix} \bar{m}_{11} & 0 & \bar{m}_{13} \\ 0 & \bar{m}_{22} & \bar{m}_{23} \\ \bar{m}_{13} & \bar{m}_{23} & \bar{m}_{33} \end{bmatrix}, \quad (52)$$

where $\bar{m}_{11}$ and $\bar{m}_{22}$ are determined by (36), $\bar{m}_{13}$, $\bar{m}_{23}$ and $\bar{m}_{33}$ can be obtained by (38) as follows:

$$\begin{aligned}\bar{m}_{11} &= \bar{m}_{22} > \frac{m^2}{m\gamma_{11} + J\varepsilon^2\gamma_{33}} > 0, \\ \bar{m}_{13} &= \frac{(\gamma_{11}\bar{m}_{11} - m)c_\theta}{\varepsilon\gamma_{33}}, \quad \bar{m}_{23} = \frac{(\gamma_{11}\bar{m}_{22} - m)s_\theta}{\varepsilon\gamma_{33}}, \\ \bar{m}_{33} &= \frac{J\varepsilon^2\gamma_{33} + \gamma_{11}(\gamma_{11}\bar{m}_{11} - m)c_\theta^2 + \gamma_{11}(\gamma_{11}\bar{m}_{22} - m)s_\theta^2}{(\varepsilon\gamma_{33})^2}.\end{aligned}$$

Step 4: Calculate $\bar{G}(q, \dot{q})$. From the definition of $H(q)$, we have

$$H(q) = \frac{1}{2} \left[-M \frac{\partial p(q)}{\partial q^T} - \frac{\partial p(q)}{\partial q} M + \bar{M}(q) \frac{\partial w(q)}{\partial q^T} + \frac{\partial w(q)}{\partial q} \bar{M}(q) \right]_{[-m]} = \frac{(m - \gamma_{11}\bar{m}_{11})}{2} \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ s_\theta & -c_\theta & 0 \end{bmatrix}. \quad (53)$$

Substituting (53) into (43) yields

$$\bar{G}(q, \dot{q}) = \frac{(m - \gamma_{11}\bar{m}_{11})\dot{\theta}}{2\varepsilon\gamma_{33}} \begin{bmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ s_\theta & -c_\theta & 0 \end{bmatrix}. \quad (54)$$

Step 5: Design $\bar{D}(q)$. First of all, two non-zero vectors $\alpha_1 = [1, 0, ac_\theta]^T$ and $\alpha_2 = [0, 1, as_\theta]^T$ are designed such that $w^T(q)\alpha_i = 0$, $i = 1, 2$,

where $a = \frac{\gamma_{11}}{\varepsilon\gamma_{33}}$. Design the dissipative matrix $\bar{D}(q)$ in terms of (44) as

$$\begin{aligned}\bar{D}(q) &= \tilde{D}(q) \text{diag}(d_1, d_2) \tilde{D}^T(q) \\ &= \begin{bmatrix} d_1 & 0 & d_1 ac_\theta \\ 0 & d_2 & d_2 as_\theta \\ d_1 ac_\theta & d_2 as_\theta & d_1 a^2 c_\theta^2 + d_2 a^2 s_\theta^2 \end{bmatrix},\end{aligned}$$

where $\tilde{D}(q) \triangleq [\alpha_1, \alpha_2]$, d_1 and d_2 are positive constants.

Finally, substituting the solved $\bar{E}_p(q)$, $\bar{M}(q)$, $\bar{G}(q, \dot{q})$ and $\bar{D}(q)$ into (12) gives the control law expression as follows:

$$\begin{aligned}v_1 &= \frac{m}{\bar{m}_{11}} [-k_1 \dot{\theta}^2 + d_1 s_\theta \dot{x}_e - d_2 c_\theta \dot{y}_e + (d_1 - d_2) as_\theta c_\theta \dot{\theta} \\ &\quad + p_1 s_\theta (x_e + as_\theta) - p_2 c_\theta (y_e + a(1 - c_\theta))] + mgc_\theta,\end{aligned} \quad (55)$$

$$\begin{aligned}v_2 &= Jk_2 [d_1 c_\theta \dot{x}_e + d_2 s_\theta \dot{y}_e + a(d_1 c_\theta^2 + d_2 s_\theta^2) \dot{\theta} \\ &\quad + p_1 c_\theta (x_e + as_\theta) + p_2 s_\theta (y_e + a(1 - c_\theta)) - \bar{m}_{11} g s_\theta],\end{aligned} \quad (56)$$

where $k_1 \triangleq \frac{(\gamma_{11}\bar{m}_{11} - m)}{\varepsilon\gamma_{33}}$, $k_2 \triangleq \frac{-\varepsilon\gamma_{33}m}{\bar{m}_{11}(m\gamma_{11} + J\varepsilon^2\gamma_{33}) - m^2}$.

The asymptotic stability of the closed-loop system consisting of (45), (55) and (56) can be proved as follows. The

Lyapunov function of the closed-loop system is assigned as its mechanical energy

$$\begin{aligned}\bar{E}(q, \dot{q}) &= \frac{1}{2} \dot{q}^T \bar{M}(q) \dot{q} + \bar{E}_p(q) = \frac{1}{2} \bar{m}_{11} \dot{x}_e^2 + \frac{1}{2} \bar{m}_{22} \dot{y}_e^2 \\ &\quad + \frac{1}{2} \bar{m}_{33} \dot{\theta}^2 + \bar{m}_{13} \dot{x}_e \dot{\theta} + \bar{m}_{23} \dot{y}_e \dot{\theta} + \frac{mg(1 - c_\theta)}{-\varepsilon\gamma_{33}} \\ &\quad + \frac{1}{2} p_1 (x_e + a_1 s_\theta)^2 + \frac{1}{2} p_2 (y_e + a_2 (1 - c_\theta))^2,\end{aligned}$$

and its derivative along the system trajectory is

$$\begin{aligned}\dot{\bar{E}}(q, \dot{q}) &= -\dot{q}^T \bar{D}(q) \dot{q} \\ &= -d_1 (\dot{x}_e + ac_\theta \dot{\theta})^2 - d_2 (\dot{y}_e + as_\theta \dot{\theta})^2 \leq 0.\end{aligned}$$

So the closed-loop system is stable.

In order to prove asymptotic stability of the closed-loop system, we should prove that there is no continuous time-varying trajectory in the set of $\bar{E}(q, \dot{q}) \equiv 0$. In fact, from

$$\dot{\bar{E}}(q, \dot{q}) = -d_1 (\dot{x}_e + ac_\theta \dot{\theta})^2 - d_2 (\dot{y}_e + as_\theta \dot{\theta})^2 \equiv 0,$$

we obtain

$$\dot{x}_e + ac_\theta \dot{\theta} \equiv 0, \quad \dot{y}_e + as_\theta \dot{\theta} \equiv 0. \quad (57)$$

Differentiating and integrating (57) yields

$$\ddot{x}_e + ac_\theta \ddot{\theta} - as_\theta \dot{\theta}^2 \equiv 0, \quad \ddot{y}_e + as_\theta \ddot{\theta} + ac_\theta \dot{\theta}^2 \equiv 0, \quad (58)$$

$$x_e + as_\theta \equiv \beta_1, \quad y_e - ac_\theta \equiv \beta_2, \quad (59)$$

where β_1 and β_2 are integral constants. Substituting (57) and (59) into control laws (55) and (56) yields

$$v_1 \equiv \frac{m}{\bar{m}_{11}} (-k_1 \dot{\theta}^2 + \beta_1 p_1 s_\theta - \beta_2 p_2 c_\theta) + mgc_\theta, \quad (60)$$

$$v_2 \equiv Jk_2 [\beta_1 p_1 c_\theta + (\beta_2 p_2 - \bar{m}_{11} g) s_\theta]. \quad (61)$$

Substituting (58), (60) and (61) into the system (45) gives

$$mac_\theta \ddot{\theta} - mas_\theta \dot{\theta}^2 - s_\theta v_1(\theta, \dot{\theta}) + \varepsilon c_\theta v_2(\theta) \equiv 0, \quad (62)$$

$$mas_\theta \ddot{\theta} + mac_\theta \dot{\theta}^2 - mg + c_\theta v_1(\theta, \dot{\theta}) + \varepsilon s_\theta v_2(\theta) \equiv 0, \quad (63)$$

$$\ddot{\theta} \equiv k_2 \beta_1 p_1 c_\theta + k_2 (\beta_2 p_2 - \bar{m}_{11} g) s_\theta. \quad (64)$$

Let (62) $\times c_\theta$ + (63) $\times s_\theta$, we get

$$\ddot{\theta} \equiv -\frac{\varepsilon}{ma} Jk_2 [\beta_1 p_1 c_\theta + (\beta_2 p_2 - \bar{m}_{11} g) s_\theta] + \frac{1}{a} g s_\theta. \quad (65)$$

Since $\varepsilon > 0$, $\gamma_{33} < 0$ and $\gamma_{11} > -\frac{J\varepsilon^2\gamma_{33}}{m}$, we know $J\varepsilon + ma \neq 0$. From (64) and (65), we have

$$\ddot{\theta} \equiv \frac{mgs_\theta}{J\varepsilon + ma}. \quad (66)$$

Substituting (66) into (64) yields

$$k_2 \beta_1 p_1 c_\theta + \left(k_2 \beta_2 p_2 - k_2 \bar{m}_{11} g - \frac{mg}{J\varepsilon + ma} \right) s_\theta \equiv 0,$$

Fig. 2 Asymptotical stabilization of VTOL aircraft.

which implies that

$$\begin{cases} \beta_1 \equiv 0 \\ \beta_2 \equiv \frac{\bar{m}_{11}k_2(J\mathcal{E}+ma)+m}{p_2k_2(J\mathcal{E}+ma)}g = \frac{m^2g}{\varepsilon\gamma_{33}p_2(J\mathcal{E}+ma)}. \end{cases} \quad (67)$$

Let (62) $\times s_\theta - (63) \times c_\theta$, we get

$$\frac{m}{\varepsilon\gamma_{33}}\dot{\theta}^2 + \beta_1 p_1 s_\theta - \beta_2 p_2 c_\theta \equiv 0. \quad (68)$$

Differentiating (68) gives

$$\ddot{\theta} \equiv -\frac{1}{2m}\varepsilon\gamma_{33}(\beta_1 p_1 c_\theta + \beta_2 p_2 s_\theta). \quad (69)$$

Substituting (67) into (69) yields

$$\ddot{\theta} \equiv -\frac{1}{2}\frac{mgs_\theta}{J\mathcal{E}+ma}. \quad (70)$$

By comparing (66) with (70), we obtain

$$-\frac{1}{2}\frac{mgs_\theta}{J\mathcal{E}+ma} \equiv \frac{mgs_\theta}{J\mathcal{E}+ma} \Rightarrow s_\theta \equiv 0. \quad (71)$$

In the range of $\theta \in (-\pi, \pi]$, only $\theta \equiv 0$ or $\theta \equiv \pi$ satisfies (71). Furthermore, we have $\dot{\theta} = \ddot{\theta} \equiv 0$. Substituting those into (57) and (58) shows that $\dot{x}_e = \ddot{x}_e = \dot{y}_e = \ddot{y}_e \equiv 0$. At this time, we can get $\beta_1 = \beta_2 \equiv 0$ from (63) and (64), which implies that $x_d \equiv 0$, $y_e \equiv 0$ or $y_e \equiv -2a$. As a result, there are only two isolated equilibrium points $(0, 0, 0, 0, 0, 0)$ and $(0, -2a, \pi, 0, 0, 0)$ in the set $\bar{E}(q, \dot{q}) \equiv 0$. However, we know that $(0, -2a, \pi)$ is not an extreme point of $\bar{E}_p(q)$, so it is an unstable equilibrium point. In summary, we conclude that there exists a unique stable equilibrium point $(0, 0, 0, 0, 0, 0)$ in the range of $(x_e, y_e) \in \mathbb{R}^2$, $\theta \in (-\pi, \pi]$ and $(\dot{x}_e, \dot{y}_e, \dot{\theta}) \in$

$\mathbb{R}^3$. According to the LaSalle invariant principle, the VTOL aircraft achieves almost global asymptotic stability.

In order to verify the effectiveness of the designed controller (55) and (56), some simulations for the VTOL aircraft are performed with MATLAB. The physical and dynamic parameters of the aircraft are introduced from [22] as follows: $m = 1\text{kg}$, $J = 1\text{kg}\cdot\text{m}^2$, $g = 9.8\text{m/s}^2$, $\varepsilon = m/J = 1$. The controller parameters are chosen as $\gamma_{11} = \gamma_{22} = 3$, $\gamma_{33} = -2$, $\bar{m}_{11} = 1.2$, $p_1 = p_2 = 0.9$ and $d_1 = d_2 = 3.8$. The simulation results are depicted in Fig.2, which describes a maneuvering process of the VTOL aircraft moving from an extreme position, turning over the head, and then flying a distance in (x, y) plane from an initial state $(5, -5, \pi, 0.1, -0.1, 0.1)^T$ to the desired state $(-5, 5, 0, 0, 0, 0)^T$. In addition, it can be seen from Fig.2 that the position and attitude of the aircraft quickly reach the desired values and remain stable after that, and $\bar{E}_p(\mathbf{q})$ converges to the minimum value of zero. Thus, the effectiveness of this controller is verified.

6.2 2-DOF Pendubot

The second example is a 2-DOF Pendubot shown in Figure 3. Its first joint (rod 1) is actuated by a driving force $u_1 \in \mathbb{R}$ and the second joint (rod 2) is unactuated ($u_2 \equiv 0$). Both the rod 1 and rod 2 are regarded as rigid bodies. Define $\mathbf{q} = [\theta_1, \theta_2]^T$, where θ_1 is the angle between the positive direction of y -axis and rod 1, and θ_2 is the relative angle between rod 1 and rod 2, both of them are positive clockwise. m_i and l_i represent mass and length of the i -th ($i = 1, 2$) rod, respectively. Let l_{Ci} denote the distance from centroid C_i to O_i and I_{O_i} denote the moment of inertia of rod i with respect to O_i . As mentioned in [29]-[32], by using the Euler-Lagrange equation, we can obtain the dynamical equation of

Fig. 3 2-DOF Pendubot.

the Pendubot as

$$\begin{bmatrix} h_1 + h_2 + 2h_3c\theta_2 & h_2 + h_3c\theta_2 \\ h_2 + h_3c\theta_2 & h_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} - h_3s\theta_2 \begin{bmatrix} 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2 \\ -\dot{\theta}_1^2 \end{bmatrix} - g \begin{bmatrix} h_4s\theta_1 + h_5s(\theta_1 + \theta_2) \\ h_5s(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}. \quad (72)$$

where the constants $h_1 \triangleq I_{O_1} + m_2l_1^2$, $h_2 \triangleq I_{O_2}$, $h_3 \triangleq m_2l_1l_{C2}$, $h_4 \triangleq m_2l_1 + m_1l_{C1}$, $h_5 \triangleq m_2l_{C2}$, $h_3 > h_2$. The control objective is to design control law u_1 with the method described in this paper so that the 2-DOF Pendubot can be locally asymptotically stabilized from an initial state near the equilibrium point $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ to the equilibrium point.

Next, follow the steps described in Section 5 to design the controller.

Step 1: Solve $\mathbf{w}(\mathbf{q}) = [w_1, w_2]$. From (72), we know $\mathbf{p} = [0, 1]^T$, and substituting it into (26) yields

$$h_3s\theta_2w_1 \left(w_1 + \frac{1}{2}w_2 \right) + \frac{1}{2}\mathbf{e}_2^T \mathbf{M}(\mathbf{q}) \frac{\partial \mathbf{w}(\mathbf{q})}{\partial \mathbf{q}^T} \mathbf{w}(\mathbf{q}) = 0, \quad (73)$$

where

$$\mathbf{M}(\mathbf{q}) \triangleq \begin{bmatrix} h_1 + h_2 + 2h_3c\theta_2 & h_2 + h_3c\theta_2 \\ h_2 + h_3c\theta_2 & h_2 \end{bmatrix}.$$

One solution of (73) is

$$\mathbf{w} = [w_1, w_2] = \left[-\frac{1}{2}\bar{w}_2, \bar{w}_2 \right], \quad (74)$$

where $\bar{w}_2$ is a constant free parameter and needs to satisfy

$$\bar{w}_2 \neq 0, \quad \mathbf{p}^T \mathbf{M}(\mathbf{q}) \mathbf{w} = -\frac{1}{2}\bar{w}_2 (h_3c\theta_2 - h_2) > 0. \quad (75)$$

Step 2: Solve $\bar{E}_p(\mathbf{q})$. Substituting (74) into (35) gives potential energy equation as follows:

$$-\frac{1}{2}\bar{w}_2 \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \theta_1} + \bar{w}_2 \frac{\partial \bar{E}_p(\mathbf{q})}{\partial \theta_2} = -gh_5s(\theta_1 + \theta_2). \quad (76)$$

Solving (76), we get a solution

$$\bar{E}_p(\mathbf{q}) = -\frac{2gh_5}{\bar{w}_2} (1 - c(\theta_1 + \theta_2)) + \frac{1}{2}p_1 \left(\theta_1 + \frac{1}{2}\theta_2 \right)^2, \quad (77)$$

where p_1 is a positive constant. Let

$$\frac{\partial \bar{E}_p(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} -\frac{2gh_5}{\bar{w}_2} s(\theta_1 + \theta_2) + p_1 \left(\theta_1 + \frac{1}{2}\theta_2 \right) \\ -\frac{2gh_5}{\bar{w}_2} s(\theta_1 + \theta_2) + \frac{1}{2}p_1 \left(\theta_1 + \frac{1}{2}\theta_2 \right) \end{bmatrix} = \mathbf{0},$$

we find that a possible extreme point of $\bar{E}_p(\mathbf{q})$ is $\mathbf{q} = \mathbf{0}$. Further, the Hessian matrix of $\bar{E}_p(\mathbf{q})$ is calculated as follows:

$$\frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}^T} = -\frac{2gh_5}{\bar{w}_2} c(\theta_1 + \theta_2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + p_1 \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

Since $p_1 > 0$ and $-gh_5/\bar{w}_2 > 0$ when $\bar{w}_2 < 0$ and $|\theta_1 + \theta_2| < \pi/2$, we have

$$\left| \frac{\partial^2 \bar{E}_p(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}^T} \right| = -\frac{p_1 g h_5}{2\bar{w}_2} c_{(\theta_1 + \theta_2)} > 0.$$

Thus, the Hessian matrix of $\bar{E}_p(\mathbf{q})$ is positive definite, which implies that $\bar{E}_p(\mathbf{q})$ has a minimum value zero only at the equilibrium point $\mathbf{q} = \mathbf{0}$. As a result, we choose constant $\bar{w}_2 < 0$, which also can ensure that (75) holds in the range of $|\theta_2| < \arccos(\frac{h_2}{h_3}) < \pi/2$.

Step 3: Design $\bar{\mathbf{M}}(\mathbf{q})$. According to (37), we design

$$\bar{\mathbf{M}}(\mathbf{q}) = \begin{bmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{12} & \bar{m}_{22} \end{bmatrix}, \quad (78)$$

where $\bar{m}_{11}$ can be designed according to (36), and both $\bar{m}_{12}$ and $\bar{m}_{22}$ can be obtained by (38) as follows:

$$\begin{aligned} \bar{m}_{11} &= -\frac{2(h_3 c_2 + h_2)^2 + 4(a_0 - 2)h_2^2}{\bar{w}_2(h_3 c_2 - h_2)} > 0, \\ \bar{m}_{12} &= \frac{1}{\bar{w}_2} \left(m_{21} + \frac{1}{2} w_2 \bar{m}_{11} \right), \\ \bar{m}_{22} &= \frac{1}{\bar{w}_2} \left(m_{22} + \frac{1}{2} m_{21} + \frac{1}{4} \bar{w}_2 \bar{m}_{11} \right), \end{aligned}$$

where the constant $a_0 > 2$.

Step 4: Calculate $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$. From the definition of $\mathbf{H}(\mathbf{q})$, we can compute

$$\begin{aligned} \mathbf{H}(\mathbf{q}) &= \frac{1}{2} \left[\frac{\partial \mathbf{M}(\mathbf{q}) \mathbf{p}}{\partial \mathbf{q}^T} + \frac{\partial \mathbf{p}^T \mathbf{M}(\mathbf{q})}{\partial \mathbf{q}} - \tilde{\mathbf{M}}(\mathbf{q}) - \tilde{\mathbf{M}}^T(\mathbf{q}) \right]_{[-m]} \\ &= \begin{bmatrix} 2h_3 s_{\theta_2} & \frac{1}{2} h_3 s_{\theta_2} \\ \frac{1}{2} h_3 s_{\theta_2} & 0 \end{bmatrix}. \end{aligned} \quad (79)$$

Then, substituting (79) into (43) yields

$$\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}}) = -\frac{1}{w_2} h_3 s_{\theta_2} \dot{\theta}_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (80)$$

Step 5: Design $\bar{\mathbf{D}}(\mathbf{q})$. Firstly, a non-zero column vector $\alpha = [1, \frac{1}{2}]^T$ is designed such that $\mathbf{w}^T \alpha = 0$. According to (44), we construct the desired dissipative matrix

$$\bar{\mathbf{D}} = d_1 \alpha \alpha^T = d_1 \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad (81)$$

where d_1 is a positive number.

Finally, substituting the solved $\bar{E}_p(\mathbf{q})$, $\bar{\mathbf{M}}(\mathbf{q})$, $\bar{\mathbf{G}}(\mathbf{q}, \dot{\mathbf{q}})$ and $\bar{\mathbf{D}}$ into (12), we can obtain the actual control law expression

as follows:

$$\begin{aligned} u_1 &= \left[\frac{h_3^2 c_{\theta_2}^2 + 2h_2 h_3 c_{\theta_2} + 2h_2^2 - h_1 h_2}{h_2^2} - \frac{a_0(h_1 - h_2)}{(h_3 c_{\theta_2} - h_2)} \right] \\ &\quad \frac{h_3 s_{\theta_2} \dot{\theta}_1^2}{(a_0 - 2)} - \left[h_3 s_{\theta_2} + \frac{a_0 h_3 s_{\theta_2} (h_3^2 c_{\theta_2}^2 - h_1 h_2)}{2(a_0 - 2)(h_3 c_{\theta_2} - h_2)^2} \right] (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \\ &\quad + \frac{\bar{w}_2 (h_3^2 c_{\theta_2}^2 - h_1 h_2)}{2(a_0 - 2)h_2^2} \left[d_1 \left(\dot{\theta}_1 + \frac{1}{2} \dot{\theta}_2 \right) + p_1 \left(\theta_1 + \frac{1}{2} \theta_2 \right) \right] \\ &\quad - \left[\frac{h_3^2 c_{\theta_2}^2 + h_2 h_3 c_{\theta_2} + h_2^2 - h_1 h_2}{h_2^2} - \frac{a_0(h_1 - h_2)}{2(h_3 c_{\theta_2} - h_2)} \right] \\ &\quad \frac{2gh_5 s_{(\theta_1 + \theta_2)}}{(a_0 - 2)} - g(h_4 s_{\theta_1} + h_5 s_{(\theta_1 + \theta_2)}). \end{aligned} \quad (82)$$

The asymptotic stability of the closed-loop system composed by (72) and (82) can be proved. The mechanical energy of the closed-loop system is chosen as a Lyapunov function as follows:

$$\begin{aligned} \bar{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2} \dot{\mathbf{q}}^T \bar{\mathbf{M}}(\mathbf{q}) \dot{\mathbf{q}} + \bar{E}_p(\mathbf{q}) = \frac{1}{2} \bar{m}_{11} \dot{\theta}_1^2 + \frac{1}{2} \bar{m}_{22} \dot{\theta}_2^2 \\ &\quad + \bar{m}_{12} \dot{\theta}_1 \dot{\theta}_2 - \frac{2gh_5}{\bar{w}_2} (1 - c_{(\theta_1 + \theta_2)}) + \frac{1}{2} p_1 \left(\theta_1 + \frac{1}{2} \theta_2 \right)^2, \end{aligned}$$

and its derivative along the system trajectory is

$$\dot{\bar{E}}(\mathbf{q}, \dot{\mathbf{q}}) = -\dot{\mathbf{q}}^T \bar{\mathbf{D}} \dot{\mathbf{q}} = -d_1 \left(\dot{\theta}_1 + \frac{1}{2} \dot{\theta}_2 \right)^2 \leq 0. \quad (83)$$

So the closed-loop system is stable. In order to prove asymptotic stability of the closed-loop system, we only need to prove that there is no non-zero trajectory in the set $\dot{\bar{E}}(\mathbf{q}, \dot{\mathbf{q}}) \equiv 0$ when $|\theta_1| < \pi$, $|\theta_2| < \arccos(\frac{h_2}{h_3}) < \pi/2$ and $|\theta_1 + \theta_2| < \pi/2$. In fact, from $\dot{\bar{E}}(\mathbf{q}, \dot{\mathbf{q}}) = -d_1 (\dot{\theta}_1 + \frac{1}{2} \dot{\theta}_2)^2 \equiv 0$, we get

$$\dot{\theta}_1 + \frac{1}{2} \dot{\theta}_2 \equiv 0. \quad (84)$$

Integrating and differentiating (84) yields

$$\theta_1 + \frac{1}{2} \theta_2 \equiv \eta, \quad (85)$$

$$\ddot{\theta}_1 + \frac{1}{2} \ddot{\theta}_2 \equiv 0, \quad (86)$$

where η is an integral constant. Substituting (84) and (85) into (82) gives control law as

$$\begin{aligned} u_1 &\equiv \left[\frac{h_3^2 c_{\theta_2}^2 + 2h_2 h_3 c_{\theta_2} + 2h_2^2 - h_1 h_2}{h_2^2} - \frac{a_0(h_1 - h_2)}{(h_3 c_{\theta_2} - h_2)} \right] \\ &\quad \frac{h_3 s_{\theta_2} \dot{\theta}_1^2}{(a_0 - 2)} + \frac{\bar{w}_2 (h_3^2 c_{\theta_2}^2 - h_1 h_2)}{2(a_0 - 2)h_2^2} p_1 \eta - \frac{2gh_5 s_{(\theta_1 + \theta_2)}}{(a_0 - 2)} \\ &\quad \left[\frac{h_3^2 c_{\theta_2}^2 + h_2 h_3 c_{\theta_2} + h_2^2 - h_1 h_2}{h_2^2} - \frac{a_0(h_1 - h_2)}{2(h_3 c_{\theta_2} - h_2)} \right] \\ &\quad - g(h_4 s_{\theta_1} + h_5 s_{(\theta_1 + \theta_2)}). \end{aligned}$$

(87)

Substituting (87) into the system equation (72) and premultiplying (72) by $\mathbf{M}^{-1}(\mathbf{q})$, we get

$$\ddot{\theta}_1 \equiv \frac{1}{h_1 h_2 - h_3^2 c_{\theta_2}^2} [h_2 u_1(\mathbf{q}, \dot{\mathbf{q}}) + h_3 s_{\theta_2} (h_2 + h_3 c_{\theta_2}) \dot{\theta}_1^2 + g (h_2 h_4 s_{\theta_1} - h_5 h_3 c_{\theta_2} s_{(\theta_1 + \theta_2)})], \quad (88)$$

$$\ddot{\theta}_2 \equiv \frac{1}{h_1 h_2 - h_3^2 c_{\theta_2}^2} [- (h_2 + h_3 c_{\theta_2}) u_1(\mathbf{q}, \dot{\mathbf{q}}) - h_3 s_{\theta_2} (h_1 + h_2 + 2 h_3 c_{\theta_2}) \dot{\theta}_1^2 - g h_4 s_{\theta_1} (h_2 + h_3 c_{\theta_2}) + g h_5 s_{(\theta_1 + \theta_2)} (h_1 + h_3 c_{\theta_2})]. \quad (89)$$

Using (86), let (88) + (89)/2, after simplification we have

$$\frac{(h_1 h_2 - h_3^2 c_{\theta_2}^2)}{(a_0 - 2) h_2^2} \left[(h_3 c_{\theta_2} + h_2) h_3 s_{\theta_2} \dot{\theta}_1^2 + \frac{1}{2} \bar{w}_2 (h_3 c_{\theta_2} - h_2) p_1 \eta - 2 g h_3 h_5 s_{(\theta_1 + \theta_2)} c_{\theta_2} \right] \equiv 0. \quad (90)$$

Since (90) holds for any $\dot{\mathbf{q}}$, $|\mathbf{M}(\mathbf{q})| = h_1 h_2 - h_3^2 c_{\theta_2}^2 > 0$ and $a_0 > 2$, then (90) is equivalent to the following equations:

$$(h_3 c_{\theta_2} + h_2) h_3 s_{\theta_2} \equiv 0, \quad (91)$$

$$-\frac{1}{2} \bar{w}_2 (h_3 c_{\theta_2} - h_2) p_1 \eta + 2 g h_3 h_5 s_{(\theta_1 + \theta_2)} c_{\theta_2} \equiv 0. \quad (92)$$

Due to $c_{\theta_2} > \frac{h_2}{h_3} > 0$, we know $h_3 c_{\theta_2} + h_2 > 0$. Hence, (91) is equivalent to

$$s_{\theta_2} \equiv 0. \quad (93)$$

(93) has a unique solution $\theta_2 \equiv 0$ in the range of $|\theta_2| < \arccos(\frac{h_2}{h_3}) < \pi/2$. Substituting $\theta_2 \equiv 0$ into (92) yields

$$-\bar{w}_2 (h_3 - h_2) p_1 \theta_1 + 4 g h_3 h_5 s_{\theta_1} \triangleq f(\theta_1) \equiv 0.$$

Obviously, $\theta_1 \equiv 0$ is a solution of $f(\theta_1) \equiv 0$. From $\bar{w}_2 < 0$, $h_3 > h_2$ and

$$\begin{aligned} \frac{\partial f(\theta_1)}{\partial \theta_1} &= -\bar{w}_2 (h_3 - h_2) p_1 + 4 g h_3 h_5 c_{\theta_1} \\ &> -\bar{w}_2 (h_3 - h_2) p_1 - 4 g h_3 h_5, \end{aligned}$$

we know that choosing parameter $p_1 > \frac{4 g h_3 h_5}{-\bar{w}_2 (h_3 - h_2)}$ can make $\frac{\partial f(\theta_1)}{\partial \theta_1} > 0$ for any $\theta_1 < |\pi|$, which indicates that the function $f(\theta_1)$ is monotonically increasing and $\theta_1 \equiv 0$ is the unique solution of $f(\theta_1)$. Furthermore, we have $\dot{\theta}_1 = \dot{\theta}_2 \equiv 0$. In summary, our conclusion is that only equilibrium point $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is included in the set of $\bar{E}(\mathbf{q}, \dot{\mathbf{q}}) \equiv 0$ when $|\theta_1| < \pi$, $|\theta_2| < \arccos(\frac{h_2}{h_3}) < \pi/2$, $|\theta_1 + \theta_2| < \pi/2$ and $p_1 >$

$\frac{4 g h_3 h_5}{-\bar{w}_2 (h_3 - h_2)}$. Therefore the 2-DOF Pendubot system is locally asymptotically stable at the equilibrium point according to the LaSalle invariant principle.

We carried out simulation studies on the 2-DOF Pendubot with MATLAB. The model parameters are introduced from [31] and shown in Table 1. The initial state is set to $(\mathbf{q}(0), \dot{\mathbf{q}}(0)) = (\pi/6, -\pi/6, 0.1, 0.1)$ and the controller parameters are chosen as $a_0 = 2.5$, $d_1 = 6$, $p_1 = 16$, $\bar{w}_2 = -1$. The change curves of the position θ_1 , θ_2 and angular velocity $\dot{\theta}_1$, $\dot{\theta}_2$ of the 2-DOF Pendubot, control input u_1 , and desired potential $\bar{E}_p(\mathbf{q})$ are depicted in Fig. 4, respectively. It can be seen from Fig.4 that the state of the 2-DOF Pendubot quickly converge to the desired equilibrium point $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ from initial values, and the closed-loop system achieves local asymptotic stability, which verifies the validity of the nonlinear smooth control law designed in this paper.

Table 1 2-DOF pendubot model parameters

Parameter	Value	Units
m_1	0.6	kg
m_2	0.6	kg
h_1	0.072	kg · m ²
h_2	0.018	kg · m ²
h_3	0.027	kg · m ²
h_4	0.27	kg · m ²
h_5	0.09	kg · m ²
g	9.8	m/s ²

7 Conclusion and Future Research

In this paper, a novel stabilization control algorithm based on the CL method is proposed for a class of mechanical systems with underactuation degree one. Our main research results include that one simpler sufficient condition satisfying the matching condition is derived, and the specific solution method for this sufficient condition, the complete control law expression, and the stability proof of closed-loop system are given in detail. Compared with the existing methods of the same type, our advantage is that the obtained sufficient condition at most need to solve two quasi-linear partial differential equations, and the rest are algebraic equations, which greatly reduces the difficulty and computational complexity of the matching condition. In addition, we only use matrix theory knowledge rather than group theory in the process of deriving matching conditions and characterizing controllers, which is more conducive to the direct application and understanding of engineers and technicians. Finally, using the method given in this paper, we design a global stabilization control strategy for a VTOL aircraft and a local stabilization control law for a 2-DOF Pendubot. Simulation results demonstrate the effectiveness of our control algorithm.

Fig. 4 Asymptotical stabilization of 2-DOF Pendubot.

In the future research, the work of this paper will be extended in following three aspects. First, explore how to extend it to mechanical systems with more under-actuation degrees. Second, apply the CL method to trajectory tracking and path following problems for underactuated mechanical systems. Third, deal with under-actuated mechanical systems with model uncertainty and external disturbance using the CL method.

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Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Data availability

All data generated or analysed during this study are included in this published article.

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Figures

Figure 1

VTOL aircraft.

Figure 2

Asymptotical stabilization of VTOL aircraft.

Figure 3

2-DOF Pendubot.

Figure 4

Asymptotical stabilization of 2-DOF Pendubot.