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Truncated-distance codes and ternary compounds

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Abstract Here, perfect truncated-distance codes (PTDC's) in the n-dimensional grid Λ_n of $\mathbb{Z}^n$ $(0 < n \in \mathbb{Z})$ and its quotient toroidal grids are obtained via the truncated distance $\rho(u,v)$ in $\mathbb{Z}^n$ given between vertices $u = (u_1, \cdots, u_n) \in \mathbb{Z}^n$ and $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ as the Hamming distance $d(u,v)$ in $\mathbb{Z}^n$ (or graph distance $d(u,v)$ in Λ_n) if $|u_i - v_i| \leq 1$, for all $i \in \{1, \ldots, n\}$, and as $n+1$, otherwise. While this ρ is related to the ℓ_p metrics, the construction of PTDC's associated with lattice tilings of $\mathbb{Z}^n$ is obtained as that of rainbow perfect dominating sets in previous work. In this work, that construction is extended to that of ternary compounds Γ_n obtained by glueing, or locking, ternary n-cubes along their codimension 1 ternary subcubes. We ascertain the existence of an isolated PTDC of radius 2 in Γ_n for $n = 2$ and conjecture that to hold for $n > 2$ with radius n. Finally, we ask whether there exists a suitable notion replacing that of quotient toroidal grids of Λ_n for the case of Γ_n.

Keywords truncated distance · truncated sphere · lattice tiling · binary cube · ternary cube

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1 Introduction

This work was motivated by computer architecture problems (cf [1]) associated to signal transmission in a finite field (cf. [10,11,14]). Related topics of coding theory and lattice domination are based on the Lee metric arising from the Minkowsky ℓ_p norm [4] with $p = 1$. These themes, related to the still unsolved Golomb-Welch Conjecture [13,15,16], were investigated via perfect Lee codes [10], diameter perfect Lee codes [14], tilings with generalized Lee spheres [2, 11], perfect dominating sets (PDS's) [20], perfect-distance dominating sets (PDDS's) [1] and efficient dominating sets [8].

Let $0 < n \in \mathbb{Z}$. The aforementioned codes or dominating sets are generally realized in lattice tilings of $\mathbb{Z}^n$ whose tiles are translates of a common generalized Lee sphere. A natural follow-up here would be to consider lattice tilings of $\mathbb{Z}^n$ with two different generalized Lee spheres. Focus on such follow-up for $n > 3$ is only possible by modifying the notion of the Lee distance d to that of a *truncated distance* ρ, defined in the next paragraph (not a standard distance, but see Remark 5) and subsequently applied in Section 2 that restates results of [12]). Furthermore, new graphs Γ_n built as compounds of ternary n-cubes, presented below in Remark 1 and Sections 3-4 that generalize the lattice graph of $\mathbb{Z}^n$, extend those applications.

Recall that the *Hamming distance* $h(u, v)$ between vectors $u = (u_1, \cdots , u_n)$ and $v = (v_1, \ldots , v_n)$ in $\mathbb{Z}^n \subset \mathbb{R}^n$ is the number of positions $i \in [n] = \{1, \ldots , n\}$ for which $u_i \neq v_i$. Now, let $\rho : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be given by:

$$\rho(u, v) = \begin{cases} h(u, v), & \text{if } |u_i - v_i| \leq 1 \text{ for all } i \in [n]; \\ n + 1, & \text{otherwise.} \end{cases}$$

Given $S \subseteq \mathbb{Z}^n$, we denote by $[S]$ the induced subgraph of S in the n-*dimensional grid* Λ_n, namely the graph whose vertex set is $\mathbb{Z}^n$ with exactly one edge between each two vertices at Euclidean distance 1. In the above definition of ρ, the Hamming distance h may be replaced alternatively by the graph distance in Λ_n and is also our choice definition to be used on the graphs Γ_n of Sections 3-4.

To each component H of $[S]$, we assign an integer $t_H > 0$ to be understood in the context below as the *radius* of a *truncated sphere* with *center* H. For every component H' of $[S]$ in the translation class $\langle H\rangle$ of H in $\mathbb{Z}^n$, we assume $t_{H'} = t_H$. This yields a correspondence κ from the set of translation classes $\langle H\rangle$ of components H of $[S]$ into $\mathbb{Z}$ such that $\kappa(\langle H\rangle) = t_H$, for every $H \in \langle H\rangle$.

Let $\rho(u, S) = \min\{\rho(u, s)|s \in S\}$. Let $(H)^{\kappa(H)} = \{u \in \mathbb{Z}^n|\rho(u, H) \leq \kappa(H)\}$. A *$\kappa$-perfect truncated-distance code*, or *κ-PTDC*, is a set $S \subseteq \mathbb{Z}^n$ such that $\forall u \in \mathbb{Z}^n$ there exists a unique $s \in S$ with $\rho(u, s) = \rho(u, S)$ and such that the collection

$$\{(H)^{\kappa(H)}|H \text{ is a connected component of } S\}$$

forms a partition of $\mathbb{Z}^n$. For each component H of S, $(H)^{\kappa(H)}$ is referred to as the *truncated sphere centered at* H (or *around* H) *with radius* $\kappa(\langle H\rangle)$. Here,

H is said to be the *truncated center of* $(H)^{\kappa(H)}$ and its vertices are said to be its *central vertices*. Given $0 < t \in \mathbb{Z}$, a κ-PTDC is said to be a *t-PTDC* if all $\kappa(\langle H\rangle)$'s are equal to t. If $|H| = 1$, then $|(H)^t| = \sum_{i=0}^{t} 2^i \binom{n}{i}$, for $0 \leq t \leq n$, (which is less than the cardinal of the corresponding ℓ_1 sphere of radius t, if $t > 1$ [4]).

Remark 1 A *binary* (resp. *ternary*) *n-cube graph* Q_n^i, succinctly called *n-cube* (resp. *n-tercube*), is a cartesian product $K_i \square \cdots \square K_i$ of n complete graphs K_i with $V(K_i) = \{0, \dots, i-1\} = F_i$, the field of order $i = 2$ (resp. 3). In Sections 3- 4, we extend the study of PTDC's to edge-disjoint unions $\Gamma_n \supset \Lambda_n$ of triangles; these are connected compounds of ternary n-cubes that we call *ternary (n-cube) compounds*, (denomination that make us think of Γ_n as *binary (n-cube) compounds*) and that cannot have *non-isolated PDS*'s or *non-isolated 1-PTDC*'s. (However, the ternary perfect single-error-correcting codes of length $\frac{3^t-1}{2}$ [18,2] are *isolated PDS*'s, or *efficient dominating sets* [8] in the ternary $\frac{3^t-1}{2}$-cubes; these also are edge-disjoint unions of triangles, for $t > 0$). We will say for $G = \Gamma_n$ in Sections 3-4 that: an edge-disjoint union G of triangles has a *non-isolated PDS S* if every vertex of G not in S is adjacent to just either one vertex of G (as in *PDS*'s) or the two end-vertices of an edge of G. Section 4 shows that Γ_2 has an infinite number of isolated 2-PTDC and conjectures that Γ_n has isolated n-PTDC's, $\forall n > 2$.

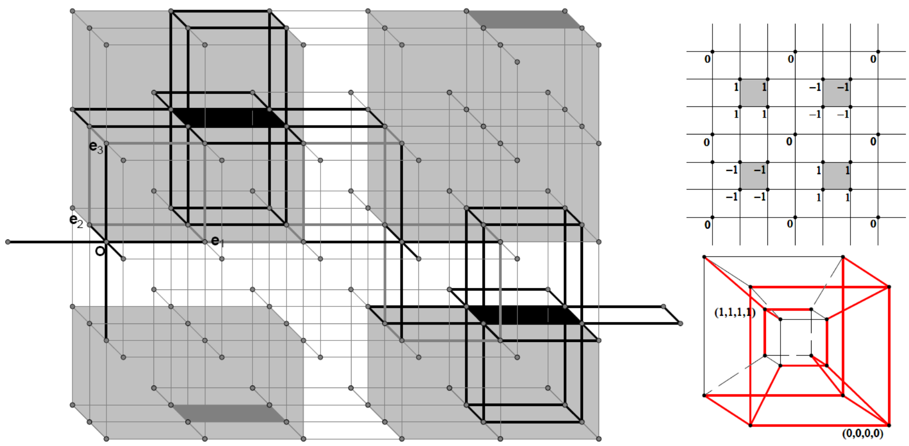

Fig. 1 Example accompanying Remark 2

Remark 2 We restate some results of [12] in Theorem 2, (resp. Theorems 3-4) below. They assert that PTDC's in Λ_n that are *lattice* as in [14] (or *lattice-like* [1]) exist having every induced component H with vertex set **(a)** whose convex hull is an n-parallelotope (resp. either a 0-cube or an $(n-1)$-cube) and **(b)** contained in a truncated sphere centered at H with radius n (resp. either

$n-2$ or 1) forming part of the associated lattice tiling of $\mathbb{Z}^n$. In the center-left of Figure 1, such 0-cubes and $(n-1)$-cubes are suggested in black and dark-gray colors for $n=3$ (Theorem 3), representing $\mathbb{Z}^n$ with dark-gray thin-edges and vertices, but dominating edges in thick black trace and other edges of a specific tile in thick dark-gray trace. This is schematized on the upper-right of the figure that represents the assignment of the last coordinate value mod 3 on the projection of this PTDC onto $\mathbb{Z}^{n-1}$. The case $n=4$ of Theorem 4 (the first PTDC which is neither PDS nor PDDS), can be visualized similarly. For example, the dominating edges, red colored in the copy of $Q_4^2 \subset \mathbb{Z}^4$ in the lower-right of Figure 1 indicate a truncated sphere of radius 2 (resp. 1) at $(0,0,0,0)$ (resp. $(1,1,1,1)$). Also in the center-left of Figure 1, light-gray 3-parallelotopes represent convex hull parts of truncated 3-spheres whose central vertices form a lattice PTDC, the simplest case of Theorem 2. A coloring-graph viewpoint of these constructions is found in [12].

Remark 3 **(I)** A t-PTDC (resp. a truncated t-sphere) of Λ_n is defined like a PDDS [1] (resp. generalized Lee sphere [11]) with ρ instead of the Lee distance d. **(II)** 1-PTDC's of Λ_n (resp. truncated 1-spheres) coincide with PDS's [20] (resp. generalized Lee 1-spheres). **(III)** An example [1], also in Remark 7, illustrated in Figure 2 below, justifies that our definition of PTDC above employs translation classes instead of isomorphism classes, for in Figure 2 the truncated spheres have common radius (namely 1) for both translation classes, but that does not exclude the eventuality of an example with differing radii.

Remark 4 Motivation for Theorems 3-4 on building "lattice" κ-PTDC's whose induced components are r-cubes ($0 \le r \le n$) of different dimensions r comes from: **(A)** the perfect covering codes with spheres of two different radii in Chapter 19 [5] and **(B)** a negative answer [19] to a conjecture [20] claiming that the induced components of every 1-PTDC S in an n-cube Q_n^2 are all r-cubes Q_r^2 (not necessarily in the same translation class [9]), where r is fixed. In fact, it was found in [19] that a 1-PTDC in Q_{13}^2 whose induced components are r-cubes Q_r^2 of two different values $r=r_1$ and $r=r_2$ exists, specifically for $r_1=4$ and $r_2=0$. This seems to be the only known counterexample to the cited conjecture.

1.1 Notation, ℓ_p metrics connections and periodicity

If no confusion arises, every $(a_1,\dots,a_n) \in \mathbb{Z}^n$ is expressed as $a_1 \cdots a_n$. Let $O = 00\cdots 0$, $e_1 = 10\cdots 0$, $e_2 = 010\cdots 0$, $\dots$, $e_{n-1} = 0\cdots 010$ and $e_n = 00\cdots 01$.

Let $z \in \mathbb{Z}^n$, let $H=(V,E)$ be a subgraph of Λ_n and let $H+z$ be the subgraph $H'=(V',E')$ whose vertex set is $V' = V+z = \{w \in \mathbb{Z}^n; \exists\, v \in V$ such that $w = v+z\}$ so that $uv \in E \Leftrightarrow (u+z)(v+z) \in E'$. For example, let H be induced in Λ_n by the vertices with entries in $\{0,1\}$. Then, every translation $H+z$ of H in Λ_n (in particular H itself) is isomorphic to Q_n^2.

Let $i \in [n]$. Each edge (resp. segment) uv of Λ_n (resp. $\mathbb{R}^n \supset \mathbb{Z}^n$) with $u \neq v$ is *parallel* to $Oe_i \in E(\Lambda_n) \Leftrightarrow u - v \in \{\pm e_i\}$ (resp. $\Leftrightarrow \exists a \in \mathbb{R} \setminus \{0\}$

with $u - v = ae_i$). A *parallelotope* $\mathcal{P}$ in $\mathbb{R}^n$ is a cartesian product of segments parallel to some of the Oe_i's ($i \in [n]$). We restrict to $\mathcal{P}$'s having their vertices in $\mathbb{Z}^n$. An *edge* of $\mathcal{P}$ is a segment of unit length parallel to some Oe_i ($i \in [n]$) separating a pair of vertices of Λ_n in $\mathcal{P}$. Recall that a *trivial* subgraph of Λ_n is composed by just one vertex. The proof of the following is similar to that of Theorem 1 [1].

Theorem 1 *Let $0 < t \in \mathbb{Z}$. If S is a t-PTDC in Λ_n, then the convex hull in $\mathbb{R}^n$ of the vertex set $V(H)$ of each nontrivial component H of $[S]$ is a parallelotope in $\mathbb{R}^n$ whose edges are parallel to some or all of the segments Oe_i in $E(\Lambda_n)$, where $i \in [n]$.*

Let H be a nontrivial induced subgraph of Λ_n with the convex hull of $V(H)$ in $\mathbb{R}^n$ as a parallelotope whose edges are parallel to some or all of the segments Oe_i in $E(\Lambda_n)$, ($i \in [n]$). If exactly r elements of $[n]$ are such values of i, then we say that H is an *r-box*. Note that H is a cartesian product $\Pi_{i=1}^n P^i$, where P^i is a finite path, $\forall i \in [n]$, with exactly r paths P^i having positive length. Clearly, the convex hull of an r-box is an r-parallelotope in $\mathbb{R}^n$. Also, every r-cube in Λ_n is an r-box, for $r \in \{0\} \cup [n]$.

Let $W_{n,H,t}$ be the truncated sphere of radius t around an H as above, where $0 \leq t \in \mathbb{Z}$. Then, H is the truncated center of $W_{n,H,t}$. A t-PTDC S of Λ_n determines a partition of $\mathbb{Z}^n$ into spheres $W_{n,H,t}$ with H running over the components of $[S]$. Such an S with the components of $[S]$ obtained by translations from a fixed finite graph H is said to be a *t-PTDC$[H]$*. (This imitates the definition of a t-PDDS$[H]$ in [1], mentioned in Remark 3(I) above). Let S be a t-PTDC$[H]$ and let H' be a component of $[S]$ obtained from H by means of a translation. Then S is said to be a *lattice t-PTDC$[H]$* if and only if there is a lattice $L \subseteq \mathbb{Z}^n$ such that: H'' is a component of $[S] \Leftrightarrow H'' = H' + z$, for some $z \in L$.

Remark 5 It is relevant to note the connections of ρ and the ℓ_p metrics [4]: A truncated sphere of truncated radius 1 centered at some vertex v is a Lee ($p = 1$) sphere of radius 1, whereas a truncated sphere of truncated radius n centered at v is a sphere of radius 1 in the maximum ($p = \infty$) distance, namely $W_{n,\{v\},1}$ (which has an n-dimensional cube for convex hull). Thus, all truncated spheres in this work are spheres in some ℓ_p metric, and the truncated distance is a convenient way to consider different ℓ_p metrics for the graph components. (When just one truncated radius t is considered, the truncated distance is properly a distance). This observation allows connections with other previous works such as (for just one t) perfect codes in the maximum distance and in the ℓ_p metric [4].

Remark 6 To finish this subsection, we rephrase graph-coloring results of [12].

Announced as Theorem 2 below, there is a construction of lattice n-PTDC's S whose induced subgraphs $[S]$ in Λ_n have their components H with:

(i) vertex sets $V(H)$ whose convex hulls are n-boxes in $\mathbb{R}^n$ with distance 3 to $[S] \setminus H$, and representatives (one per component H) forming a lattice with generators along the coordinate directions;

(ii) each $V(H)$ contained in a truncated sphere $(H)^{\kappa(H)}$ whose radius is n.

(We do not know whether similar lattice t-PTDC's S exist with $[S]$ having r-parallelotopes as their components, for r fixed such that $0 < r < n$.)

Extending the meaning of the adjective *lattice* in Subsection 1.2 below, as in [12] we have in Theorems 3-4 that a lattice κ-PTDC S exists in Λ_n whose induced subgraph $[S]$ has its components H with vertex sets $V(H)$:

(i′) having both $(n-1)$-cubes and 0-cubes as their convex hulls in $\mathbb{R}^n$;
(ii′) contained in truncated spheres $(H)^{\kappa(H)}$ whose radii, namely 1 and $(n-2)$, correspond respectively to the $(n-1)$-cubes and 0-cubes in item (i′).

A set $S \subset \mathbb{Z}^n = E(\Lambda_n)$ is *periodic* if and only if there exist $p_1, \ldots, p_n$ in $\mathbb{Z}$ such that $v \in S$ implies $v \pm p_i e_i \in S, \forall i \in [n]$. Since each lattice t-PTDC$[H]$ S is periodic [1], then every canonical projection from Λ_n onto a toroidal grid $\mathcal{T}$, i.e. a cartesian product $\mathcal{T} = C_{k_1 p_1} \square C_{k_2 p_2} \square \cdots \square C_{k_n p_n}$ of n cycles $C_{k_i p_i}$ $(0 < k_i \in \mathbb{Z}, \forall i \in [n])$, takes S onto a t-PTDC$[H]$ in $\mathcal{T}$. This observation adapts to respective situations that complement the statements of Theorems 2-4 via canonical projections from $\mathbb{Z}^n$ onto adequate toroidal grids $\mathcal{T}$.

Question 1 Do lattice t-PTDC's S exist with $[S]$ having r-parallelotopes as their components, for r fixed such that $0 < r < n$?

1.2 **Further specifications**

Given a lattice L in $\mathbb{Z}^n$, a subset $T \subseteq \mathbb{Z}^n$ that contains exactly one vertex in each class mod L (so that T is a complete system of coset representatives of L in Λ_n) is said to be an *FR* (acronym suggesting "fundamental region") of L. A partition of $\mathbb{Z}^n$ into FR's of L is said to be a *tiling* of $\mathbb{Z}^n$. Those FR's are said to be its *tiles*.

We extend the notion of lattice κ-PTDC so that the associated *fundamental region* (see [3], pg 26), or FR (see above) of every new lattice contains a finite number of (in our applications, just two) members H of each $\langle H \rangle$.

In the literature, existing constructions of lattice t-PTDC's in Λ_n $(t < n)$ concern just $t = 1$ (see [1,2,11,14]) but there are not many known lattice 1-PTDC's. For example, [7] shows that there is only one lattice 1-PTDC$[Q_2^2]$ and no non-lattice 1-PTDC$[Q_2^2]$. In addition, there is a lattice 2-PDDS$[Q_1^2]$ in Λ_3 arising from a tiling of Minkowsky cited in [1].

Conjecture 1 There is no t-PTDC lattice in Λ_n, for $1 < t < n$.

This conjecture has an analogous form for perfect codes in the ℓ_p metrics in [4], and together with the conjecture mentioned in Remark 4(B), produces a contrast with the constructions in Theorems 3-4, below.

If S is a periodic non-lattice t-PTDC$[H]$ in Λ_n, then there exists $0 < m \in \mathbb{Z}$ and a tiling of Λ_n with tiles that are disjoint copies of the vertex set $V(H^*)$ of a connected subgraph H^* induced in Λ_n by the union of:

(a) the vertex sets of m disjoint copies $H^1, \ldots, H^m$ of H (components of $[S]$);
(b) the sets $(H^j)^{\kappa(H)}$ of vertices $v \in \mathbb{Z}^n$ with $\rho(v, H^j) \leq t$, for $j \in [m]$, where $(H^1)^{\kappa(H)}, \ldots, (H^m)^{\kappa(H)}$ are pairwise disjoint copies of $(H)^{\kappa(H)}$ in $\mathbb{Z}^n$.

By taking such an m as small as possible, we say that S is a t-*PTDC*$[H; m]$.

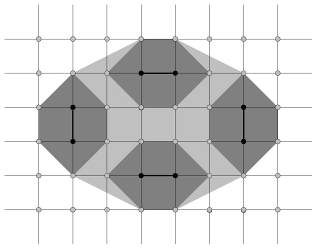

Fig. 2 FR of a lattice L_S for a 1-PTDC$[Q_1^2; 4]$ S in Λ_3.

Remark 7 In Section 5 [1], a non-lattice 1-PTDC$[Q_1^2; 4]$ S is shown to exist with a lattice L_S based on it, each of the FR's of L_S containing two copies of Q_2^2 parallel to Oe_1 and two more copies of Q_2^2 parallel to Oe_2, these four copies being components of $[S]$. This is represented in Figure 2, where the convex hulls of the truncated 1-spheres of such four components (in thick black trace) are shaded in dark-gray color and the remaining area completing their convex hull is shaded in light-gray color. A fixed vertex v_T can be taken in each resulting tile T so that all the resulting vertices v_T constitute L_S.

Thus, even for a non-lattice t-PTDC S in a Λ_n, a lattice can be recovered and formed by selected vertices v_T in the corresponding tiles T associated with S. We say that such set S is a *lattice* t-*PTDC*$[H; m]$, indicating the number m of isomorphic components of $[S]$ in a typical tile T in which to fix a distinguished vertex v_T.

We further specify the developments above as follows. A t-PTDC S in Λ_n with the components of $[S]$ obtained by translations from two non-parallel subgraphs H_0, H_1 of Λ_n is said to be a t-*PTDC*$[H_0, H_1]$. The case of this in Figure 2 clearly is a κ-PTDC, with two translation classes of components of $[S]$ formed by the copies of Q_2^2 parallel to each of the directions coordinates. Here, κ sends those copies onto $t = 1$. Each tile of this κ-PTDC contains two copies of Q_2^2 parallel to Oe_1 and two copies of Q_2^2 parallel to Oe_2, accounting for $m = 4$.

More generally, let $t_i \in [n]$, for $i = 0, 1$. We say that a set $S \subset V$ is a (t_0, t_1)-*PTDC*$[H_0, H_1]$ in Λ_n if for each $v \in V$ there is:

(i″) a unique index $i \in \{0, 1\}$ and a unique component H_v^i of $[S]$ obtained by means of a translation from H_i such that the truncated distance $\rho(v, H_v^i)$ from v to H_v^i satisfies $\rho(v, H_v^i) \leq t_i$ and
(ii″) a unique vertex w in H_v^i such that $\rho(v, w) = \rho(v, H_v^i)$.

Even though such a set S is not lattice as in [14] (or lattice-like [1]), it may happen that there exists a lattice $L_S \subset \mathbb{Z}^n$ such that for $0 < m_0, m_1 \in \mathbb{Z}$ there exists an FR of L_S in Λ_n given by the union of two disjoint subgraphs H_0^*, H_1^*, where H_i^* $(i = 0, 1)$ is induced in Λ_n by the disjoint union of:

(a′) the vertex sets of m_i disjoint copies $H_i^1, \dots, H_i^{m_i}$ of H_i, (components of $[S]$);

(b′) the sets $(H_i^j)^{\kappa(H)}$ of vertices $v \in \mathbb{Z}^n$ for which $0 < \rho(v, H_i^j) \leq t_i$, for $j \in [m_i]$, where $(H_i^1)^{\kappa(H)}, \dots, (H_i^{m_i})^{\kappa(H)}$ are pairwise disjoint copies of $H^\kappa(H))$ in $\mathbb{Z}^n$.

Section 2 contains a restatement of results of [12], translating its graph-coloring context into our present one. In fact, in Theorem 3 below, a lattice 1-PTDC$[H_0, H_1; m_0, m_1]$ is obtained; in addition, a family of lattice (t_0, t_1)-PTDC$[H_0, H_1; m_0, m_1]$'s in the graphs Λ_n that extend the lattice 1-PTDC of Theorem 3 is obtained in Theorem 4.

2 Restatement of known results

A particular case of lattice t-PTDC$[H]$ is that in which H is an n-box of Λ_n. For each such H, Theorem 2 below says that there is a lattice n-PTDC$[H]$ in Λ_n. (In [2], n-boxes of unit volume in Λ_n are shown to determine 1-PDDS$[H]$'s if and only if either $n = 2^r - 1$ or $n = 3^r - 1$).

How to construct a lattice n-PTDC$[H]$ S in Λ_n? From Subsection 1.1 we know that S determines a partition of $\mathbb{Z}^n$ into spheres $W_{n,H',n}$, where H' runs over the components of $[S[$. These spheres must conform a tiling associated to a lattice $L_S \subset \mathbb{Z}^n$ as follows: In each $W_{n,H',n}$, let $b_1 b_2 \cdots b_n$ be the vertex $a_1 a_2 \cdots a_n$ for which $a_1 + a_2 + \cdots + a_n$ is minimal. We say that this $b_1 b_2 \cdots b_n$ is the *anchor* of $W_{n,H',n}$. The anchors of the spheres $W_{n,H',n}$ form the lattice $L = L_S$. Without loss of generality we can assume that O is the anchor of a $W_{n,H_0,n}$ whose truncated center H_0 is a component of $[S]$. Let $c_1 c_2 \cdots c_n$ be the vertex $a_1 a_2 \cdots a_n$ in $W_{n,H_0,n}$ for which $a_1 + a_2 + \cdots + a_n$ is maximal. Then L_S has generating set $\{(1 + c_1)e_1, (1 + c_2)e_2, \dots, (1 + c_n)e_n\}$ and is formed by all linear combinations of those $(1 + c_i)e_i$, $(i \in [n])$. This insures that S exists and is lattice [14] (or lattice-like [1]) via L_S.

Theorem 2 *[12] For each $i \in [n]$, let P_i be a path of length $c_i - 2$, parallel to Oe_i, so $H = \Pi_{i=1}^n P_i$ is an n-box in Λ_n. Then, there is a lattice n-*PTDC$[H]$ *S of Λ_n with minimum ℓ_1-distance 3 between the components of* $[S]$.

Proof An argument for this precedes indeed the statement, and a similar proof is in [12] from a graph-coloring point of view. The claim can be proved alternatively by an additive-group epimorphism technique [1, 14] modified in [12] and containing the tool used to prove Theorems 3-4.

As in Theorem 2 and commented in Remark 2, Figure 1 displays a representation of the convex hulls of two induced components, in light-gray color, of a lattice t-PTDC$[H]$ in Λ_n, where $t = n = 3$ and $H = Q_0^2$. The components of this PTDC are just 0-cubes, the simplest case of Theorem 2 whose proof, discussed below, is contained in [12] from a graph-coloring perspective.

Theorem 3 *There exists a lattice* 1-PTDC$[Q_2^2, Q_0^2; 2, 2]$ *(in particular a* PDS*)* S *in* Λ_3 *with minimum* ℓ_1*-distance 3 between the components of* $[S]$.

As also commented in Remark 2, Theorem 3 is illustrated in Figure 1, showing in black color the components of $[S]$ in a typical FR of L_S, namely: two copies of Q_2^2 and two copies of Q_0^2, with the edges in the corresponding truncated 1-spheres shown in thick black trace; the other edges induced in the union of these 4 components are shown in thick dark-gray trace. As also commented in Remark 2, truncated 3-spheres of the 3-PTDC$[Q_0^2]$ resulting from Theorem 2 are shaded in light-gray color. Also, dark-gray color was used to indicate two other copies of Q_2^2 appearing in the figure that are components of $[S]$. Notice that the vertices O, e_1, e_2, e_3 are indicated in the figure.

To state Theorem 4, let $H_0 = Q_{n-1}^2$, $H_1 = Q_0^2$, $m_0 = m_1 = 2$, $t_0 = 1$, $t_1 = n - 2$.

Theorem 4 *There exists a lattice* $(1, n-2)$-PTDC$[Q_{n-1}^2, Q_0^2; 2, 2]$ S *in* Λ_n.

Remark 8 From the last observation in Remark 6, it can be deduced that the respective PTDC S covers (via canonical projections) in Theorem:

- 2, an n-PTDC$[H]$ in a toroidal grid $\mathcal{T} =$ $C_{c_1k_1}\Box C_{c_2k_2}\Box\cdots\Box C_{c_nk_n}, (1 < k_i \in \mathbb{Z}, \forall i \in [n])$;
- 3, a 1-PTDC$[Q_2^2, Q_0^2; 2, 2]$ in a cartesian product $\mathcal{T} =$ $C_{6k_1}\Box C_{6k_2}\Box C_{3k_3}, (0 < k_i, \forall i \in [3])$;
- 4, a $(1, n-2)$-PTDC$[Q_{n-1}^2, Q_0^2; 2, 2]$ in a cartesian product $\mathcal{T} =$ $C_{6k_1}\Box\ldots\Box C_{6k_{n-1}}\Box C_{3k_n}, (0 < k_i, \forall i \in [n])$.

3 PDS's in ternary cube compounds

Let the 2-tercube (or *tersquare*) Q_2^3 be denoted $[\emptyset]$, with vertices given by the 2-tuples xy, $(x, y \in F_3 = \{0, 1, 2\})$. As a graph, $[\emptyset] = (00, 10, 20)\Box(00, 01, 02)$, the cartesian product of two triangles whose vertex sets are both F_3. The *1-sub-tercubes* Q_1^3 of the 2-tercube $[\emptyset]$ have vertex sets $\{0y\}_{y\in F_3}$, $\{1y\}_{y\in F_3}$, $\{2y\}_{y\in F_3}$, $\{x0\}_{x\in F_3}$, $\{x1\}_{x\in F_3}$, $\{x2\}_{x\in F_3}$. They will be indicated x^0, x^1, x^2, y^0, y^1, y^2, respectively. To each of these triangles t^s $(t \in \{x, y\}; s \in F_3)$ of $[\emptyset]$ we glue, or lock, a corresponding tersquare $[{t \atop s}]$ intersecting $[\emptyset]$ exactly in t^s.

This way, six tersquares $[{x \atop 0}], \ldots, [{y \atop 2}]$ are obtained (that we call *subcentral tersquares*) that intersect $[\emptyset]$ respectively in the triangles $x^0, \ldots, y^2$. Next, we glue, or lock, to the remaining new triangles (i.e., other than those already in $[\emptyset]$), a set of nine new tersquares that we denote $[{xy \atop 00}], \ldots, [{xy \atop 22}]$ and call *corner tersquares*. These nine tersquares share merely a vertex with $[\emptyset]$. The resulting graph $[[\emptyset]]$, that will be referred to as a 2-*hive*, contains a total of $1+6+9 = 16$ tersquares, where the *central tersquare* $[\emptyset]$ of $[[\emptyset]]$ shares just one triangle with each of the six subcentral tercubes and just one vertex with each of the nine corner tersquares.

The tersquare $[\emptyset]$ is given as a subgraph at the center of Figure 3, and also results in the representation at the lower-right quarter of Table by identifying the four vertices labelled 22, and also the two vertices labelled 20, resp. 21, resp. 02, resp. 12.

TABLE I

$\begin{array}{ccccccccccc} 01 & - & 11 & - & 01 & - & 11 & - & 01 & - & 11 \\ \vert & \genfrac{[}{]}{0pt}{}{xxyy}{0110} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{010} & \vert & \genfrac{[}{]}{0pt}{}{yy}{10} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{110} & \vert & \genfrac{[}{]}{0pt}{}{xxyy}{1010} & \vert \\ 00 & - & 10 & - & 00 & - & 10 & - & 00 & - & 10 \\ \vert & \genfrac{[}{]}{0pt}{}{xxy}{011} & \vert & \genfrac{[}{]}{0pt}{}{xy}{01} & \vert & \genfrac{[}{]}{0pt}{}{y}{1} & \vert & \genfrac{[}{]}{0pt}{}{xy}{11} & \vert & \genfrac{[}{]}{0pt}{}{xxy}{101} & \vert \\ 01 & - & 11 & - & 01 & - & 11 & - & 01 & - & 11 \\ \vert & \genfrac{[}{]}{0pt}{}{xx}{01} & \vert & \genfrac{[}{]}{0pt}{}{x}{0} & \vert & [\emptyset] & \vert & \genfrac{[}{]}{0pt}{}{x}{1} & \vert & \genfrac{[}{]}{0pt}{}{xx}{10} & \vert \\ 00 & - & 10 & - & 00 & - & 10 & - & 00 & - & 10 \\ \vert & \genfrac{[}{]}{0pt}{}{xxy}{010} & \vert & \genfrac{[}{]}{0pt}{}{xy}{00} & \vert & \genfrac{[}{]}{0pt}{}{y}{0} & \vert & \genfrac{[}{]}{0pt}{}{xy}{10} & \vert & \genfrac{[}{]}{0pt}{}{xxy}{100} & \vert \\ 10 & - & 11 & - & 10 & - & 11 & - & 10 & - & 11 \\ \vert & \genfrac{[}{]}{0pt}{}{xxyy}{0101} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{001} & \vert & \genfrac{[}{]}{0pt}{}{yy}{01} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{101} & \vert & \genfrac{[}{]}{0pt}{}{xxyy}{1001} & \vert \\ 00 & - & 01 & - & 00 & - & 01 & - & 00 & - & 01 \end{array}$	$\begin{array}{ccccccccccc} 02 & - & 22 & - & 02 & - & 22 & - & 02 & - & 22 \\ \vert & \genfrac{[}{]}{0pt}{}{xxyy}{0220} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{020} & \vert & \genfrac{[}{]}{0pt}{}{yy}{20} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{220} & \vert & \genfrac{[}{]}{0pt}{}{xxyy}{2020} & \vert \\ 00 & - & 20 & - & 00 & - & 20 & - & 00 & - & 20 \\ \vert & \genfrac{[}{]}{0pt}{}{xxy}{022} & \vert & \genfrac{[}{]}{0pt}{}{xy}{02} & \vert & \genfrac{[}{]}{0pt}{}{y}{2} & \vert & \genfrac{[}{]}{0pt}{}{xy}{22} & \vert & \genfrac{[}{]}{0pt}{}{xxy}{202} & \vert \\ 02 & - & 22 & - & 02 & - & 22 & - & 02 & - & 22 \\ \vert & \genfrac{[}{]}{0pt}{}{xx}{02} & \vert & \genfrac{[}{]}{0pt}{}{x}{0} & \vert & [\emptyset] & \vert & \genfrac{[}{]}{0pt}{}{x}{2} & \vert & \genfrac{[}{]}{0pt}{}{xx}{20} & \vert \\ 00 & - & 20 & - & 00 & - & 20 & - & 00 & - & 20 \\ \vert & \genfrac{[}{]}{0pt}{}{xxy}{020} & \vert & \genfrac{[}{]}{0pt}{}{xy}{00} & \vert & \genfrac{[}{]}{0pt}{}{y}{0} & \vert & \genfrac{[}{]}{0pt}{}{xy}{20} & \vert & \genfrac{[}{]}{0pt}{}{xxy}{200} & \vert \\ 20 & - & 22 & - & 20 & - & 22 & - & 20 & - & 22 \\ \vert & \genfrac{[}{]}{0pt}{}{xxyy}{0202} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{002} & \vert & \genfrac{[}{]}{0pt}{}{yy}{02} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{202} & \vert & \genfrac{[}{]}{0pt}{}{xxyy}{2002} & \vert \\ 00 & - & 02 & - & 00 & - & 02 & - & 00 & - & 02 \end{array}$
$\begin{array}{ccccccccccc} 12 & - & 22 & - & 12 & - & 22 & - & 12 & - & 22 \\ \vert & \genfrac{[}{]}{0pt}{}{xxyy}{1221} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{121} & \vert & \genfrac{[}{]}{0pt}{}{yy}{21} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{221} & \vert & \genfrac{[}{]}{0pt}{}{xxyy}{2121} & \vert \\ 11 & - & 21 & - & 11 & - & 21 & - & 11 & - & 21 \\ \vert & \genfrac{[}{]}{0pt}{}{xxy}{122} & \vert & \genfrac{[}{]}{0pt}{}{xy}{12} & \vert & \genfrac{[}{]}{0pt}{}{y}{2} & \vert & \genfrac{[}{]}{0pt}{}{xy}{22} & \vert & \genfrac{[}{]}{0pt}{}{xxy}{212} & \vert \\ 12 & - & 22 & - & 12 & - & 22 & - & 12 & - & 22 \\ \vert & \genfrac{[}{]}{0pt}{}{xx}{12} & \vert & \genfrac{[}{]}{0pt}{}{x}{1} & \vert & [\emptyset] & \vert & \genfrac{[}{]}{0pt}{}{x}{2} & \vert & \genfrac{[}{]}{0pt}{}{xx}{21} & \vert \\ 11 & - & 21 & - & 11 & - & 21 & - & 11 & - & 21 \\ \vert & \genfrac{[}{]}{0pt}{}{xxy}{121} & \vert & \genfrac{[}{]}{0pt}{}{xy}{11} & \vert & \genfrac{[}{]}{0pt}{}{y}{1} & \vert & \genfrac{[}{]}{0pt}{}{xy}{21} & \vert & \genfrac{[}{]}{0pt}{}{xxy}{211} & \vert \\ 12 & - & 22 & - & 12 & - & 22 & - & 12 & - & 22 \\ \vert & \genfrac{[}{]}{0pt}{}{xxyy}{1212} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{112} & \vert & \genfrac{[}{]}{0pt}{}{xy}{12} & \vert & \genfrac{[}{]}{0pt}{}{xyy}{212} & \vert & \genfrac{[}{]}{0pt}{}{xxyy}{2112} & \vert \\ 11 & - & 21 & - & 11 & - & 21 & - & 11 & - & 21 \end{array}$	$\begin{array}{ccccccc} 22 & - & 02 & - & 12 & - & 22 \\ \vert & & \vert & & \vert & & \vert \\ 21 & - & 01 & - & 11 & - & 21 \\ \vert & & \vert & [\emptyset] & \vert & & \vert \\ 20 & - & 00 & - & 10 & - & 20 \\ \vert & & \vert & & \vert & & \vert \\ 22 & - & 02 & - & 12 & - & 22 \end{array}$

This construction goes ahead with the further iterative gluing of successive tersquares. In the end (or in the limit), a total compound $\varGamma_2$ of glued tersquares results that we call a ternary square compound. $\varGamma_2$ is suggested in Table I, containing just one 4-cycle (of the 9 in Q_2^3) per participating (but not totally shown) glued tersquare. This yields 9 sub-lattice types in $\varGamma_2$ (one per 4-cycle of Q_2^3, three shown in Table I), each isomorphic to $\varLambda_2$. Thus, $\varGamma_2$ is a superset of each of the copies of $\varLambda_2$ generated by some 4-cycle of $\varGamma_2$. We further specify $\varGamma_2$ after presenting a non-isolated PDS in $[[\emptyset]]$ (Theorem 5).

The large connected graph in Figure 3 represents the 2-hive $[[\emptyset]]$, with $\genfrac{[}{]}{0pt}{}{x}{0}$, $\genfrac{[}{]}{0pt}{}{x}{1}$ and $\genfrac{[}{]}{0pt}{}{x}{2}$ in light-blue background, $\genfrac{[}{]}{0pt}{}{y}{0}$, $\genfrac{[}{]}{0pt}{}{y}{1}$ and $\genfrac{[}{]}{0pt}{}{x}{2}$ in yellow background, and the nine tersquares $\genfrac{[}{]}{0pt}{}{xy}{i\,j}$ in green background if $y \neq 1$ and light-gray background if $y = 1$, where the apparent colored areas spatial disposition results in the partial hiding of some of these areas. The thick red edges induce a subgraph of $[[\emptyset]]$ whose vertex set is a non-isolated PDS S, as defined in Remark 1, of $[[\emptyset]]$.

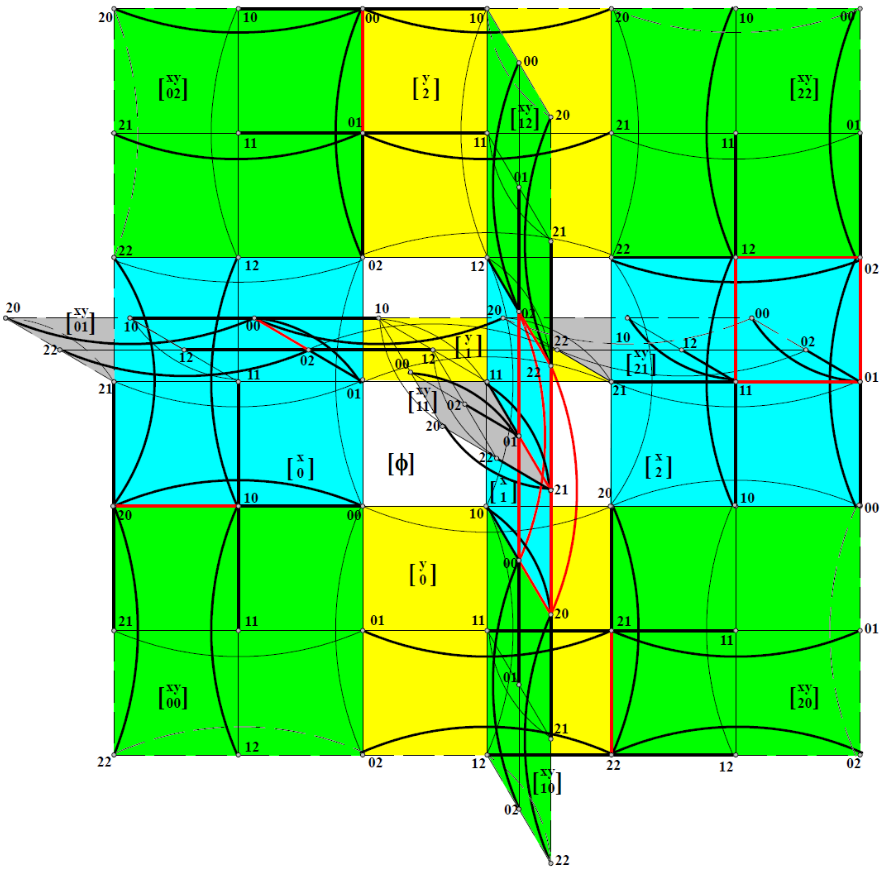

Fig. 3 Representing a PDS of $[[\emptyset]]$ in Γ_2

The dominating edges are drawn in thick dark trace, allowing the reader to verify that all vertices of $[[\emptyset]]$ not in S are dominated as indicated in Remark 1. (It is elementary to verify that no isolated PDS in $[[\emptyset]]$ exists). The remaining edges of $[[\emptyset]]$, not in thick (either ‘ or black) trace, are in dashed black trace if they are "exterior" edges of $[[\emptyset]]$ (i.e., those shared by 2-hives other than $[[\emptyset]]$ in Γ_2) and in thin black trace otherwise (i.e., if they are "interior" edges of $[[\emptyset]]$). Motivation for having pursued such a non-isolated PDS arose as a palliative remedy for the impossibility of having here a PDS like the Livingston-Stout PDS in the grid $P_4\Box P_4$ [17], which happens to be the only isolated PDS in grid graphs $P_m\Box P_n$, for $2 < min\{m,n\}$ [6].

Theorem 5 *There exists a non-isolated PDS, or 1-PTDC*$[Q_1^2, Q_2^2, Q_1^3\Box Q_1^2; 4, 1, 1]$*, in the 2-hive* $[[\emptyset]] \subset \Gamma_2$*.*

Proof The claimed PDS in $[[\emptyset]]$ is formed by the vertex sets of the following components (of its induced subgraphs):

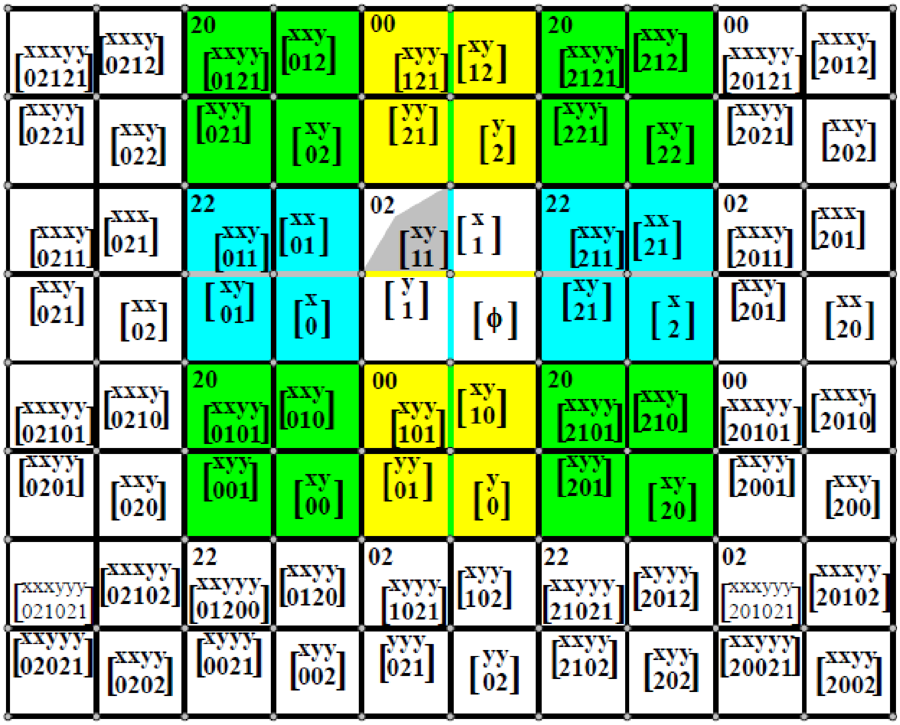

Fig. 4 Planar projection of a portion of Γ_2 larger than in Figure 3

(a) the edge $(00,01)$ of triangle $(00,01,02)$ shared by tersquares $[{}^{y}_{2}]$ and $[{}^{xy}_{02}]$;
(b) the edge $(10,20)$ of triangle $(00,10,20)$ shared by tersquares $[{}^{x}_{0}]$ and $[{}^{xy}_{00}]$;
(c) the edge $(21,22)$ of triangle $(20,21,22)$ shared by tersquares $[{}^{y}_{0}]$ and $[{}^{xy}_{20}]$;
(d) the edge $(00,02)$ of triangle $(00,02,01)$ shared by tersquares $[{}^{y}_{1}]$ and $[{}^{xy}_{01}]$;
(e) the 4-cycle $(02,12,11,01)$ of tersquare $[{}^{x}_{2}]$, where its edge $(02,12)$ (of triangle $\Delta = (02,12,22)$) is shared (with Δ) with tersquare $[{}^{xy}_{22}]$;
(f) the triangular prism in $[{}^{x}_{1}]$ formed by triangles $(00,01,02)$ and $(20, 21, 22)$ and edges $(00,20)$, $(02,22)$ (of triangle $(02,22,12)$, shared by tersquare $[{}^{xy}_{22}]$) and $(01,21)$ (of triangle $(01,21,11)$, shared by tersquare $[{}^{xy}_{11}]$).

As can be verified in Figure 3, the edges departing from these red-edge components cover all other vertices of $[[\emptyset]]$, proving the theorem.

Figure 3 may be augmented by adding the tersquares $[{}^{xx\,y}_{i\,j\,k}]$, with $i \neq j$ in F_3, etc. One may continue by adding tersquares $[{}^{xx\cdots xx\,yy\cdots yy}_{i\,j\cdots kl\,mn\cdots pq}]$ with $i \neq j \neq \ldots \neq k \neq l$ (resp. $m \neq n \neq \ldots \neq p \neq q$), i.e. not two contiguous equal values under the x's (resp. y's). Taking $i, j, \ldots, k, l, m, n, \ldots p, q$ in F_3 to be contiguously different with just two values, (e.g. 0,1; resp. 0,2; resp. 1,2, in the upper-left; resp. upper-right; resp. lower-left quarter in Table I) is a way of obtaining 3 sub-lattices of Γ_2, each isomorphic to Λ_2, as in Table I. But there is an infinite family of "parallel" sub-lattices in Γ_2 for each of the 9 sublattice types, one per each 4-cycle of a typical tersquare as in the lower-left quarter of Table I. For example, the type with values 0,2 represents "parallel" sub-lattices having "shortest" tersquares $[\emptyset]$, $\left[{x \atop 1}\right]$, $\left[{y \atop 1}\right]$, $\left[{xx \atop 01}\right]$, $\left[{yy \atop 01}\right]$, $\left[{xx \atop 21}\right]$, $\left[{yy \atop 21}\right]$, etc., including every $\left[{xx\cdots xx \atop i\,j\cdots k1}\right]$, $\left[{yy\cdots yy \atop mn\cdots p1}\right]$ and $\left[{xx\cdots xx\,yy\cdots yy \atop i\,j\cdots k1\,mn\cdots p1}\right]$.

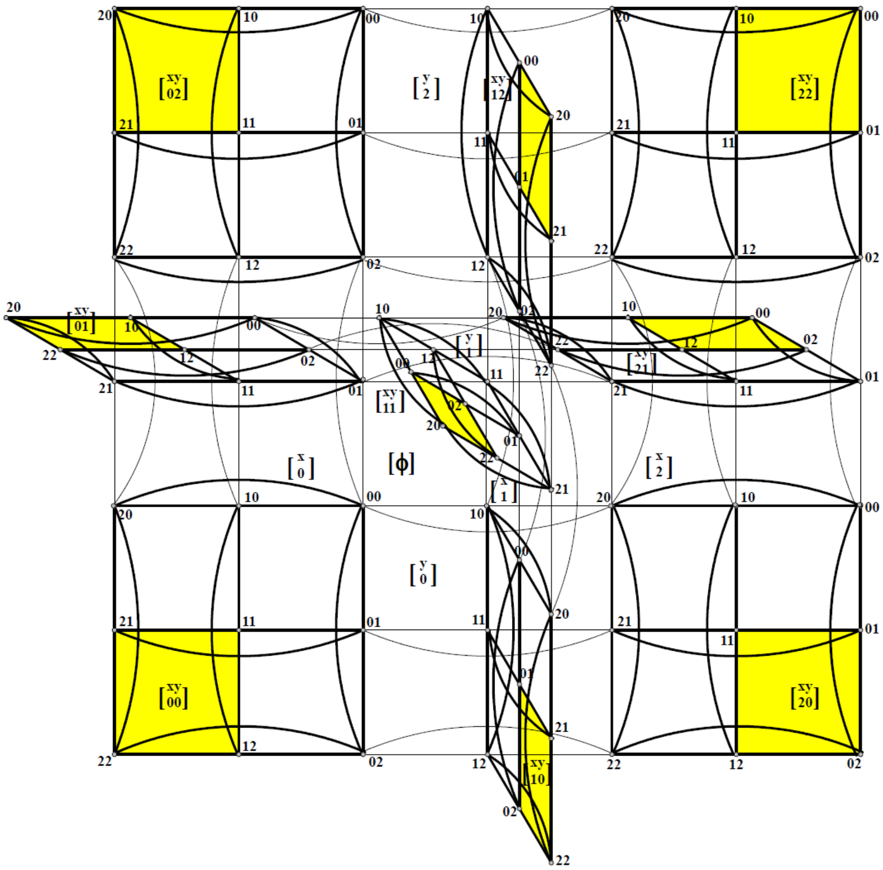

Fig. 5 Components of a 2-PTDC$[Q_2^2]$ of $[[\emptyset]]$ in Γ_2

These sub-lattices can be refined by replacing each edge e in them by the 2-path closing a triangle with e and adding more vertices and additional edges out of them to the new vertices obtained by the said replacement, in order to distinguish the glued or locked 2-tercubes in Γ_2. In particular, Figure 3 is obtained from such a procedure by starting from a plane containing the upper-right quarter of Table I. Figure 4 shows a projection of a partial extension of Figure 3 in such a plane, where tersquare colors are kept as in Figure 3, including if they are projected into a 2-path. Colors here still are light-blue, blue, green and light-gray. Each tersquare in Figure 4, represented by four squares with the common central vertex 11, is just recognizable by its upper-left vertex denomination presence. Of the four such squares, the lower-right one has the corresponding tersquare denomination. The other tersquare denominations in the three remaining squares correspond to the counterpart denominations in the representation of $[\emptyset]$.

Similarly, a graph Γ_n may be defined as a compound of glued n-tercubes along their codimension 1 ternary cubes in a likewise manner, for any $2 < n \in \mathbb{Z}$. Such a compound may be referred to as a ternary n-cube compound. The n-tuples representing the vertices of such Γ_n are given in terms of the n-tercubes $[\begin{smallmatrix}\cdots\cdots\\ \cdots\cdots\end{smallmatrix}]$ containing them; these n-tuples are assigned locally by reflection on the $(n-1)$-subtercubes, as done in Figure 3 for $n=2$, which represents the 2-hive $[[\emptyset]]$ with $[\emptyset]$ at its center, and its neighboring tersquares $[\begin{smallmatrix}x\\0\end{smallmatrix}]$, $[\begin{smallmatrix}x\\1\end{smallmatrix}]$, $[\begin{smallmatrix}x\\2\end{smallmatrix}]$ in light-blue background, $[\begin{smallmatrix}y\\0\end{smallmatrix}]$, and $[\begin{smallmatrix}y\\1\end{smallmatrix}]$, $[\begin{smallmatrix}y\\2\end{smallmatrix}]$ in yellow background, and the tersquares neighboring those tersquares, namely $[\begin{smallmatrix}x\,y\\i\,j\end{smallmatrix}]$, $(i,j \in F_3)$, in green and light-gray backgrounds. It can be seen that Γ_n may be considered as a superset of Λ_n in three different ways, as was commented above in relation to Table I for $n=2$.

We pose the following.

Question 2 Does there exist a non-isolated PDS S in Γ_n, for $n \geq 2$? If so, could S behave like a lattice, for example by restricting itself to a lattice PDS over any sub-lattice of Γ_n, as exemplified in Table I?

4 Isolated 2-PTDC's in the ternary square compound $\boldsymbol{\Gamma_2}$

In this section, we keep working in Γ_2, but slightly modifying the definition of a κ-PTDC by replacing the used Hamming distance H by the graph distance distance of Γ_2. After all, it is clear that the Hamming distance in our graph-theoretical context is nothing else than the graph distance. Then, we have the following result.

Theorem 6 *There exists 262144 isolated 2-PTDC's in the 2-hive* $[[\emptyset]] \subset \Gamma_2$.

Proof In Figure 5, the nine copies $[\begin{smallmatrix}xy\\ij\end{smallmatrix}]$, $(i,j \in F_3)$, of the tersquare are shown with its edges in thick black trace, against the thin black trace of the remaining edges of the 2-hive $[[\emptyset]]$. These nine copies happen to be the truncated 2-spheres centered at the vertices of an isolated PTDC of $[[\emptyset]]$ constituted as a selection of one vertex per yellow-faced 4-cycle in the figure. We may say that these yellow-faced 4-cycles are the "external" 4-cycles of $[[\emptyset]]$. So, there are $4^9 = 262144$ 2-PTDC's in Γ_2.

Corollary 1 *There exists a 2-PTDC*$[Q_2^2]$ *in the 2-hive* $[[\emptyset]]$.

Proof The yellow-faced 4-cycles are copies of Q_2^2 and induce the components of the claimed 2–PTDC-$[Q_2^2]$.

Theorem 7 *There exists a 2-PTDC*$[Q_2^2]$ *in* Γ_2*, as well as an infinite number of isolated 2-PTDC in* Γ_2*.*

TABLE II

${x \brack 0}$	${xy \brack 00}$	${xy \brack 01}$	${xy \brack 02}$	${x \brack 0}$	${xy \brack 00}$	${xy \brack 01}$	${xy \brack 02}$
${xx \brack 01}$	${xxy \brack 010}$	${xxy \brack 011}$	${xxy \brack 012}$	${xx \brack 02}$	${xxy \brack 020}$	${xxy \brack 021}$	${xxy \brack 022}$
${xxx \brack 010}$	${xxxy \brack 0100}$	${xxxy \brack 0101}$	${xxxy \brack 0102}$	${xxx \brack 020}$	${xxxy \brack 0200}$	${xxxy \brack 0201}$	${xxxy \brack 0202}$
${xxx \brack 012}$	${xxxy \brack 0120}$	${xxxy \brack 0121}$	${xxxy \brack 0122}$	${xxx \brack 021}$	${xxxy \brack 0210}$	${xxxy \brack 0211}$	${xxxy \brack 0212}$
${y \brack 0}$	${xy \brack 00}$	${xy \brack 10}$	${xy \brack 20}$	${y \brack 0}$	${xy \brack 00}$	${xy \brack 10}$	${xy \brack 20}$
${yy \brack 01}$	${xyy \brack 001}$	${xyy \brack 101}$	${xyy \brack 201}$	${yy \brack 02}$	${xyy \brack 002}$	${xyy \brack 102}$	${xyy \brack 202}$
${yyy \brack 010}$	${xyyy \brack 0010}$	${xyyy \brack 1010}$	${xyyy \brack 1010}$	${yyy \brack 020}$	${xyyy \brack 0020}$	${xyyy \brack 1020}$	${xyyy \brack 1020}$
${yyy \brack 012}$	${xyyy \brack 0012}$	${xyyy \brack 1012}$	${xyyy \brack 1012}$	${yyy \brack 012}$	${xyyy \brack 0021}$	${xyyy \brack 1021}$	${xyyy \brack 1021}$
${x \brack 1}$	${xy \brack 10}$	${xy \brack 11}$	${xy \brack 12}$	${x \brack 1}$	${xy \brack 10}$	${xy \brack 11}$	${xy \brack 12}$
${xx \brack 10}$	${xxy \brack 100}$	${xxy \brack 101}$	${xxy \brack 102}$	${xx \brack 12}$	${xyy \brack 120}$	${xxy \brack 121}$	${xxy \brack 122}$
${xxx \brack 101}$	${xxxy \brack 1010}$	${xxxy \brack 1011}$	${xxxy \brack 1012}$	${xxx \brack 120}$	${xxxy \brack 1200}$	${xxxy \brack 1201}$	${xxxy \brack 1202}$
${xxx \brack 102}$	${xxxy \brack 1020}$	${xxxy \brack 1021}$	${xxxy \brack 1022}$	${xxx \brack 121}$	${xxxy \brack 1210}$	${xxxy \brack 1211}$	${xxxy \brack 1212}$
${y \brack 1}$	${xy \brack 01}$	${xy \brack 11}$	${xy \brack 21}$	${y \brack 1}$	${xy \brack 01}$	${xy \brack 11}$	${xy \brack 21}$
${yy \brack 10}$	${xyy \brack 010}$	${xyy \brack 110}$	${xyy \brack 210}$	${yy \brack 12}$	${xyy \brack 012}$	${xyy \brack 112}$	${xyy \brack 212}$
${yyy \brack 101}$	${xyyy \brack 0101}$	${xyyy \brack 1101}$	${xyyy \brack 2101}$	${yyy \brack 120}$	${xyyy \brack 0120}$	${xyyy \brack 1120}$	${xyyy \brack 2120}$
${yyy \brack 102}$	${xyyy \brack 0102}$	${xyyy \brack 1102}$	${xyyy \brack 2102}$	${yyy \brack 121}$	${xyyy \brack 0121}$	${xyyy \brack 1121}$	${xyyy \brack 2121}$
${x \brack 2}$	${xy \brack 20}$	${xy \brack 21}$	${xy \brack 22}$	${x \brack 2}$	${xy \brack 20}$	${xy \brack 21}$	${xy \brack 22}$
${xx \brack 20}$	${xxy \brack 200}$	${xxy \brack 201}$	${xxy \brack 202}$	${xx \brack 21}$	${xxy \brack 210}$	${xxy \brack 211}$	${xxy \brack 212}$
${xxx \brack 201}$	${xxxy \brack 2010}$	${xxxy \brack 2011}$	${xxxy \brack 2012}$	${xxx \brack 210}$	${xxxy \brack 2100}$	${xxxy \brack 2101}$	${xxxy \brack 2102}$
${xxx \brack 202}$	${xxxy \brack 2020}$	${xxxy \brack 2021}$	${xxxy \brack 2022}$	${xxx \brack 212}$	${xxxy \brack 2120}$	${xxxy \brack 2121}$	${xxxy \brack 2122}$
${y \brack 2}$	${xy \brack 02}$	${xy \brack 12}$	${xy \brack 22}$	${y \brack 2}$	${xy \brack 02}$	${xy \brack 12}$	${xy \brack 22}$
${yy \brack 20}$	${xyy \brack 020}$	${xyy \brack 120}$	${xyy \brack 220}$	${yy \brack 21}$	${xyy \brack 021}$	${xyy \brack 121}$	${xyy \brack 221}$
${yyy \brack 201}$	${xyyy \brack 0201}$	${xyyy \brack 1201}$	${xyyy \brack 2201}$	${yyy \brack 210}$	${xyyy \brack 0210}$	${xyyy \brack 1210}$	${xyyy \brack 2210}$
${yyy \brack 202}$	${xyyy \brack 0202}$	${xyyy \brack 1202}$	${xyyy \brack 2202}$	${yyy \brack 212}$	${xyyy \brack 0212}$	${xyyy \brack 1212}$	${xyyy \brack 2212}$

Proof We consider the following 18 triple unions of tercubes in the 2-hive $[[\emptyset]]$:

$$\begin{array}{l}
({xy \brack 00} \cup {x \brack 0} \cup {xy \brack 01}),\ ({xy \brack 00} \cup {x \brack 0} \cup {xy \brack 02}),\ ({xy \brack 01} \cup {x \brack 0} \cup {xy \brack 02}),\\
({xy \brack 00} \cup {y \brack 0} \cup {xy \brack 10}),\ ({xy \brack 00} \cup {x \brack 0} \cup {xy \brack 20}),\ ({xy \brack 10} \cup {x \brack 0} \cup {xy \brack 20}),\\
({xy \brack 10} \cup {x \brack 1} \cup {xy \brack 11}),\ ({xy \brack 10} \cup {x \brack 1} \cup {xy \brack 12}),\ ({xy \brack 11} \cup {x \brack 1} \cup {xy \brack 12}),\\
({xy \brack 01} \cup {y \brack 1} \cup {xy \brack 11}),\ ({xy \brack 01} \cup {y \brack 1} \cup {xy \brack 21}),\ ({xy \brack 11} \cup {y \brack 0} \cup {xy \brack 21}),\\
({xy \brack 20} \cup {x \brack 2} \cup {xy \brack 21}),\ ({xy \brack 20} \cup {x \brack 2} \cup {xy \brack 22}),\ ({xy \brack 21} \cup {x \brack 2} \cup {xy \brack 22}),\\
({xy \brack 02} \cup {y \brack 2} \cup {xy \brack 12}),\ ({xy \brack 02} \cup {y \brack 2} \cup {xy \brack 22}),\ ({xy \brack 12} \cup {y \brack 2} \cup {xy \brack 22}),
\end{array}$$

where the union of the three graphs in each line will be called a *quarter* of $[[\emptyset]]$, as well as six pairs of new 2-hives overlapping $[[\emptyset]]$ in the six shown quarters, having these pairs of new 2-hives the following respective pairs of central tersquares:

$$({xx \brack 01}, {xx \brack 02}), ({yy \brack 01}, {yy \brack 02}), ({xx \brack 10}, {xx \brack 12}), ({yy \brack 10}, {yy \brack 12}), ({xx \brack 20}, {xx \brack 21}), ({yy \brack 20}, {yy \brack 21}).$$

These six pairs account for 12 2-hives, each having 3 corner tersquares in common with $[[\emptyset]]$ separated by a common subcentral tersquare; the remaining 6 corner tersquares in each of the said 12 2-hives contain each a yellow-faced

4-cycles; in each such yellow-faced 4-cycle, we can select a vertex so that each involved 2-hive has an isolated 2-PTDC formed by such vertices; the union of all the isolated 2-PTDC's so obtained constitutes an isolated 2-PTDC of the union of all those 2-hives while the vertices of all yellow-faced 4-cycles constitute a 2-PTDC$[Q_2^2]$.

Let us check the 12 2-hives mentioned above, ordered in the form they were already presented. They show up in Table II, where each such 2-hive $[[U]]$ with

$$[U] \in \{\genfrac{[}{]}{0pt}{}{xx}{01}, \genfrac{[}{]}{0pt}{}{xx}{02}, \genfrac{[}{]}{0pt}{}{yy}{01}, \genfrac{[}{]}{0pt}{}{yy}{02}, \genfrac{[}{]}{0pt}{}{xx}{10}, \genfrac{[}{]}{0pt}{}{xx}{12}, \genfrac{[}{]}{0pt}{}{yy}{10}, \genfrac{[}{]}{0pt}{}{yy}{12}, \genfrac{[}{]}{0pt}{}{xx}{20}, \genfrac{[}{]}{0pt}{}{xx}{21}, \genfrac{[}{]}{0pt}{}{yy}{20}, \genfrac{[}{]}{0pt}{}{yy}{21}\}$$

is given as a corresponding 4×4-array of tersquares. In such a 4×4-array, the tersquares cited in the first row are shared with the 2-hive $[[\emptyset]]$; the central tersquare $[U]$ is given as the first entry of the second row; the subcentral tersquares are given as the six remaining entries of the second row and the first column, and the corner tersquares that are not in $[[\emptyset]]$ are the six remaining entries, in the third and fourth rows. In all, there are 72 such corner tersquares, that added to the 9 of $[[\emptyset]]$ yields 81 corner tersquares in the 12 2-hives of Table II. It is clear that this 81 corner tersquares are distinct and separated, so that by selecting a vertex in each of their corresponding yellow-faced 4-cycles, the claimed 2-PTDC is initiated. To continue iteratively, starting with the upper-left 2-hive $[\genfrac{[}{]}{0pt}{}{xx}{00}]$ and ending with the lower-right 2-hive $[\genfrac{[}{]}{0pt}{}{yy}{21}]$, both in Table II, we have the following (vertically disposed) correspondences from their new subcentral tersquares (other than $[x_0]$ and $[y_2]$) to the new pairs of 2-hives to be considered in the iteration for considerarion of new yellow-faced 4-cycles and candidates for vertices in PDS's:

$$\left|\begin{array}{c|c|c}
\genfrac{[}{]}{0pt}{}{xxx}{010} \to \{[\genfrac{[}{]}{0pt}{}{xxxx}{0101}], [\genfrac{[}{]}{0pt}{}{xxxx}{0102}]\} & \cdots\cdots & \genfrac{[}{]}{0pt}{}{yyy}{210} \to \{[\genfrac{[}{]}{0pt}{}{yyyy}{2101}], [\genfrac{[}{]}{0pt}{}{xxxx}{2102}]\} \\
\genfrac{[}{]}{0pt}{}{xxx}{012} \to \{[\genfrac{[}{]}{0pt}{}{xxxx}{0120}], [\genfrac{[}{]}{0pt}{}{xxxx}{0121}]\} & \cdots\cdots & \genfrac{[}{]}{0pt}{}{yyy}{212} \to \{[\genfrac{[}{]}{0pt}{}{yyyy}{2120}], [\genfrac{[}{]}{0pt}{}{xxxx}{2121}]\} \\
\genfrac{[}{]}{0pt}{}{xxy}{010} \to \{[\genfrac{[}{]}{0pt}{}{xxyy}{0101}], [\genfrac{[}{]}{0pt}{}{xxyy}{0102}]\} & \cdots\cdots & \genfrac{[}{]}{0pt}{}{xyy}{021} \to \{[\genfrac{[}{]}{0pt}{}{xxyy}{0121}], [\genfrac{[}{]}{0pt}{}{xxyy}{0221}]\} \\
\genfrac{[}{]}{0pt}{}{xxy}{011} \to \{[\genfrac{[}{]}{0pt}{}{xxyy}{0110}], [\genfrac{[}{]}{0pt}{}{xxyy}{0112}]\} & \cdots\cdots & \genfrac{[}{]}{0pt}{}{xyy}{121} \to \{[\genfrac{[}{]}{0pt}{}{xxyy}{1021}], [\genfrac{[}{]}{0pt}{}{xxyy}{1221}]\} \\
\genfrac{[}{]}{0pt}{}{xxy}{012} \to \{[\genfrac{[}{]}{0pt}{}{xxyy}{0120}], [\genfrac{[}{]}{0pt}{}{xxyy}{0121}]\} & \cdots\cdots & \genfrac{[}{]}{0pt}{}{xyy}{221} \to \{[\genfrac{[}{]}{0pt}{}{xxyy}{2021}], [\genfrac{[}{]}{0pt}{}{xxyy}{2121}]\}
\end{array}\right| \qquad (1)$$

and so on. In (1), each target pair in the last 3 lines has each of its two 2-hives arriving from two different source tersquares; for example, $[\genfrac{[}{]}{0pt}{}{xxyy}{0121}]$ is in the pair $\{[\genfrac{[}{]}{0pt}{}{xxyy}{0120}], [\genfrac{[}{]}{0pt}{}{xxyy}{0121}]\}$ in the fifth row on the left of (1) as the target of the tersquare $\genfrac{[}{]}{0pt}{}{xxy}{012}$ and also in the pair $\{[\genfrac{[}{]}{0pt}{}{xxyy}{0121}], [\genfrac{[}{]}{0pt}{}{xxyy}{0221}]\}$ in the third row on the right of (1) as the targer of the tersquare $\genfrac{[}{]}{0pt}{}{xyy}{021}$. We observe that in this iterative process of reconstructing Γ_2 via overlapping of 2-hives away from $[[\emptyset]]$, the central tersquares of participating 2-hives are those of the form $\genfrac{[}{]}{0pt}{}{x\cdots xy\cdots y}{\cdots\cdots\cdots}$, where the number of x's and the number of y's are both even.

By iteration, we obtain both a 2-PTDC$[Q_2^2]$ formed by the vertices of all yellow-faced 4-cycles in successive corner tersquares and an infinite number of isolated 2-PTDC's of Γ_2, (262144 per 2-hive).

Conjecture 2 There exists an isolated n-PTDC in Γ_n, $\forall n > 2$.

Question 3 Does there exist a suitable way of declaring a susbset $S \subset V(\Gamma_n)$ to be *periodic*, or *periodic-like*, that extends the notion of periodicity of Λ_n

at the end of Remark 6 to one in Γ_n? so that a replacement of the notion of "quotient" or "toroidal" graphs of Λ_n can be found that way for Γ_n?

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