Algebraic Properties of Intuitionistic L-fuzzy Multiset Finite Automata

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Algebraic Properties of Intuitionistic L -fuzzy Multiset Finite Automata

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Abstract Algebraic properties and structures of intuitionistic L -fuzzy multiset finite automata (ILFMA) are discussed through congruences on a semigroup in this paper. Firstly, we put forward the notion of the intuitionistic L -fuzzy compatible relation, the compatible monoid associated to the intuitionistic L -fuzzy compatible relation can be effectively constructed, and we construct two finite monoids through two different congruence relations on a given ILFMA, then we also prove that they are isomorphic. Furthermore, using the quotient structure of ILFMA, algebraic properties of intuitionistic L -fuzzy multiset transformation semigroup are discussed. According to intuitionistic L -admissible relation and homomorphism of ILFMA, we show that there is a bijective correspondence between an ILFMA and the quotient structure of another ILFMA.

Keywords Multiset · Intuitionistic L -fuzzy multiset finite automata · Congruence relation · Quotient structure · Monoid · Homomorphism

1 Introduction

Fuzzy automata theory has been an active research area. Wee [27] proposed the concept of fuzzy state machines and pointed out that they can be used in system verification. Santos [19] dealt with various fuzzy automata. Mordeson and Malik [17] studied finite fuzzy automata and regular fuzzy languages. Malik et al.[16] investigated some algebraic structures of fuzzy automata. Atanassov [1] further proposed the definition of intuitionistic fuzzy set, and Jun in [9] generalized the fuzzy automata to intuitionistic fuzzy automata. Jun also presented standard algebraic concepts related to intuitionistic fuzzy finite state machine including congruences relations and homomorphisms in [10]. Zhang

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and Li [29] studied some properties of intuitionistic fuzzy recognizers. In [20], the authors defined intuitionistic general fuzzy automaton, and showed that how to construct the minimal intuitionistic fuzzy automaton with the same language. It is well-known that algebraic concepts such as congruences, transformation semigroup and homomorphisms of automata are essential for the investigation of the properties and structures of intuitionistic fuzzy finite automata, efficiency of their applications, and optimization algorithms. For more information about congruence relations, homomorphisms and transformation semigroup of fuzzy finite automata we refer to ([11, 22, 28, 23, 7, 15, 8]).

In the last decade, Multiset theory has been receiving much attention from many researchers. Multiset has recently been widely applied in many fields, including mathematics and computer science, particularly in biochemical and molecular computing (cf., [3, 18]). For example, Csuhajvarjú et al. [4] proposed multiset automata and multiset grammars and in [6], the authors studied some properties of mealy multiset automata. Afterwards, Kudlek [13] investigated multiset pushdown automata and the languages they can recognize. More recently, Wang et al. [26] employed fuzzy information to multiset finite automata and multiset grammars. Tiwari and Sharma et al. discussed minimization problem of fuzzy multiset automata in [21, 24] and the algebraic properties of fuzzy multiset finite automata, respectively. In fact, as Ciobanu and Gontineac [5] puts it, “multiset automata become very powerful.” In the present paper, we will utilize intuitionistic fuzzy sets to establish a fundamental framework of multiset finite automata theory, and discuss algebraic properties and structures of intuitionistic fuzzy multiset finite automata through congruences on a semigroup.

The remainder of this paper is as follows. Section 2 introduces the notions of multiset, intuitionistic L -fuzzy set and intuitionistic L -fuzzy multiset finite automata (LFMFA). Following the ideas of the intuitionistic fuzzy information, Section 3 discusses intuitionistic L -fuzzy compatible relation and the compatible monoid. Using the quotient structure of ILFMA, algebraic properties of intuitionistic L -fuzzy multiset transformation semigroup are discussed. According to intuitionistic L -admissible relation and homomorphism of ILFMA, we establish an isomorphism between an LFMFA and the quotient of another LFMFA.

2 Intuitionistic L -fuzzy multiset finite automata

This section reviews some basic notions of intuitionistic L -fuzzy set and multiset finite automata already defined.

Definition 1 [2] Let L be a lattice, a *negation operation* is a mapping $f : L \rightarrow L$ such that

- (1) $f(0) = 1, f(1) = 0$,
- (2) f is strict decreasing,
- (3) $f(f(l)) = l$ for all $l \in L$.

Definition 2 [2] Let L be a complete lattice, and f be an involutive order reversing function defined on L . An *intuitionistic L -fuzzy set* (ILFS) A in E is:

$$A = \{\langle e, \mu_A(e), \nu_A(e) \rangle | e \in E\},$$

for simplicity we write it as $A = (\mu_A, \nu_A)$, $\mu_A(e)$ and $\nu_A(e)$ are the values in L and they represent membership degrees and nonmembership degrees satisfying $\mu(e) \leq f(\nu(e))$ for every element $e \in E$.

Definition 3 [3, 4] Let X be a finite alphabet. A finite *multiset* is a mapping θ from X to the natural numbers $\mathbb{N}$, The norm of θ on X is given by $\|\theta\| = \sum_{b \in X} \theta(b)$.

The set of all finite nonempty multiset on X can be expressed by $X^\oplus$, the identity of $X^\oplus$ is labeled by $0_X \in X^\oplus$, that is, $0_X(b) = 0$ for any $b \in X$. And for every $b \in X$, the notation $\langle b \rangle$ is used for a singleton multiset, and is given by

$$\langle b \rangle(b') = \begin{cases} 1, & \text{if } b = b', \\ 0, & \text{if } b \neq b' \end{cases}$$

for any $b' \in X$. For example, let $X = \{b_1, b_2, b_3\}$, the multiset $\theta = \{1/b_1, 0/b_2, 2/b_3\}$ can be rewritten as the following simple form $\theta = \langle b_1 \rangle \oplus \langle b_3 \rangle \oplus \langle b_3 \rangle$.

Now we define several operations on $X^\oplus$. For any $\theta, \sigma \in X^\oplus$, and $b \in X$, we have

(**Inclusion**) $\theta \subseteq \sigma$ if and only if $\theta(b) \leq \sigma(b)$.

(**Addition**) $(\theta \oplus \sigma)(b) = \theta(b) + \sigma(b)$.

(**Difference**) $(\theta \ominus \sigma)(b) = \max\{0, \theta(b) - \sigma(b)\}$.

Clearly, $(X^\oplus, \oplus)$ is a commutative monoid.

Definition 4 Let L be a complete lattice. An *intuitionistic L -fuzzy multiset finite automata* (or ILFMFA, for short) $\mathcal{M}$ on L is defined as: $\mathcal{M} = (Q, X, A)$, where Q is a set of states, X is a input alphabet, $A = (\mu_A, \nu_A)$ is an ILFS with the above mappings, i.e., $\mu_A, \nu_A : Q \times X^\oplus \times Q \rightarrow L$, and μ_A, ν_A are called intuitionistic L -fuzzy transition relation of states.

Since the operation $\oplus$ for multisets is commutative, intuitively, the instruction $\mu_A(p_1, \theta \oplus \sigma, p_2)$ ($\nu_A(p_1, \theta \oplus \sigma, p_2)$) stands for that the current state p_1 , with inputting the multiset θ or σ being scanned, can turn to the state p_2 , that is, the choice of intuitionistic L -fuzzy transition relation between applicable instructions (p_1, θ, p_2) and (p_1, σ, p_2) , possibly being scanned different parts of the inputting multiset is non-deterministic.

We next show the operation of the intuitionistic L -fuzzy transition relation by using the notion of a *configuration*. A configuration of an LFMFA is described by $(p, \theta) \in (Q, X^\oplus)$. The LFMFA is non-deterministic, so there may be several transition relations that are possible in a given configuration. Thus, for an LFMFA $\mathcal{M}$ and its two configurations (p_1, θ) and (p_2, σ) , we

define a move from (p_1, θ) to (p_2, σ) with degree of membership and non-membership $u, v \in L$, write as $(p_1, \theta) \xrightarrow{(u,v)} (p_2, \sigma)$, if there exists a multi-set $\omega \in X^\oplus$ with $\omega \subseteq \theta$ such that $\mu_A(p_1, \omega, p_2) = u$ and $\nu_A(p_1, \omega, p_2) = v$. We use $\rightarrow^*$ to denote reflexive and transitive closure of $\rightarrow$. Let $(p_1, \theta), (p_2, \sigma) \in (Q, X^\oplus)$, we have $(p_1, \theta) \xrightarrow{(l,k)}^* (p_2, \sigma)$ if there exists $n+1 (n \geq 1)$ configurations $(p_1, \theta), (q_1, \theta_1), \dots, (q_{n-1}, \theta_{n-1}), (p_2, \sigma)$, such that $(p_1, \theta) \xrightarrow{(l_1, k_1)} (q_1, \theta_1) \xrightarrow{(l_2, k_2)} \dots \xrightarrow{(l_{n-1}, k_{n-1})} (q_{n-1}, \theta_{n-1}) \xrightarrow{(l_n, k_n)} (p_2, \sigma)$, $l_i, k_i \in L$, then $l = l_1 \wedge l_2 \wedge \dots \wedge l_{n-1} \wedge l_n$ and $k = k_1 \vee k_2 \vee \dots \vee k_{n-1} \vee k_n$. Naturally the membership degrees and nonmembership degrees of a configuration (p_1, θ) deriving another configuration (p_2, σ) are expressed as follows:

$$\begin{aligned} \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, \sigma)) &= \bigvee_{s \in Q} \{ \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (s, \theta \ominus \omega)) \wedge \mu_{\mathcal{M}}((s, \theta \ominus \omega) \rightarrow^* (p_2, \sigma)) : \omega \in X^\oplus, \text{ and } \omega \subseteq \theta \}, \\ \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, \sigma)) &= \bigwedge_{s \in Q} \{ \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (s, \theta \ominus \omega)) \vee \nu_{\mathcal{M}}((s, \theta \ominus \omega) \rightarrow^* (p_2, \sigma)) : \omega \in X^\oplus, \text{ and } \omega \subseteq \theta \}, \end{aligned}$$

and

$$\begin{aligned} \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, \theta)) &= \begin{cases} 1, & \text{if } p_1 = p_2; \\ 0, & \text{if } p_1 \neq p_2; \end{cases} \\ \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, \theta)) &= \begin{cases} 0, & \text{if } p_1 = p_2; \\ 1, & \text{if } p_1 \neq p_2. \end{cases} \end{aligned}$$

Example 1 Given an LFMFA $\mathcal{M} = (Q, X, A)$, where $Q = \{p_0, p_1, p_2, p_3\}$, $X = \{a, b\}$, and the intuitionistic L -fuzzy transition relations are given as follows:

$$\begin{aligned} \mu_A(p_0, \langle a \rangle \oplus \langle a \rangle, p_1) &= 0.4, & \mu_A(p_0, \langle a \rangle, p_2) &= 0.5, \\ \mu_A(p_0, \langle b \rangle \oplus \langle b \rangle, p_3) &= 0.8, & \mu_A(p_0, \langle b \rangle \oplus \langle b \rangle \oplus \langle b \rangle, p_2) &= 0.5, \\ \mu_A(p_1, \langle a \rangle, p_0) &= 0.9, & \mu_A(p_1, \langle a \rangle \oplus \langle a \rangle \oplus \langle b \rangle, p_3) &= 0.7, \\ \mu_A(p_1, \langle b \rangle \oplus \langle b \rangle, p_2) &= 0.6, & \mu_A(p_2, \langle a \rangle, p_3) &= 0.3, \\ \mu_A(p_2, \langle b \rangle, p_1) &= 0.6, \end{aligned}$$

and

$$\begin{aligned} \nu_A(p_0, \langle a \rangle \oplus \langle a \rangle, p_1) &= 0.5, & \nu_A(p_0, \langle a \rangle, p_2) &= 0.4, \\ \nu_A(p_0, \langle b \rangle \oplus \langle b \rangle, p_3) &= 0.1, & \nu_A(p_0, \langle b \rangle \oplus \langle b \rangle \oplus \langle b \rangle, p_2) &= 0.4, \\ \nu_A(p_1, \langle a \rangle, p_0) &= 0.1, & \nu_A(p_1, \langle a \rangle \oplus \langle a \rangle \oplus \langle b \rangle, p_3) &= 0.2, \\ \nu_A(p_1, \langle b \rangle \oplus \langle b \rangle, p_2) &= 0.3, & \nu_A(p_2, \langle a \rangle, p_3) &= 0.6, \\ \nu_A(p_2, \langle b \rangle, p_1) &= 0.3. \end{aligned}$$

If $\theta = \langle a \rangle \oplus \langle a \rangle \oplus \langle a \rangle \oplus \langle b \rangle \oplus \langle b \rangle$, $\sigma = 0_X$, then $\mu_{\mathcal{M}}((p_0, \theta) \rightarrow^* (p_3, \sigma))$ are calculated as follows:

$$\begin{aligned} (1) & (p_0, \theta) \xrightarrow{(0.4, 0.5)} (p_1, \langle a \rangle \oplus \langle b \rangle \oplus \langle b \rangle) \xrightarrow{(0.6, 0.3)} (p_2, \langle a \rangle) \xrightarrow{(0.3, 0.6)} (p_3, 0_X); \\ (2) & (p_0, \theta) \xrightarrow{(0.4, 0.5)} (p_1, \langle a \rangle \oplus \langle b \rangle \oplus \langle b \rangle) \xrightarrow{(0.9, 0.1)} (p_0, \langle b \rangle \oplus \langle b \rangle) \xrightarrow{(0.8, 0.1)} (p_3, 0_X); \\ (3) & (p_0, \theta) \xrightarrow{(0.4, 0.5)} (p_2, \langle a \rangle \oplus \langle a \rangle \oplus \langle b \rangle \oplus \langle b \rangle) \xrightarrow{(0.6, 0.3)} (p_1, \langle a \rangle \oplus \langle a \rangle \oplus \langle b \rangle) \xrightarrow{(0.7, 0.2)} (p_3, 0_X). \end{aligned}$$

Thus

$$\begin{aligned} \mu_{\mathcal{M}}((p_0, \theta) \rightarrow^* (p_3, \sigma)) &= \vee \{0.4 \wedge 0.6 \wedge 0.3, 0.4 \wedge 0.9 \wedge 0.8, 0.5 \wedge 0.6 \wedge 0.7\} = 0.5. \\ \nu_{\mathcal{M}}((p_0, \theta) \rightarrow^* (p_3, \sigma)) &= \wedge \{0.5 \vee 0.3 \vee 0.6, 0.5 \vee 0.1 \vee 0.1, 0.4 \vee 0.3 \vee 0.2\} = 0.4. \end{aligned}$$

3 Quotient structures

3.1 Monoid of L -intuitionistic fuzzy multiset finite automata

We will always use L to denote a complete lattice in the following discussion. Suppose that $\mathcal{M} = (Q, X, A)$ is an LFMFA, for any $\theta, \sigma \in X^\oplus$, we defined a relation $\simeq$ on $X^\oplus$, that is,

$$\theta \simeq \sigma \Leftrightarrow \begin{cases} \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) = \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X)), \\ \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) = \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X)) \end{cases}$$

for some $p_1, p_2 \in Q$. Clearly, $\simeq$ is a congruence relation on $X^\oplus$. We call $\simeq$ intuitionistic L -fuzzy compatible relation. The quotient set $X^\oplus / \simeq$ with respect to the relation $\simeq$ is the equivalence class $X^\oplus / \simeq = \{[\theta] | \theta \in X^\oplus\}$ with $[\theta] = \{\sigma \in X^\oplus | \theta \simeq \sigma\}$. The finiteness of $E(\mathcal{M})$ is closely related with Q and the relation of $\simeq$. Now define a binary operation $\odot$ on $E(\mathcal{M})$ by $[\theta] \odot [\sigma] = [\theta \oplus \sigma]$ for all $[\theta], [\sigma] \in E(\mathcal{M})$. Clearly, $\odot$ is well defined. Then we get the following result.

Theorem 1 *Let $\mathcal{M} = (Q, X, A)$ be an ILFMA. Then $(E(\mathcal{M}), \odot)$ is finite monoid (called compatible monoid here).*

Example 2 Let $\mathcal{M} = (Q, X, A)$ be an ILFMA, where $Q = \{p\}$, $X = \{a, b\}$. Define an ILFS $A = \{\mu_{\mathcal{M}}, \nu_{\mathcal{M}}\}$ on $Q \times X^\oplus \times Q$ by

$$\begin{aligned} \mu_{\mathcal{M}}((p, \langle a \rangle) \rightarrow^* (p, 0_X)) &= 0.7, \nu_{\mathcal{M}}((p, \langle a \rangle) \rightarrow^* (p, 0_X)) = 0.2, \\ \mu_{\mathcal{M}}((p, \langle b \rangle) \rightarrow^* (p, 0_X)) &= 0.3, \nu_{\mathcal{M}}((p, \langle b \rangle) \rightarrow^* (p, 0_X)) = 0.6. \end{aligned}$$

Then for every $\theta \in X^\oplus$, we know that $\mu_{\mathcal{M}}((p, \theta) \rightarrow^* (p, 0_X)) = 0.3$, $\nu_{\mathcal{M}}((p, \theta) \rightarrow^* (p, 0_X)) = 0.6$ if and only if θ contains at least a singleton multiset $\langle b \rangle$. Hence, $E(\mathcal{M}) = \{[0_X], [\theta], [\sigma]\}$, where $[0_X]$ is the identity element of $E(\mathcal{M})$, $[\theta]$ contains at least a singleton multiset $\langle b \rangle$, and $[\sigma]$ does not contains the singleton multiset $\langle b \rangle$ for all $\theta, \sigma \in X^\oplus$.

Motivated by the relation $\simeq$, we shall define another congruence relation on $X^\oplus$. For any $\theta, \sigma \in X^\oplus$, $\theta \equiv \sigma$ is defined as follow, for any $l, k \in L$ with $l < f(k)$,

$$\begin{aligned} \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) &> l \Leftrightarrow \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X)) > l, \\ \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) &< k \Leftrightarrow \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X)) < k \end{aligned}$$

for some $p_1, p_2 \in Q$. It is easy to verify that $\equiv$ is an equivalence on $X^\oplus$. Furthermore, let $\theta, \sigma \in X^\oplus$, if $\theta \equiv \sigma$, then for any $\omega \in X^\oplus$, and $p_1, p_2 \in Q$, we get

$$\begin{aligned} &\mu_{\mathcal{M}}((p_1, \theta \oplus \omega) \rightarrow^* (p_2, 0_X)) > l \\ \Leftrightarrow &\bigvee_{q' \in Q} \{\mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (q', \omega)) \wedge \mu_{\mathcal{M}}((q', \omega) \rightarrow^* (p_2, 0_X))\} > l \\ \Leftrightarrow &\text{for some } q' \in Q, \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (q', \omega)) \wedge \mu_{\mathcal{M}}((q', \omega) \rightarrow^* (p_2, 0_X)) > l \\ \Leftrightarrow &\text{for some } q' \in Q, \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (q', \omega)) \wedge \mu_{\mathcal{M}}((q', \omega) \rightarrow^* (p_2, 0_X)) > l \\ \Leftrightarrow &\mu_{\mathcal{M}}((p_1, \sigma \oplus \omega) \rightarrow^* (p_2, 0_X)) > l, \end{aligned}$$

and

$$\begin{aligned}
& \nu_{\mathcal{M}}((p_1, \theta \oplus \omega) \rightarrow^* (p_2, 0_X)) < k \\
& \iff \bigwedge_{q' \in Q} \{ \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (q', \omega)) \vee \nu_{\mathcal{M}}((q', \omega) \rightarrow^* (p_2, 0_X)) \} < k \\
& \iff \text{for some } q' \in Q, \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (q', \omega)) \vee \nu_{\mathcal{M}}((q', \omega) \rightarrow^* (p_2, 0_X)) < k \\
& \iff \text{for some } q' \in Q, \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (q', \omega)) \vee \nu_{\mathcal{M}}((q', \omega) \rightarrow^* (p_2, 0_X)) < k \\
& \iff \nu_{\mathcal{M}}((p_1, \sigma \oplus \omega) \rightarrow^* (q, 0_X)) < k.
\end{aligned}$$

Thus $\theta \oplus \omega \equiv \sigma \oplus \omega$. $\omega \oplus \theta \equiv \omega \oplus \sigma$ can be proved similarly, that is, $\equiv$ is a congruence relation. Let $\tilde{\theta} = \{\sigma \in X^\oplus \mid \theta \equiv \sigma\}$, the quotient set $\{\tilde{\theta} \mid \theta \in X^\oplus\}$ associated to the relation $\equiv$ is denoted by $\tilde{E}(\mathcal{M})$.

Theorem 2 Let $\mathcal{M} = (Q, X, A)$ be an LFMFA. Define an operation $\tilde{\odot}$ on $\tilde{E}(\mathcal{M})$ as $\tilde{\theta} \tilde{\odot} \tilde{\sigma} = \widetilde{\theta \oplus \sigma}$ for any $\tilde{\theta}, \tilde{\sigma} \in \tilde{E}(\mathcal{M})$. Then $(\tilde{E}(\mathcal{M}), \tilde{\odot})$ is a finite monoid, and the mapping from $E(\mathcal{M})$ to $\tilde{E}(\mathcal{M})$ given by $[\theta] \mapsto \tilde{\theta}$ is a onto homomorphism.

Proof It is straightforward to verify that $(\tilde{E}(\mathcal{M}), \tilde{\odot})$ is a monoid. Define a mapping $\rho : E(\mathcal{M}) \rightarrow \tilde{E}(\mathcal{M})$ as $\rho([\theta]) = \tilde{\theta}$ for any $[\theta] \in E(\mathcal{M})$. Let $\theta, \sigma \in X^\oplus$ such that $[\theta] = [\sigma]$, then $\mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) = \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X))$ and $\nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) = \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X))$ for some $p_1, p_2 \in Q$. Let $l, k \in L$ with $l < f(k)$, we have

$$\begin{aligned}
(\forall p, q \in Q) \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) > l & \iff \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X)) > l, \\
(\forall p, q \in Q) \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) < k & \iff \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (p_2, 0_X)) < k,
\end{aligned}$$

then we get $\theta \equiv \sigma$, that is, $\tilde{\theta} = \tilde{\sigma}$. So ρ is well defined, and ρ is onto. Furthermore, for any $[\theta], [\sigma] \in E(\mathcal{M})$, then

$$\rho([\theta] \odot [\sigma]) = \rho([\theta \oplus \sigma]) = \widetilde{\theta \oplus \sigma} = \tilde{\theta} \tilde{\odot} \tilde{\sigma} = \rho([\theta]) \tilde{\odot} \rho([\sigma]).$$

We know that $E(\mathcal{M})$ is finite by Theorem 1, so $\tilde{E}(\mathcal{M})$ is also finite.

Example 3 Consider the LFMFA $\mathcal{M} = (Q, X, A)$ in Example 2. Suppose that $k = 0.1$ and $l = 0.7$. Then we get $\theta \equiv \sigma$ for any multisets $\theta, \sigma \in X^\oplus$, and $\tilde{E}(\mathcal{M}) = \{\tilde{\theta}\}$.

Assume $\mathcal{M} = (Q, X, A)$ is an LFMFA. Now we define an ILFS $\theta^{\mathcal{M}} = (\mu_{\theta^{\mathcal{M}}}, \nu_{\theta^{\mathcal{M}}})$ from $Q \times Q$ to L by, for any $p_1, p_2 \in Q, \theta \in X^\oplus$,

$$\begin{aligned}
\mu_{\theta^{\mathcal{M}}}(p_1, p_2) &= \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)), \\
\nu_{\theta^{\mathcal{M}}}(p_1, p_2) &= \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)).
\end{aligned}$$

Theorem 3 Let $\mathcal{M} = (Q, X, A)$ be an LFMFA, $S_{\mathcal{M}} = \{\theta^{\mathcal{M}} \mid \theta \in X^\oplus\}$, and $\circ$ is a binary operation $S_{\mathcal{M}}$. Then we have

- (1) $\theta^{\mathcal{M}} \circ \sigma^{\mathcal{M}} = (\theta \oplus \sigma)^{\mathcal{M}}$ for any $\theta, \sigma \in X^\oplus$;
- (2) $(S_{\mathcal{M}}, \circ)$ is a finite monoid, and $(S_{\mathcal{M}}, \circ)$ is isomorphic to $(\tilde{E}(\mathcal{M}), \tilde{\odot})$.

Proof (1) For any $p_1, p_2 \in Q$ and $\theta, \sigma \in X^\oplus$, we have

$$\begin{aligned}\mu_{(\theta \oplus \sigma)^\mathcal{M}}(p_1, p_2) &= \bigvee_{s \in Q} \mu_{\mathcal{M}}((p_1, \theta \oplus \sigma) \rightarrow^* (s, \sigma)) \wedge \mu_{\mathcal{M}}((s, \sigma) \rightarrow^* (p_2, 0_X)) \\ &= \bigvee_{s \in Q} (\mu_{(\theta)^\mathcal{M}}(p_1, s) \wedge \mu_{(\sigma)^\mathcal{M}}(s, p_2)) \\ &= \mu_{\theta^\mathcal{M} \circ \sigma^\mathcal{M}}(p_1, p_2),\end{aligned}$$

and

$$\begin{aligned}\nu_{(\theta \oplus \sigma)^\mathcal{M}}(p_1, p_2) &= \bigwedge_{s \in Q} \nu_{\mathcal{M}}((p_1, \theta \oplus \sigma) \rightarrow^* (s, \sigma)) \vee \nu_{\mathcal{M}}((s, \sigma) \rightarrow^* (p_2, 0_X)) \\ &= \bigwedge_{s \in Q} (\nu_{(\theta)^\mathcal{M}}(p_1, s) \vee \nu_{(\sigma)^\mathcal{M}}(s, p_2)) \\ &= \nu_{\theta^\mathcal{M} \circ \sigma^\mathcal{M}}(p_1, p_2).\end{aligned}$$

Hence, $\theta^\mathcal{M} \circ \sigma^\mathcal{M} = (\theta \oplus \sigma)^\mathcal{M}$.

(2) It is clear that $(S_\mathcal{M}, \circ)$ is a monoid, and $S_\mathcal{M}$ is finite because Q is finite. Let $\rho : S_\mathcal{M} \rightarrow E(\mathcal{M})$ be a mapping defined by $\rho(\theta^\mathcal{M}) = [\theta]$ for any $\theta^\mathcal{M} \in S_\mathcal{M}$. It is straightforward to show ρ is well defined, and ρ is one-to-one and onto. For any $\theta^\mathcal{M}, \sigma^\mathcal{M} \in S_\mathcal{M}$, we have

$$\rho(\theta^\mathcal{M} \circ \sigma^\mathcal{M}) = \rho((\theta \oplus \sigma)^\mathcal{M}) = [\theta \oplus \sigma] = [\theta] \odot [\sigma] = \rho(\theta^\mathcal{M}) \odot \rho(\sigma^\mathcal{M}).$$

Hence, ρ is a homomorphism.

Let $\approx$ be a congruence relation on $X^\oplus$ with finite index, and let $\overline{Q} = \{[[\theta]] : \theta \in X^\oplus\}$, where $[[\theta]] = \{\sigma \in X^\oplus \mid \sigma \approx \theta\}$. Starting from the congruence relation $\approx$, we can construct an LFMFA $\overline{\mathcal{M}} = (\overline{Q}, X, \overline{A})$ over a given an LFMFA $\mathcal{M} = (Q, X, A)$, the configurations of $\overline{\mathcal{M}}$ are defined as follows:

$$\mu_{\overline{\mathcal{M}}}([[\theta]], \sigma) \rightarrow ([w], 0_X) = \begin{cases} \mu_{\mathcal{M}}([[\theta]], \sigma) \rightarrow ([w], 0_X), & \text{if } w \approx \theta \oplus \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{\overline{\mathcal{M}}}([[\theta]], \sigma) \rightarrow ([w], 0_X) = \begin{cases} \nu_{\mathcal{M}}([[\theta]], \sigma) \rightarrow ([w], 0_X), & \text{if } w \approx \theta \oplus \sigma, \\ 1, & \text{otherwise} \end{cases}$$

for all $[[\theta]], [[w]] \in \overline{Q}$ and $\sigma \in X$. Obviously, $\mu_{\overline{\mathcal{M}}}$ and $\nu_{\overline{\mathcal{M}}}$ are single valued.

Lemma 1 *If the LFMFA $\overline{\mathcal{M}} = (\overline{Q}, X, \overline{A})$ is effectively constructed as above, then for any $l, k \in L$ with $l < N(k)$, $k > 0$, we have*

(1) $\mu_{\overline{\mathcal{M}}}([[\theta]], \sigma) \rightarrow^* ([w], 0_X) > l$, $\nu_{\overline{\mathcal{M}}}([[\theta]], \sigma) \rightarrow^* ([w], 0_X) < k$ for all $\theta, \sigma \in X^\oplus$.

(2) $\mu_{\overline{\mathcal{M}}}([[\theta]], \sigma) \rightarrow^* ([\gamma], 0_X) > l$, $\nu_{\overline{\mathcal{M}}}([[\theta]], \sigma) \rightarrow^* ([\gamma], 0_X) < k$ imply that $[[\theta \oplus \sigma]] = [[\gamma]]$ for all $\theta, \sigma, \gamma \in X^\oplus$.

Proof (1) For any $\theta, \sigma \in X^\oplus$, suppose that $|\sigma| = n$. If $n = 0$, then $\sigma = 0_X$, we have

$$\mu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow ([\theta \oplus \sigma], 0_X) = \mu_{\overline{\mathcal{M}}}([\theta], 0_X) \rightarrow ([\theta], 0_X) = 1 > l,$$

$$\nu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow ([\theta \oplus \sigma], 0_X) = \nu_{\overline{\mathcal{M}}}([\theta], 0_X) \rightarrow ([\theta], 0_X) = 0 < k.$$

Hence this result is true for $n = 0$. Suppose the result is true for all $\gamma \in X^\oplus$ with $|\gamma| = n - 1$, $n > 0$. Let $\sigma = \gamma \oplus b$, $b \in X$, then we have

$$\begin{aligned} & \mu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([\theta \oplus \sigma], 0_X) \\ &= \mu_{\overline{\mathcal{M}}}([\theta], \gamma \oplus b) \rightarrow^* ([\theta \oplus \gamma \oplus b], 0_X) \\ &= \bigvee_{[[q']] \in \overline{Q}} \mu_{\overline{\mathcal{M}}}([\theta], \gamma) \rightarrow^* ([q'], b) \wedge \mu_{\overline{\mathcal{M}}}([q'], b) \rightarrow ([\theta \oplus \gamma \oplus b], 0_X) \\ &\geq \mu_{\overline{\mathcal{M}}}([\theta], \gamma) \rightarrow^* ([\theta \oplus \gamma], 0_X) \wedge \mu_{\overline{\mathcal{M}}}([\theta \oplus \gamma], b) \rightarrow ([\theta \oplus \gamma \oplus b], 0_X) > l; \end{aligned}$$

and

$$\begin{aligned} & \nu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([\theta \oplus \sigma], 0_X) \\ &= \nu_{\overline{\mathcal{M}}}([\theta], \gamma \oplus b) \rightarrow^* ([\theta \oplus \gamma \oplus b], 0_X) \\ &= \bigwedge_{[[q']] \in \overline{Q}} \nu_{\overline{\mathcal{M}}}([\theta], \gamma) \rightarrow^* ([q'], b) \vee \nu_{\overline{\mathcal{M}}}([q'], b) \rightarrow ([\theta \oplus \gamma \oplus b], 0_X) \\ &\leq \nu_{\overline{\mathcal{M}}}([\theta], \gamma) \rightarrow^* ([\theta \oplus \gamma], 0_X) \vee \nu_{\overline{\mathcal{M}}}([\theta \oplus \gamma], b) \rightarrow ([\theta \oplus \gamma \oplus b], 0_X) < k. \end{aligned}$$

Thus (1) is valid.

(2) Let $\theta, \sigma, \gamma \in X^\oplus$, and $|\sigma| = n$ such that

$$\mu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([\gamma], 0_X) > l, \nu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([\gamma], 0_X) < k,$$

then we have $[\theta \oplus \sigma] = [\gamma]$. If $n = 0$, that is $\sigma = 0_X$. Hence

$$\mu_{\overline{\mathcal{M}}}([\theta], 0_X) \rightarrow^* ([\gamma], \sigma) = \mu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([\gamma], 0_X) > l;$$

$$\nu_{\overline{\mathcal{M}}}([\theta], 0_X) \rightarrow^* ([\gamma], \sigma) = \nu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([w], 0_X) < k.$$

Thus $[\theta] = [\gamma]$, and $[\theta \oplus \sigma] = [\gamma]$. Suppose the result is true for all $w \in X^\oplus$ with $|w| = n - 1$, $n > 0$. Let $\sigma = w \oplus b$, $b \in X$, and let

$$\mu_{\overline{\mathcal{M}}}([\theta], w \oplus b) \rightarrow^* ([\gamma], 0_X) = \mu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([\gamma], 0_X) > l;$$

$$\nu_{\overline{\mathcal{M}}}([\theta], w \oplus b) \rightarrow^* ([\gamma], 0_X) = \nu_{\overline{\mathcal{M}}}([\theta], \sigma) \rightarrow^* ([\gamma], 0_X) < k.$$

Then

$$\begin{aligned} & \mu_{\overline{\mathcal{M}}}([\theta], w \oplus b) \rightarrow^* ([\gamma], 0_X) \\ &= \bigvee_{[[\varpi]] \in \overline{Q}} \mu_{\overline{\mathcal{M}}}([\theta], w) \rightarrow^* ([\varpi], b) \wedge \mu_{\overline{\mathcal{M}}}([\varpi], b) \rightarrow ([\gamma], 0_X) > l; \end{aligned}$$

and

$$\begin{aligned} & \nu_{\overline{\mathcal{M}}}([\theta], w \oplus b) \rightarrow^* ([\gamma], 0_X) \\ &= \bigwedge_{[[\varpi]] \in \overline{Q}} \nu_{\overline{\mathcal{M}}}([\theta], w) \rightarrow^* ([\varpi], b) \vee \nu_{\overline{\mathcal{M}}}([\varpi], b) \rightarrow ([w], 0_X) < k, \end{aligned}$$

which imply that $\mu_{\overline{\mathcal{M}}}([\theta], w) \rightarrow^* ([\varpi], b) > l$, $\mu_{\overline{\mathcal{M}}}([\varpi], b) \rightarrow ([\gamma], 0_X) > l$, and $\nu_{\overline{\mathcal{M}}}([\theta], w) \rightarrow^* ([\varpi], b) < k$, $\nu_{\overline{\mathcal{M}}}([\varpi], b) \rightarrow ([w], 0_X) < k$ for

some $[[\varpi]] \in \overline{Q}$. Since $[[\theta \oplus w]] = [[\varpi]]$ and $[[\varpi \oplus b]] = [[\gamma]]$, then $[[\theta \oplus \sigma]] = [[\theta \oplus w \oplus b]] = [[\varpi \oplus b]] = [[\gamma]]$. This completes the proof.

For any $\theta, \sigma \in X^\oplus$, $\theta \approx \sigma$ holds, let $[[\theta']], [[\sigma']] \in \overline{Q}$ such that $\mu_{\overline{\mathcal{M}}}([[\theta']], \theta) \rightarrow^* ([[\sigma']], 0_X) > l$ and $\nu_{\overline{\mathcal{M}}}([[\theta']], \theta) \rightarrow^* ([[\sigma']], 0_X) < k$. By Lemma 1, it follows that $[[\theta' \oplus \sigma]] = [[\theta' \oplus \theta]] = [[\sigma']]$, we have $\mu_{\overline{\mathcal{M}}}([[\theta']], \sigma) \rightarrow^* ([[\sigma']], 0_X) > l$ and $\nu_{\overline{\mathcal{M}}}([[\theta']], \sigma) \rightarrow^* ([[\sigma']], 0_X) < k$. Also, if

$\mu_{\overline{\mathcal{M}}}([[\theta']], \sigma) \rightarrow^* ([[\sigma']], 0_X) > l$ and $\nu_{\overline{\mathcal{M}}}([[\theta']], \sigma) \rightarrow^* ([[\sigma']], 0_X) < k$, then

$$\mu_{\overline{\mathcal{M}}}([[\theta']], \theta) \rightarrow^* ([[\sigma']], 0_X) > l \text{ and } \nu_{\overline{\mathcal{M}}}([[\theta']], \theta) \rightarrow^* ([[\sigma']], 0_X) < k.$$

Hence $\theta \equiv \sigma$. Conversely, for any $\theta \equiv \sigma$, and any $[[\theta']] \in \overline{Q}$, let

$$\mu_{\overline{\mathcal{M}}}([[\theta']], \theta) \rightarrow^* ([[\theta' \oplus \theta]], 0_X) > l, \nu_{\overline{\mathcal{M}}}([[\theta']], \theta) \rightarrow^* ([[\theta' \oplus \theta]], 0_X) < k.$$

Then we obtain

$$\mu_{\overline{\mathcal{M}}}([[\theta']], \sigma) \rightarrow^* ([[\theta' \oplus \theta]], 0_X) > l, \nu_{\overline{\mathcal{M}}}([[\theta']], \sigma) \rightarrow^* ([[\theta' \oplus \theta]], 0_X) < k.$$

Thus $[[\theta' \oplus \sigma]] = [[\theta' \oplus \theta]]$. Choose $\theta' = 0_X$, we get $\theta \approx \sigma$. Then $\theta \equiv \sigma$ if and only if $\theta \approx \sigma$.

Summing up, we get the following result.

Theorem 4 An LFMFA $\mathcal{M}$ can be effectively constructed by using a given congruence relation $\approx$ on $X^\oplus$ with finite index, and the relation $\approx$ is the same as congruence relation $\equiv$.

3.2 Intuitionistic L -fuzzy multiset transformation semigroup

Definition 5 Let S be a finite semigroup. An intuitionistic L -fuzzy multiset semigroup (ILFMS) $\mathcal{T}$ is defined as: $\mathcal{T} = (Q, S, \mathcal{A})$, where the configurations of $\mathcal{T}$ is shown below:

- (1) Let $p_1, p_2 \in Q$, and $\iota, j \in S$, satisfying the following conditions:
$$\mu_{\mathcal{T}}((p_1, \iota \oplus j) \rightarrow^* (p_2, 0_S)) = \bigvee_{s \in Q} \{ \mu_{\mathcal{T}}((p_1, \iota) \rightarrow^* (s, j)) \wedge \mu_{\mathcal{T}}((s, j) \rightarrow^* (p_2, 0_S)) \},$$

$$\nu_{\mathcal{T}}((p_1, \iota \oplus j) \rightarrow^* (p_2, 0_S)) = \bigwedge_{s \in Q} \{ \nu_{\mathcal{T}}((p_1, \iota) \rightarrow^* (s, j)) \vee \nu_{\mathcal{T}}((s, j) \rightarrow^* (p_2, 0_S)) \}.$$

- (2) If the semigroup S contains an identity 0_S , then we have

$$\mu_{\mathcal{T}}((p_1, 0_S) \rightarrow (p_2, 0_S)) = \begin{cases} 1, & \text{if } p_1 = p_2, \\ 0, & \text{if } p_1 \neq p_2, \end{cases}$$

$$\nu_{\mathcal{T}}((p_1, 0_S) \rightarrow (p_2, 0_S)) = \begin{cases} 0, & \text{if } p_1 = p_2, \\ 1, & \text{if } p_1 \neq p_2. \end{cases}$$

$\mathcal{T} = (Q, S, \mathcal{A})$ is called *faithful* if it fulfils the following:

- (3) If $\mu_{\mathcal{T}}((p_1, \iota) \rightarrow^* (p_2, 0_S)) = \mu_{\mathcal{T}}((p_1, j) \rightarrow^* (p_2, 0_S))$ and $\nu_{\mathcal{T}}((p_1, \iota) \rightarrow^* (p_2, 0_S)) = \nu_{\mathcal{T}}((p_1, j) \rightarrow^* (p_2, 0_S))$, then $\iota = j$.

Let $\mathcal{T} = (Q, S, \mathcal{A})$ be an ILFMS. This ILFMS may not faithful. Now let us construct a faithful ILFMS starting from $\mathcal{T}$ as follows.

For any $i, j \in S$, we defined a relation $\sim$ on S , that is,

$$i \sim j \Leftrightarrow \begin{cases} \mu_{\mathcal{T}}((p_1, i) \rightarrow^* (p_2, 0_S)) = \mu_{\mathcal{T}}((p_1, j) \rightarrow^* (p_2, 0_S)), \\ \nu_{\mathcal{T}}((p_1, i) \rightarrow^* (p_2, 0_S)) = \nu_{\mathcal{T}}((p_1, j) \rightarrow^* (p_2, 0_S)) \end{cases}$$

for some $p_1, p_2 \in Q$. Clearly, $\sim$ is a congruence relation on S .

We denote the class containing i by $\prec i \succ$, let $S/\sim = \{\prec i \succ \mid i \in S\}$. Similarly, a binary operation $\diamond$ is defined on $S/\sim$ by $\prec i \succ \diamond \prec j \succ = \prec i \oplus j \succ$. Then $(S/\sim, \diamond)$ is a monoid with identity $0_{S/\sim}$. Starting from the congruence relation $\sim$ we construct an *ILFMS* $\tilde{\mathcal{T}} = (Q, S/\sim, \tilde{\mathcal{A}})$, the configurations of $\tilde{\mathcal{T}}$ are defined by

$$\mu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ) \rightarrow^* (p_2, 0_{S/\sim})) = \mu_{\mathcal{T}}((p_1, i) \rightarrow^* (p_2, 0_S)),$$

$$\nu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ) \rightarrow^* (p_2, 0_{S/\sim})) = \nu_{\mathcal{T}}((p_1, i) \rightarrow^* (p_2, 0_S))$$

for every $\prec i \succ \in S/\sim$ and $p_1, p_2 \in Q$. Furthermore, for any $\prec i \succ, \prec j \succ \in S/\sim$, then

$$\begin{aligned} & \mu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ \diamond \prec j \succ) \rightarrow^* (p_2, 0_{S/\sim})) \\ &= \mu_{\tilde{\mathcal{T}}}((p_1, \prec i \oplus j \succ) \rightarrow^* (p_2, 0_{S/\sim})) \\ &= \mu_{\mathcal{T}}((p_1, i \oplus j) \rightarrow^* (p_2, 0_S)) \\ &= \bigvee_{s \in Q} \{\mu_{\mathcal{T}}((p_1, i) \rightarrow^* (s, j)) \wedge \mu_{\mathcal{T}}((s, j) \rightarrow^* (p_2, 0_S))\} \\ &= \bigvee_{s \in Q} \{\mu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ) \rightarrow^* (s, \prec j \succ)) \wedge \mu_{\tilde{\mathcal{T}}}((s, \prec j \succ) \rightarrow^* (p_2, 0_{S/\sim}))\}, \end{aligned}$$

and

$$\begin{aligned} & \nu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ \diamond \prec j \succ) \rightarrow^* (p_2, 0_{S/\sim})) \\ &= \nu_{\tilde{\mathcal{T}}}((p_1, \prec i \oplus j \succ) \rightarrow^* (p_2, 0_{S/\sim})) \\ &= \nu_{\mathcal{T}}((p_1, i \oplus j) \rightarrow^* (p_2, 0_S)) \\ &= \bigwedge_{s \in Q} \{\mu_{\mathcal{T}}((p_1, i) \rightarrow^* (s, j)) \vee \nu_{\mathcal{T}}((s, j) \rightarrow^* (p_2, 0_S))\} \\ &= \bigwedge_{s \in Q} \{\nu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ) \rightarrow^* (s, \prec j \succ)) \vee \nu_{\tilde{\mathcal{T}}}((s, \prec j \succ) \rightarrow^* (p_2, 0_{S/\sim}))\}. \end{aligned}$$

Also

$$\mu_{\tilde{\mathcal{T}}}((p_1, 0_{S/\sim}) \rightarrow (p_2, 0_{S/\sim})) = \begin{cases} 1, & p_1 = p_2, \\ 0, & p_1 \neq p_2 \end{cases}$$

and

$$\nu_{\tilde{\mathcal{T}}}((p_1, 0_{S/\sim}) \rightarrow (p_2, 0_{S/\sim})) = \begin{cases} 0, & p_1 = p_2, \\ 1, & p_1 \neq p_2, \end{cases}$$

where $p_1, p_2 \in Q$. And if $\mu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ) \rightarrow^* (p_2, 0_{S/\sim})) = \mu_{\tilde{\mathcal{T}}}((p_1, \prec j \succ) \rightarrow^* (p_2, 0_{S/\sim}))$ and $\nu_{\tilde{\mathcal{T}}}((p_1, \prec i \succ) \rightarrow^* (p_2, 0_{S/\sim})) = \nu_{\tilde{\mathcal{T}}}((p_1, \prec j \succ) \rightarrow^* (p_2, 0_{S/\sim}))$, then $\mu_{\mathcal{T}}((p_1, i) \rightarrow^* (p_2, 0_S)) = \mu_{\mathcal{T}}((p_1, j) \rightarrow^* (p_2, 0_S))$, $\nu_{\mathcal{T}}((p_1, i) \rightarrow^* (p_2, 0_S)) = \nu_{\mathcal{T}}((p_1, j) \rightarrow^* (p_2, 0_S))$. Hence, $i \sim j$ and so $\prec i \succ = \prec j \succ$. Thus $\tilde{\mathcal{T}}$ is a faithful *ILFMS*.

Theorem 5 Let $\mathcal{M} = (Q, X, A)$ be an ILFMA. $E(\mathcal{M})$ is the monoid as constructed above. Then $\tilde{\mathcal{T}} = (Q, E(\mathcal{M}), \tilde{A})$ is a faithful ILFMS, where the configurations of $\tilde{\mathcal{T}}$ are defined by

$$\begin{aligned}\mu_{\tilde{\mathcal{T}}}((p_1, [\theta]) \rightarrow^* (p_2, [0_X])) &= \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X)), \\ \nu_{\tilde{\mathcal{T}}}((p_1, [\theta]) \rightarrow^* (p_2, [0_X])) &= \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p_2, 0_X))\end{aligned}$$

for all $p_1, p_2 \in Q$ and $\theta \in X^\oplus$.

Proof For any $p_1, p_2 \in Q$, and $[\theta], [\sigma] \in E(\mathcal{M})$, by Theorem 1, then we have

$$\begin{aligned}\mu_{\tilde{\mathcal{T}}}((p_1, [\theta] \odot [\sigma]) \rightarrow^* (p_2, [0_X])) \\ &= \mu_{\tilde{\mathcal{T}}}((p_1, [\theta \oplus \sigma]) \rightarrow^* (p_2, [0_X])) \\ &= \mu_{\mathcal{M}}((p_1, \theta \oplus \sigma) \rightarrow^* (p_2, 0_X)) \\ &= \bigvee_{s \in Q} \{ \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (s, \sigma)) \wedge \mu_{\mathcal{M}}((s, \sigma) \rightarrow^* (p_2, 0_X)) \} \\ &= \bigvee_{s \in Q} \{ \mu_{\tilde{\mathcal{T}}}((p_1, [\theta]) \rightarrow^* (s, [\sigma])) \wedge \mu_{\tilde{\mathcal{T}}}((s, [\sigma]) \rightarrow^* (p_2, [0_X])) \},\end{aligned}$$

and

$$\begin{aligned}\nu_{\tilde{\mathcal{T}}}((p_1, [\theta] \odot [\sigma]) \rightarrow^* (p_2, [0_X])) \\ &= \nu_{\tilde{\mathcal{T}}}((p_1, [\theta \oplus \sigma]) \rightarrow^* (p_2, [0_X])) \\ &= \nu_{\mathcal{M}}((p_1, \theta \oplus \sigma) \rightarrow^* (p_2, 0_X)) \\ &= \bigwedge_{s \in Q} \{ \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (s, \sigma)) \vee \nu_{\mathcal{M}}((s, \sigma) \rightarrow^* (p_2, 0_X)) \} \\ &= \bigwedge_{s \in Q} \{ \nu_{\tilde{\mathcal{T}}}((p_1, [\theta]) \rightarrow^* (s, [\sigma])) \vee \nu_{\tilde{\mathcal{T}}}((s, [\sigma]) \rightarrow^* (p_2, [0_X])) \}.\end{aligned}$$

Also

$$\mu_{\tilde{\mathcal{T}}}((p_1, [0_X]) \rightarrow^* (p_2, [0_X])) = \mu_{\mathcal{M}}((p_1, 0_X) \rightarrow^* (p_2, 0_X)) = \begin{cases} 1, & \text{if } p_1 = p_2, \\ 0, & \text{if } p_1 \neq p_2, \end{cases}$$

and

$$\nu_{\tilde{\mathcal{T}}}((p_1, [0_X]) \rightarrow^* (p_2, [0_X])) = \nu_{\mathcal{M}}((p_1, 0_X) \rightarrow^* (p_2, 0_X)) = \begin{cases} 0, & \text{if } p_1 = p_2, \\ 1, & \text{if } p_1 \neq p_2. \end{cases}$$

Now, suppose that $\mu_{\tilde{\mathcal{T}}}((p_1, [\theta]) \rightarrow^* (p_2, [0_X])) = \mu_{\tilde{\mathcal{T}}}((p_1, [\sigma]) \rightarrow^* (p_2, [0_X]))$ and $\nu_{\tilde{\mathcal{T}}}((p_1, [\theta]) \rightarrow^* (p_2, [0_X])) = \nu_{\tilde{\mathcal{T}}}((p_1, [\sigma]) \rightarrow^* (p_2, [0_X]))$ for any $p_1, p_2 \in Q$, then $\mu_{\mathcal{T}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) = \mu_{\mathcal{T}}((p_1, \sigma) \rightarrow^* (p_2, 0_X))$ and $\nu_{\mathcal{T}}((p_1, \theta) \rightarrow^* (p_2, 0_X)) = \nu_{\mathcal{T}}((p_1, \sigma) \rightarrow^* (p_2, 0_X))$. Then $\theta \simeq \sigma$ and $[\theta] = [\sigma]$. Therefore, $\tilde{\mathcal{T}}$ is a faithful ILFMS.

3.3 Intuitionistic L -admissible relation

Definition 6 Let $\mathcal{M} = (Q, X, A)$ be an LFMFA, and $\mathfrak{R}$ be an equivalence relation on Q . Suppose $l, k \in L$ with $l < N(k)$. We say $\mathfrak{R}$ is an intuitionistic

L -admissible relation (ILA-relaiton) if there exist $p_1, p_2, p' \in Q$ with $p_1 \mathfrak{R} p_2$ such that

$$(1) \mu_{\mathcal{M}}((p_1, b) \rightarrow (p', 0_X)) > l \Rightarrow (\exists q' \in Q)(q' \mathfrak{R} p') \mu_{\mathcal{M}}((p_2, b) \rightarrow (q', 0_X)) \geq \mu_{\mathcal{M}}((p_1, b) \rightarrow (q', 0_X));$$

$$(2) \nu_{\mathcal{M}}((p_1, b) \rightarrow (p', 0_X)) < k \Rightarrow (\exists q' \in Q)(q' \mathfrak{R} p') \nu_{\mathcal{M}}((p_2, b) \rightarrow (q', 0_X)) \leq \nu_{\mathcal{M}}((p_1, b) \rightarrow (q', 0_X))$$

hold for any $b \in X$.

Theorem 6 Let $\mathcal{M} = (Q, X, A)$ be an LFMFA, and $\mathfrak{R}$ be an equivalence relation on Q . Suppose $l, k \in L$ with $l < N(k)$. Then $\mathfrak{R}$ is an ILA-relaiton iff there exist $p_1, p_2, p' \in Q$ with $p_1 \mathfrak{R} p_2$ such that

$$(1) \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) > l \Rightarrow (\exists q' \in Q)(q' \mathfrak{R} p') \mu_{\mathcal{M}}((p_2, \theta) \rightarrow^* (q', 0_X)) \geq \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (q', 0_X));$$

$$(2) \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) < k \Rightarrow (\exists q' \in Q)(q' \mathfrak{R} p') \nu_{\mathcal{M}}((p_2, \theta) \rightarrow^* (q', 0_X)) \leq \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (q', 0_X))$$

hold for any $\theta \in X^{\oplus}$.

Proof (1) Let $\mathfrak{R}$ be an ILA-relaiton, and let $p_1, p_2, p' \in Q$ with $p_1 \mathfrak{R} p_2$, $\theta \in X^{\oplus}$ $\mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) > l$. If $n = 0$, then $p' = p_1$. Hence the result holds for $q' = p_2$. Assume that it also holds for any $\sigma \in X^{\oplus}$ with $|\sigma| = n - 1$. Now suppose that $\theta = \sigma \oplus b$, where $b \in X$. Thus

$$\begin{aligned} & \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) \\ &= \mu_{\mathcal{M}}((p_1, \sigma \oplus b) \rightarrow^* (p', 0_X)) \\ &= \bigvee_{s \in Q} \{ \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (s, b)) \wedge \mu_{\mathcal{M}}((s, b) \rightarrow (p', 0_X)) \} > l \end{aligned}$$

Then there exists $s' \in Q$ satisfying the following conditions:

$$\begin{aligned} & \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (s', b)) \wedge \mu_{\mathcal{M}}((s', b) \rightarrow^* (p', 0_X)) \\ &= \bigvee_{s \in Q} \{ \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (s, b)) \wedge \mu_{\mathcal{M}}((s, b) \rightarrow (p', 0_X)) \}. \end{aligned}$$

Hence, $\mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (s, b)) > l, \mu_{\mathcal{M}}((s, b) \rightarrow (p', 0_X)) > l$. Also, there exists q'_s such that $\mu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'_s, b)) \geq \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (s, b))$. Since $\mu_{\mathcal{M}}((s, b) \rightarrow (p', 0_X)) > l$, and $q'_s \mathfrak{R} s$, there exists $q' \in Q$ such that $\mu_{\mathcal{M}}((q'_s, b) \rightarrow^* (q', 0_X)) \geq \mu_{\mathcal{M}}((s, b) \rightarrow^* (p', 0_X))$, and $q' \mathfrak{R} p'$. Thus,

$$\begin{aligned} & \mu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'_s, b)) \wedge \mu_{\mathcal{M}}((q'_s, b) \rightarrow^* (q', 0_X)) \geq \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (s, b)) \wedge \\ & \mu_{\mathcal{M}}((s, b) \rightarrow (p', 0_X)) \end{aligned}$$

for some $q' \in Q$, and so

$$\begin{aligned} \mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) &= \mu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (s', b)) \wedge \mu_{\mathcal{M}}((s', b) \rightarrow (p', 0_X)) \\ &\leq \mu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'_s, b)) \wedge \mu_{\mathcal{M}}((q'_s, b) \rightarrow (q', 0_X)) \\ &\leq \bigvee_{q'' \in Q} \{ \mu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'', b)) \wedge \mu_{\mathcal{M}}((q'', b) \rightarrow (q', 0_X)) \} \\ &= \mu_{\mathcal{M}}((p_2, \sigma \oplus b) \rightarrow^* (q', 0_X)) \\ &= \mu_{\mathcal{M}}((p_2, \theta) \rightarrow^* (q', 0_X)). \end{aligned}$$

$$\mu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) > l.$$

(2) Let $\mathfrak{R}$ be an ILA-relation, and let $p_1, p_2, p' \in Q$ with $p_1 \mathfrak{R} p_2$, $\theta \in X^\oplus$, $\nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) < k$. If $n = 0$, then $p' = p_1$. Hence the result holds for $q' = p_2$. Assume that it also holds for all $\sigma \in X^\oplus$ with $|\sigma| = n - 1$. Let $\theta = \sigma \oplus b$, where $b \in X$. Thus

$$\begin{aligned} & \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) \\ &= \nu_{\mathcal{M}}((p_1, \sigma \oplus b) \rightarrow^* (p', 0_X)) \\ &= \bigwedge_{t \in Q} \{ \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (t, b)) \vee \nu_{\mathcal{M}}((t, b) \rightarrow (p', 0_X)) \} < k, \end{aligned}$$

Thus, there exists $t' \in Q$ satisfying the following conditions:

$$\begin{aligned} & \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (t', b)) \vee \nu_{\mathcal{M}}((t', b) \rightarrow (p', 0_X)) \\ &= \bigwedge_{t \in Q} \{ \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (t, b)) \vee \nu_{\mathcal{M}}((t, b) \rightarrow (p', 0_X)) \}. \end{aligned}$$

Therefore, $\{ \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (t, b)) < k, \nu_{\mathcal{M}}((t, b) \rightarrow (p', 0_X)) \} < k$. By induction assumption, there exists q'_t such that $\nu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'_t, b)) \leq \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (t, b))$. Since $\nu_{\mathcal{M}}((t, b) \rightarrow (p', 0_X)) < k$, and $q'_t \mathfrak{R} t$, there exists $q' \in Q$ such that $\nu_{\mathcal{M}}((q'_t, b) \rightarrow (p', 0_X)) \leq \nu_{\mathcal{M}}((t, b) \rightarrow (q', 0_X))$, and $q' \mathfrak{R} p'$. Thus,

$$\begin{aligned} & \nu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'_t, b)) \vee \nu_{\mathcal{M}}((q'_t, b) \rightarrow (q', 0_X)) \leq \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (t, b)) \vee \\ & \nu_{\mathcal{M}}((t, b) \rightarrow (p', 0_X)) \end{aligned}$$

for some $q' \in Q$, and so

$$\begin{aligned} \nu_{\mathcal{M}}((p_1, \theta) \rightarrow^* (p', 0_X)) &= \nu_{\mathcal{M}}((p_1, \sigma) \rightarrow^* (t', b)) \wedge \nu_{\mathcal{M}}((t', b) \rightarrow (p', 0_X)) \\ &\leq \nu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'_t, b)) \wedge \nu_{\mathcal{M}}((q'_t, b) \rightarrow (p', 0_X)) \\ &\leq \bigwedge_{q'' \in Q} \{ \nu_{\mathcal{M}}((p_2, \sigma) \rightarrow^* (q'', b)) \vee \nu_{\mathcal{M}}((q'', b) \rightarrow (q', 0_X)) \} \\ &= \nu_{\mathcal{M}}((p_2, \sigma \oplus b) \rightarrow^* (q', 0_X)) \\ &= \nu_{\mathcal{M}}((p_2, \theta) \rightarrow^* (q', 0_X)). \end{aligned}$$

The converse is trivial.

Definition 7 An *ILFMA-homomorphism* defined by (ϕ, φ) is a pair of mappings from an ILFMA $\mathcal{M}_1 = (Q_1, X_1, A_1)$ to another $\mathcal{M}_2 = (Q_2, X_2, A_2)$, where $\phi : Q_1 \rightarrow Q_2$, $\varphi : X_1 \rightarrow X_2$ satisfying the following conditions:

$$\mu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q_1, \sigma)) \leq \mu_{\mathcal{M}_2}((\phi(p_1), \varphi(\theta)) \rightarrow^* (\phi(q_1), \varphi(\sigma)))$$

and

$$\nu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q_1, \sigma)) \geq \nu_{\mathcal{M}_2}((\phi(p_1), \varphi(\theta)) \rightarrow^* (\phi(q_1), \varphi(\sigma))).$$

We say $(\phi, \varphi) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a *strong homomorphism* if

$$\begin{aligned} & \mu_{\mathcal{M}_2}((\phi(p_1), \varphi(\theta)) \rightarrow^* (\phi(q_1), 0_{X_1})) \\ &= \vee \{ \mu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q', 0_X)) \mid q' \in Q_1, \phi(q') = \phi(q_1) \}, \end{aligned}$$

and

$$\begin{aligned} & \nu_{\mathcal{M}_2}((\phi(p_1), \varphi(\theta)) \rightarrow^* (\phi(q_1), 0_{X_1})) \\ &= \wedge \{ \nu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q', 0_X)) | q' \in Q_1, \phi(q') = \phi(q_1) \} \end{aligned}$$

for any $p_1, q_1 \in Q$ and $\theta \in X_1^\oplus$.

By the above definition, we $\varphi(\theta_1 \oplus \theta_2) = \varphi(\theta_1) \oplus \varphi(\theta_2)$ for any $\theta_1, \theta_2 \in X_1^\oplus$. We say an ILFMA-homomorphism (strong homomorphism) $(\phi, \varphi) : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is an *ILFMA-isomorphism* (*strong isomorphism*) if ϕ and φ are bijective and onto.

Example 4 Let $\mathcal{M}_i = (Q_i, X, A_i)$ be an LFMFA, $i = 1, 2$, where $Q_1 = \{p_1, p_2, p_3\}$, $Q_2 = \{q_1, q_2, q_3\}$, $X_1 = X_2 = \{a, b\}$, the configurations of $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined by

$$\begin{array}{ll} \mu_{\mathcal{M}_1}((p_1, a) \rightarrow (p_1, 0_{X_1})) = 0.3 & \mu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_1})) = 0.3 \\ \mu_{\mathcal{M}_1}((p_1, b) \rightarrow (p_2, 0_{X_1})) = 0.6 & \mu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_1})) = 0.6 \\ \mu_{\mathcal{M}_1}((p_2, a) \rightarrow (p_1, 0_{X_1})) = 0.3 & \mu_{\mathcal{M}_2}((q_2, a) \rightarrow (q_1, 0_{X_1})) = 0.3 \\ \mu_{\mathcal{M}_1}((p_2, b) \rightarrow (p_3, 0_{X_1})) = 0.6 & \mu_{\mathcal{M}_2}((q_2, b) \rightarrow (q_1, 0_{X_1})) = 0.6 \\ \mu_{\mathcal{M}_1}((p_3, a) \rightarrow (p_3, 0_{X_1})) = 0.3 & \mu_{\mathcal{M}_2}((q_3, b) \rightarrow (q_1, 0_{X_1})) = 0.5 \\ \mu_{\mathcal{M}_1}((p_3, b) \rightarrow (p_2, 0_{X_1})) = 0.6 & \mu_{\mathcal{M}_2}((q_3, b) \rightarrow (q_2, 0_{X_1})) = 0.5, \end{array}$$

and

$$\begin{array}{ll} \nu_{\mathcal{M}_1}((p_1, a) \rightarrow (p_1, 0_{X_1})) = 0.6 & \nu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_1})) = 0.6 \\ \nu_{\mathcal{M}_1}((p_1, b) \rightarrow (p_2, 0_{X_1})) = 0.3 & \nu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_1})) = 0.3 \\ \nu_{\mathcal{M}_1}((p_2, a) \rightarrow (p_1, 0_{X_1})) = 0.6 & \nu_{\mathcal{M}_2}((q_2, a) \rightarrow (q_1, 0_{X_1})) = 0.6 \\ \nu_{\mathcal{M}_1}((p_2, b) \rightarrow (p_3, 0_{X_1})) = 0.3 & \nu_{\mathcal{M}_2}((q_2, b) \rightarrow (q_1, 0_{X_1})) = 0.3 \\ \nu_{\mathcal{M}_1}((p_3, a) \rightarrow (p_3, 0_{X_1})) = 0.6 & \nu_{\mathcal{M}_2}((q_3, b) \rightarrow (q_1, 0_{X_1})) = 0.4 \\ \nu_{\mathcal{M}_1}((p_3, b) \rightarrow (p_2, 0_{X_1})) = 0.3 & \nu_{\mathcal{M}_2}((q_3, b) \rightarrow (q_2, 0_{X_1})) = 0.4. \end{array}$$

For all other configurations are defined by $\mu_{\mathcal{M}_1}((p_i, x) \rightarrow (p_j, y)) = 0$, $\nu_{\mathcal{M}_1}((p_i, x) \rightarrow (p_j, y)) = 1$, and $\mu_{\mathcal{M}_2}((q_i, x) \rightarrow (q_j, y)) = 0$, $\nu_{\mathcal{M}_2}((q_i, x) \rightarrow (q_j, y)) = 1$, $i, j = 1, 2, 3$, and $x, y \in X_1^\oplus$. Now define $\phi : Q_1 \rightarrow Q_2$ and $\varphi : X_1 \rightarrow X_2$ as follows, $\phi(p_1) = \phi(p_3) = q_1$, $\phi(p_2) = q_2$, and $\varphi(a) = a$, $\varphi(b) = b$. Thus, we have the following equalities.

$$\left\{ \begin{array}{l} \mu_{\mathcal{M}_2}((\phi(p_1), \varphi(a)) \rightarrow (\phi(p_1), 0_{X_2})) = \mu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_2})) = 0.3 = \mu_{\mathcal{M}_1}((p_1, a) \rightarrow (p_1, 0_{X_1})) \\ \mu_{\mathcal{M}_2}((\phi(p_1), \varphi(b)) \rightarrow (\phi(p_2), 0_{X_2})) = \mu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_2})) = 0.6 = \mu_{\mathcal{M}_1}((p_1, b) \rightarrow (p_2, 0_{X_1})) \\ \mu_{\mathcal{M}_2}((\phi(p_2), \varphi(a)) \rightarrow (\phi(p_1), 0_{X_2})) = \mu_{\mathcal{M}_2}((q_2, a) \rightarrow (q_1, 0_{X_2})) = 0.3 = \mu_{\mathcal{M}_1}((p_2, a) \rightarrow (p_1, 0_{X_1})) \\ \mu_{\mathcal{M}_2}((\phi(p_2), \varphi(b)) \rightarrow (\phi(p_3), 0_{X_2})) = \mu_{\mathcal{M}_2}((q_2, b) \rightarrow (q_1, 0_{X_2})) = 0.6 = \mu_{\mathcal{M}_1}((p_2, b) \rightarrow (p_3, 0_{X_1})) \\ \mu_{\mathcal{M}_2}((\phi(p_3), \varphi(a)) \rightarrow (\phi(p_3), 0_{X_2})) = \mu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_2})) = 0.3 = \mu_{\mathcal{M}_1}((p_3, a) \rightarrow (p_3, 0_{X_1})) \\ \mu_{\mathcal{M}_2}((\phi(p_3), \varphi(b)) \rightarrow (\phi(p_2), 0_{X_2})) = \mu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_2})) = 0.6 = \mu_{\mathcal{M}_1}((p_3, b) \rightarrow (p_2, 0_{X_1})), \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \nu_{\mathcal{M}_2}((\phi(p_1), \varphi(a)) \rightarrow (\phi(p_1), 0_{X_2})) = \nu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_2})) = 0.6 = \mu_{\mathcal{M}_1}((p_1, a) \rightarrow (p_1, 0_{X_1})) \\ \nu_{\mathcal{M}_2}((\phi(p_1), \varphi(b)) \rightarrow (\phi(p_2), 0_{X_2})) = \nu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_2})) = 0.3 = \nu_{\mathcal{M}_1}((p_1, b) \rightarrow (p_2, 0_{X_1})) \\ \nu_{\mathcal{M}_2}((\phi(p_2), \varphi(a)) \rightarrow (\phi(p_1), 0_{X_2})) = \nu_{\mathcal{M}_2}((q_2, a) \rightarrow (q_1, 0_{X_2})) = 0.6 = \nu_{\mathcal{M}_1}((p_2, a) \rightarrow (p_1, 0_{X_1})) \\ \nu_{\mathcal{M}_2}((\phi(p_2), \varphi(b)) \rightarrow (\phi(p_3), 0_{X_2})) = \nu_{\mathcal{M}_2}((q_2, b) \rightarrow (q_1, 0_{X_2})) = 0.3 = \nu_{\mathcal{M}_1}((p_2, b) \rightarrow (p_3, 0_{X_1})) \\ \nu_{\mathcal{M}_2}((\phi(p_3), \varphi(a)) \rightarrow (\phi(p_3), 0_{X_2})) = \nu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_2})) = 0.6 = \nu_{\mathcal{M}_1}((p_3, a) \rightarrow (p_3, 0_{X_1})) \\ \nu_{\mathcal{M}_2}((\phi(p_3), \varphi(b)) \rightarrow (\phi(p_2), 0_{X_2})) = \nu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_2})) = 0.3 = \nu_{\mathcal{M}_1}((p_3, b) \rightarrow (p_2, 0_{X_1})). \end{array} \right.$$

Clearly, (ϕ, φ) is an ILFMA-strong homomorphism.

If $X_1 = X_2$ in Definition 7, that is, φ is the identity map. In that case we write it as $\phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$.

Given an LFMFA, $\mathcal{M} = (Q, X, A)$, suppose $\Re$ is an ILA-relaiton on Q . Let $\underline{Q} = \{[p] | p \in Q\}$. Now we construct an LFMFA $\underline{\mathcal{M}} = (\underline{Q}, X, \underline{A})$ by using the LFMFA $\mathcal{M}$, the configurations of $\underline{\mathcal{M}}$ are given by

$$\mu_{\underline{\mathcal{M}}}([p_1], b) \rightarrow ([p_1], 0_X) = \bigvee_{p_2 \in [p_1]} \mu_{\mathcal{M}}(p_1, b) \rightarrow (p_2, 0_X),$$

$$\nu_{\underline{\mathcal{M}}}([p_1], b) \rightarrow ([p_2], 0_X) = \bigwedge_{p_2 \in [p_2]} \nu_{\mathcal{M}}(p_1, b) \rightarrow (p_2, 0_X)$$

for any $p_1, p_2 \in Q$, $b \in X$. Also, we define a mapping $\underline{\phi} : Q \rightarrow \underline{Q}$ by $\underline{\phi}(p_1) = [p_1]$ for any $p_1 \in Q$. Clearly, $\underline{\phi}$ is a well-defined and onto mapping. Let $p_1, s \in Q$, $\theta \in X^\oplus$, we have $\mu_{\underline{\mathcal{M}}}((\phi(p_1), \theta) \rightarrow^* (\underline{\phi}(s), 0_X)) = \mu_{\underline{\mathcal{M}}}([p_1], \theta) \rightarrow^* ([s], 0_X) = \bigvee_{s \in [s]} \mu_{\mathcal{M}}(p_1, \theta) \rightarrow^* (s, 0_X) \geq \mu_{\mathcal{M}}(p_1, \theta) \rightarrow^* (s, 0_X)$, and $\nu_{\underline{\mathcal{M}}}((\phi(p_1), \theta) \rightarrow^* (\underline{\phi}(s), 0_X)) \leq \nu_{\mathcal{M}}(p_1, \theta) \rightarrow^* (s, 0_X)$. Hence $\underline{\phi}$ is a onto homomorphism from $\mathcal{M}$ to $\underline{\mathcal{M}}$.

Lemma 2 Let $\mathcal{M}_i = (Q_i, X, A_i)$ be an LFMFA, $i = 1, 2$. Suppose that ϕ is an LFMFA-strong homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$. The kernel of ϕ is the relation

$$\text{Ker}\phi = \{(p_1, p_2) | \phi(p_1) = \phi(p_2)\}.$$

Then $\text{Ker}\phi$ is an ILA-relaiton on Q_1 .

Proof Clearly, $\text{Ker}\phi$ is an equivalence relation. For any $p_1, p_2, p' \in Q_1$ satisfying $(p_1, p_2) \in \text{Ker}\phi$ and $\mu_{\mathcal{M}_1}((p_1, b) \rightarrow (p', 0_X)) > l$ for $b \in X$, $l \in L$. Then $\phi(p_1) = \phi(p_2)$, $\mu_{\mathcal{M}_2}((\phi(p_2), b) \rightarrow (\phi(p'), 0_X)) = \mu_{\mathcal{M}_2}((\phi(p_1), b) \rightarrow (\phi(p'), 0_X)) \geq \mu_{\mathcal{M}_1}((p_1, b) \rightarrow (p', 0_X))$. Also, we have

$$\mu_{\mathcal{M}_2}((\phi(p_2), b) \rightarrow (\phi(p'), 0_X)) = \bigvee \{\mu_{\mathcal{M}_1}((p_2, b) \rightarrow (q', 0_X)) | \phi(q') = \phi(p')\}$$

$$\nu_{\mathcal{M}_2}((\phi(p_2), b) \rightarrow (\phi(p'), 0_X)) = \bigwedge \{\nu_{\mathcal{M}_1}((p_2, b) \rightarrow (q', 0_X)) | \phi(q') = \phi(p')\}.$$

Thus there exists $q_1 \in Q_1$ such that $\phi(q_1) = \phi(q')$ and $\mu_{\mathcal{M}_1}((p_2, b) \rightarrow (q_1, 0_X)) \geq \mu_{\mathcal{M}_1}((p_2, b) \rightarrow (q', 0_X))$. Also, there exists $q_2 \in Q_1$ such that $\phi(q_2) = \phi(q')$ and $\nu_{\mathcal{M}_1}((p_2, b) \rightarrow (q_2, 0_X)) \leq \nu_{\mathcal{M}_1}((p_2, b) \rightarrow (q', 0_X))$. Hence, $\text{Ker}\phi$ is an ILA-relaiton on Q_1 .

Theorem 7 Let $\mathcal{M}_i = (Q_i, X, A_i)$ be an LFMFA, $i = 1, 2$. Suppose that ϕ is an LFMFA-strong homomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$. Then there exists an LFMFA isomorphism ρ from $\underline{\mathcal{M}}_1 = (Q_1/\text{Ker}\phi, X, \underline{A}_1)$ to $\mathcal{M}_2$ such that $\phi = \rho \circ \underline{\phi}$.

Proof Define $\rho : \underline{\mathcal{M}}_1 \rightarrow \mathcal{M}_2$ by $\rho([p_1]) = \phi(p_1)$ for any $[p_1] \in Q_1/\text{Ker}\phi$. Let $p_1, p_2 \in Q_1$ with $[p_1] = [p_2]$. Then $(p_1, p_2) \in \text{Ker}\phi$, also $\rho([p_1]) = \phi(p_1) = \phi(p_2) = \rho([p_2])$. That is, ρ is well-defined. Clearly, ρ is a both bijective and onto mapping. Then for any $\theta \in X^\oplus$, we have

$$\begin{aligned} \mu_{\underline{\mathcal{M}}_1}([p_1], \theta) \rightarrow^* ([p_2], 0_X) &= \bigvee_{q' \in [p_2]} \mu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q', 0_X)) \\ &= \bigvee_{q' \in Q_1} \{\mu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q', 0_X)) | \phi(q') = \phi(p_2)\} \end{aligned}$$

$$\begin{aligned}
&= \mu_{\mathcal{M}_2}((\phi(p_1), \theta) \rightarrow^* (\phi(p_2), 0_X)) \\
&= \mu_{\mathcal{M}_2}((\rho([p_1]), \theta) \rightarrow^* (\rho([p_2]), 0_X))
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\underline{\mathcal{M}_1}}((\phi(p_1), \theta) \rightarrow^* (\phi(p_2), 0_X)) &= \bigwedge_{q' \in [p_1]} \nu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q', 0_X)) \\
&= \bigwedge_{q' \in Q_1} \{\nu_{\mathcal{M}_1}((p_1, \theta) \rightarrow^* (q', 0_X)) \mid \phi(q') = \phi(p_2)\} \\
&= \nu_{\mathcal{M}_2}((\phi(p_1), \theta) \rightarrow^* (\phi(p_2), 0_X)) \\
&= \nu_{\mathcal{M}_2}((\rho([p_1]), \theta) \rightarrow^* (\rho([p_2]), 0_X)).
\end{aligned}$$

Hence ρ is a homomorphism. Thus, we have $\phi = \rho \circ \underline{\phi}$.

Example 5 Let $\mathcal{M}_1 = (Q_1, X_1, A_1)$ be the LFMFA of Example 4, and let $\mathcal{M}_2 = (Q_2, X_2, A_2)$ be an LFMFA, where $Q_2 = \{q_1, q_2\}$, $X_2 = \{a, b\}$, the configurations of $\mathcal{M}_2$ are expressed by

$$\begin{aligned}
\mu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_2})) &= 0.3 & \mu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_2})) &= 0.6 \\
\mu_{\mathcal{M}_2}((q_2, a) \rightarrow (q_1, 0_{X_2})) &= 0.3 & \mu_{\mathcal{M}_2}((q_2, b) \rightarrow (q_1, 0_{X_2})) &= 0.6
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\mathcal{M}_2}((q_1, a) \rightarrow (q_1, 0_{X_2})) &= 0.6 & \nu_{\mathcal{M}_2}((q_1, b) \rightarrow (q_2, 0_{X_2})) &= 0.3 \\
\nu_{\mathcal{M}_2}((q_2, a) \rightarrow (q_1, 0_{X_2})) &= 0.6 & \nu_{\mathcal{M}_2}((q_2, b) \rightarrow (q_1, 0_{X_2})) &= 0.3,
\end{aligned}$$

Now define $\phi : Q_1 \rightarrow Q_2$ and $\varphi : X_1 \rightarrow X_2$ as follows: $\phi(p_1) = \phi(p_3) = q_1$, $\phi(p_2) = q_2$, and $\varphi(a) = a$, $\varphi(b) = b$. So we know that (ϕ, φ) is an LFMFA-strong homomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$. We also note that $\text{Ker}\phi = \{(p_i, p_i), (p_1, p_3), (p_3, p_1) \mid i = 1, 2, 3\}$. Thus, $[p_1] = \{p_1, p_3\}$ and $[p_2] = \{p_2\}$. Then $Q_1/\text{Ker}\phi = \{[p_1], [p_2]\}$. Now we can construct an LFMFA $\underline{\mathcal{M}_1} = (Q_1/\text{Ker}\phi, X, A_1)$, the configurations of $\underline{\mathcal{M}_1}$ are defined as follows:

$$\begin{aligned}
\mu_{\underline{\mathcal{M}_1}}([p_1], a) \rightarrow ([p_1], 0_X) &= 0.3 & \mu_{\underline{\mathcal{M}_1}}([p_1], b) \rightarrow ([p_2], 0_X) &= 0.6 \\
\mu_{\underline{\mathcal{M}_1}}([p_2], a) \rightarrow ([p_1], 0_X) &= 0.3 & \mu_{\underline{\mathcal{M}_1}}([p_2], b) \rightarrow ([p_1], 0_X) &= 0.6
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\underline{\mathcal{M}_1}}([p_1], a) \rightarrow ([p_1], 0_X) &= 0.6 & \nu_{\underline{\mathcal{M}_1}}([p_1], b) \rightarrow ([p_2], 0_X) &= 0.3 \\
\nu_{\underline{\mathcal{M}_1}}([p_2], a) \rightarrow ([p_1], 0_X) &= 0.6 & \nu_{\underline{\mathcal{M}_1}}([p_2], b) \rightarrow ([p_1], 0_X) &= 0.3.
\end{aligned}$$

Then there exists an isomorphism ρ from $\underline{\mathcal{M}_1}$ to $\mathcal{M}_2$ such that $\phi = \rho \circ \underline{\phi}$.

4 Conclusion

Quotient algebras are a basic tool for exploring the structures of the algebraic theory of automata. There are close correlations among congruences, homomorphisms and quotient algebras. In this paper, algebraic properties and structures of intuitionistic L -fuzzy multiset finite automata were discussed through congruences on a semigroup. We also discussed the intuitionistic L -fuzzy compatible relation and the compatible monoid. Furthermore, we constructed two finite monoids through two different congruence relations on a given ILFMA,

then we proved that they are isomorphic. Finally, the quotient structure of LFMFA with respect to intuitionistic L -admissible relation was discussed, then we showed that there is a bijective correspondence between an LFMFA and the quotient structure of another LFMFA.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Authorship contributions

YbW and LxZ jointly drafted the manuscript. YbW conceived this paper and commented critically on several drafts of the manuscript. All authors approved the final test of the manuscript.

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