Inverse Fuzzy Multigraphs and Planarity with Application in Decision-making

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Inverse fuzzy multigraphs and planarity with application in decision-making

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Abstract

Recently, in [3], we defined the concept of inverse fuzzy graph as a generalization of graph, which is able to answer some problems that graph theory and fuzzy graph theory can not explain. Now, in this paper we define the notion of inverse fuzzy multigraph and the concept of planarity on it by using the concepts of intersecting value and inverse fuzzy planarity value. Then we introduce some related theorems which determining upper bounds and lower bounds for the inverse fuzzy planarity value. After that we define the strong (weak) planarity of an inverse fuzzy multigraph and investigate related results. Finally, we give an application of inverse fuzzy multigraphs to decision-making how to reduce the cost of travel tours.

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Keywords: Inverse fuzzy (multi)graphs, planarity, decision making.

1 Introduction

Graph theory is considered as one of the branches in modern mathematics which has experienced a lot of development during the recent years because of its large applications. One of the most important topics in graphs is planarity, which was based on Euler's polyhedral formula, which is related to the polyhedron edges, vertices and faces. The applications of planar graphs occur naturally such as designing and structuring complex radio electronic circuits, railway maps, planetary gearbox and chemical molecules. While modeling an urban city, pipelines, railway lines, subway tunnels, electric transmission lines and metro lines are extremely important. Crossing is beneficial as it helps in utilizing less space and is inexpensive, but there are some drawbacks too. As the crossing of such lines is quite dangerous for human lives, but, by taking certain amount of security measures, it can be made. The crossing between two uncrowded route is less risk than the crossing between two crowded routes. The allowance of such crossings leads to the concept of planarity and planarity value in fuzzy and inverse fuzzy graph.

On the other hand, the fuzzy concept is considered as a modern concept which appeared recently in 1965 [16] when Zadeh defined the fuzzy subset of a crisp set, as a generalization of the classical subset. According to classical concept, any element either belongs to the studied subset which can be given one as the membership degree, or is not related to this subset, which can be given zero as the membership degree. However, according to the fuzzy concept, any element belongs to the studied fuzzy subset with a membership degree between zero and one, which is closer to the human thought so that our minds are working with the approximate type and we do not have an exact understanding for everything. For example, we can recognize the white and black colors, along with all of their gray gradients. The fuzzy concept has attracted a lot of attention due to its various applications in many areas such as industries, communications, and artificial intelligence although it is considered as a new branch of mathematics. Thus, a large number of scientists and

researchers have focused on improving and applying this concept in every field of mathematics. Graph theory is regarded as one of these fields where the fuzzy concept can be implemented. This theory was suggested in 1975 by Rosenfeld [13], when he considered fuzzy relations on fuzzy sets and developed the structure of fuzzy graphs, through obtaining the analogues related to several graph theoretical concepts. In addition, he introduced and examined such concepts as paths, connectedness, bridges, cut vertices, trees, and forests. Further, Mordeson proposed the concept of fuzzy graphs and fuzzy hypergraphs. In another study, Rakesh Jaiswal [6] analyzed the concept of fuzzy graph coloring in traffic light problem. Furthermore, Lavanya [7] analyzed the concept of fuzzy total coloring and its application for job scheduling. In another study, Eslahchi [5] identified the vertex strength of fuzzy graphs. In the last years, fuzzy graph theory was enriched with deep structure [4, 8, 9, 10, 11, 12].

Recently, in [3], we defined the concept of inverse fuzzy graph as a generalization of graph, which is different with fuzzy graph.

Now, in this paper we defined the notions of inverse fuzzy multigraph and planarity of an inverse fuzzy multigraph by using the concepts of intersecting value and inverse fuzzy planarity value and we investigated related results.

2 Preliminaries

In this section, we reminder the definitions of inverse fuzzy graph $G^I = (\sigma, \mu)$, underlying crisp graph respect to G^I , partial I-fuzzy graph and I-fuzzy subgraph. Then we present the concepts of degree and total degree of a vertex, also order, size of G^I and some related definitions.

Definition 2.1. [3] Let $G = (V, E)$ be a finite simple graph and $\sigma : V \rightarrow [0, 1]$ and $\mu : E \rightarrow [0, 1]$ be two fuzzy subsets of V and E , respectively. Then $G^I = (\sigma, \mu)$ is called an *inverse fuzzy graph* or shortly *I-fuzzy graph* on G if for any $xy \in E$,

$$\mu(xy) \geq \min\{\sigma(x), \sigma(y)\}.$$

Note 2.2. [3] It is clear that the inverse fuzzy graph is completely different with the fuzzy graph which required to be $\mu(xy) \leq \min\{\sigma(x), \sigma(y)\}$, also it is completely different with the complement of fuzzy graph since every complement is itself a fuzzy graph, and the name inverse fuzzy graph does not mean that G^I is a special case of fuzzy graph but we use this name because the $(\leq)$ in the condition of fuzzy graphs inversed and becomes $(\geq)$.

Also it is obvious that the inverse fuzzy graph and all of fuzzy structures like vague, intuitionistic, soft and neutrosophic graphs are different.

Definition 2.3. [3] Let $G^I = (\sigma, \mu)$ be an I-fuzzy graph on $G = (V, E)$. We defined the *underlying crisp graph* $G^* = (V^*, E^*)$ respect to G^I as follows

$$V^* = V \setminus \{x \in V : \sigma(x) = 0 \text{ and } \mu(xy) = 0, \text{ for all } xy \in E\},$$

$$E^* = \{xy \in E : \mu(xy) > 0\}.$$

Definition 2.4. [3] Let $G^I = (\sigma, \mu)$ and $H^I = (\tau, \nu)$ be two I-fuzzy graphs on $G = (V, E)$ with underlying crisp graphs $G^* = (V_G^*, E_G^*)$ and $H^* = (V_H^*, E_H^*)$, respectively. Then

(i) H^I is called a *partial I-fuzzy graph* of G^I , if for any $x \in V_H^*$ and $xy \in E_H^*$,

$$V_H^* \subseteq V_G^*, \tau(x) \leq \sigma(x), \nu(xy) \leq \mu(xy).$$

(ii) H^I is called an *I-fuzzy subgraph* of G^I , if for any $x \in V_H^*$ and $xy \in E_H^*$,

$$V_H^* \subseteq V_G^*, \tau(x) = \sigma(x), \nu(xy) = \mu(xy).$$

It is obvious that every I-fuzzy subgraph is a partial I-fuzzy graph, but the converse is not true in general.

Definition 2.5. [3] Let $G^I = (\sigma, \mu)$ be an I-fuzzy graph on $G = (V, E)$. Then the *degree* of any vertex $x \in V$ is defined by $d(x) = \sum_{xy \in E} \mu(xy)$. *Total degree* of any vertex $x \in V$ is defined by $d_T(x) = d(x) + \sigma(x)$. *Order* of G^I is defined by $O(G^I) = \sum_{x \in V} \sigma(x)$ and *size* of G^I is defined by $S(G^I) = \sum_{xy \in E} \mu(xy)$.

Theorem 2.6. [3] For any I-fuzzy graph G^I on $G = (V, E)$,

$$\sum_{x \in V} d(x) = 2 \sum_{xy \in E} \mu(xy).$$

Definition 2.7. [3] Let $G^I = (\sigma, \mu)$ be an I-fuzzy graph on G . Then G^I is called a *complete I-fuzzy graph* on G , if $E = V \times V$.

Definition 2.8. [3] Let $G^I = (\sigma, \mu)$ be an I-fuzzy graph on G . Then G is called a *vertex stable* if for any $xy \in E$, $\sigma(x) = \sigma(y)$, an *edge stable* if for any $xy, zt \in E$, $\mu(xy) = \mu(zt)$ and *stable* if it is both vertex stable and edge stable.

3 Inverse fuzzy multigraphs

In this section, we define the notion of inverse fuzzy multigraph and we present some definitions related to it.

Definition 3.1. Let V be a non empty set, $\sigma : V \rightarrow [0, 1]$ be a fuzzy set and $E = \{((x, y), \mu_i(x, y)), i = 1, 2, \dots, m \mid (x, y) \in V \times V\}$ be a fuzzy multiset of $V \times V$, such that $\mu_i(x, y) \geq \min\{\sigma(x), \sigma(y)\}$ for $i = 1, 2, \dots, m$ where $m = \max\{i : \mu_i(x, y) \neq 0\}$. Then $\psi^I = (V, \sigma, E)$ is denoted as *inverse fuzzy multigraph* where $\sigma(x)$ and $\mu_i(x, y)$ represent the membership degree of the vertex x and the membership degree of the edge xy in ψ^I , respectively.

Example 3.2. Let $\sigma : V \rightarrow [0, 1]$ be a fuzzy set of $V = \{x_1, x_2, x_3\}$ defined such that

$$\sigma(x_1) = 0.7, \sigma(x_2) = 0.4 \text{ and } \sigma(x_3) = 0.6.$$

Let E be a fuzzy multiset of $V \times V$ defined such that

$$E = \{((x_1, x_2), 0.5), ((x_2, x_3), 0.4), ((x_2, x_3), 0.8)\}.$$

Then $\psi^I = (V, \sigma, E)$ is an inverse fuzzy multigraph represented in Figure 1.

Figure 1: The inverse fuzzy multigraph ψ^I

Definition 3.3. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph. Then the *degree* of any vertex $x \in V$ is defined by $d(x) = \sum_{i=1}^m \mu_i(x, y)$, for all $y \in V$.

Example 3.4. In the inverse fuzzy multigraph ψ^I which represented in Figure 1, degree of the vertices x_1, x_2, x_3 are $d(x_1) = 0.5, d(x_2) = 1.7$ and $d(x_3) = 1.2$.

Definition 3.5. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph. Then ψ^I is called:
Vertex stable if for any inverse fuzzy edge $((x, y), \mu_k(x, y)) \in E, \sigma(x) = \sigma(y)$.
Edge stable if for any inverse fuzzy edges $((x, y), \mu_k(x, y)), ((z, t), \mu_l(z, t)) \in E, \mu_k(x, y) = \mu_l(z, t)$.
Stable, if it is both vertex stable and edge stable.

Definition 3.6. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph. Then an inverse fuzzy edge $((x, y), \mu_k(x, y))$ is called an *effective edge* of ψ^I if

$$\mu_k(x, y) \geq \max\{\sigma(x), \sigma(y)\}.$$

Otherwise, we called the inverse fuzzy edge *non-effective edge*.

4 Planarity of inverse fuzzy multigraphs

In this section, we define the concept of planarity of an inverse fuzzy multigraph depending on some important related definitions like intersecting value and inverse fuzzy planarity value.

Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph and for a certain geometric representation it has two inverse fuzzy edges $((x, y), \mu_k(x, y))$ and $((z, t), \mu_l(z, t))$ which are intersected at a point P , where k, l are fixed integers. In fuzzy sense, if one at least of $\mu_k(x, y)$ and $\mu_l(z, t)$ is near to 0, then the crossing will not be important in the representation. But if both of $\mu_k(x, y)$ and $\mu_l(z, t)$ are near to 1, then the crossing will be important in the representation. Depending on this idea we will define the intersecting value and inverse fuzzy planarity value.

Definition 4.1. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph with a certain geometric representation which has the intersecting point P between the inverse fuzzy edges $((x, y), \mu_k(x, y))$ and $((z, t), \mu_l(z, t))$. Then we define the *intersecting value* at the point P by

$$I_P = \frac{\mu_k(x, y) + \mu_l(z, t)}{2}.$$

If the number of intersecting points in an inverse fuzzy multigraph increases, then planarity decreases. Also, if the value of I_P increases, then planarity decreases. Therefore, I_P is inversely proportional to the planarity. Depending on this idea we define the concept of planarity value of an inverse fuzzy multigraph.

Definition 4.2. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph with a certain geometric representation which has n intersecting points $P_1, P_2, \dots, P_n$. Then we define the *inverse fuzzy planarity value* P' of this geometric representation of ψ^I by

$$P' = \frac{1}{1 + I_{P_1} + I_{P_2} + \dots + I_{P_n}}.$$

It is clear that $0 < P' \leq 1$ and we can say that every geometric representation of any inverse fuzzy multigraph is planar with certain planarity value P' .

Figure 2: The inverse fuzzy multigraph ψ^I

Then this geometric representation of ψ^I has 2 intersecting points, P_1 and P_2 . The intersecting value at the point P_1 is

$$I_{P_1} = \frac{0.3 + 0.3}{2} = 0.3,$$

and the intersecting value at the point P_2 is

$$I_{P_2} = \frac{0.5 + 0.7}{2} = 0.6.$$

So the inverse fuzzy planarity value of this geometric representation of ψ^I is

$$P' = \frac{1}{1 + 0.3 + 0.6} \approx 0.53.$$

Theorem 4.4. For any inverse fuzzy multigraph ψ^I with geometric representation which has n intersecting points:

$$\frac{1}{1 + n} \leq P' \leq \frac{1}{1 + n\alpha},$$

where $\alpha = \min\{\sigma(x) : x \in V\}$.

Proof. According to Definition 4.2:

$$P' = \frac{1}{1 + I_{P_1} + I_{P_2} + \dots + I_{P_n}},$$

and by Definition 4.1:

$$I_P = \frac{\mu_k(x, y) + \mu_l(z, t)}{2}.$$

Since $\mu_k(x, y) \leq 1$, for any inverse fuzzy edge $((x, y), \mu_k(x, y))$ in ψ^I , we have

$$I_{P_i} \leq \frac{1 + 1}{2} = 1.$$

For $i = \{1, 2, \dots, n\}$, we get $I_{P_1} + I_{P_2} + \dots + I_{P_n} \leq n$ and so

$$P' = \frac{1}{1 + I_{P_1} + I_{P_2} + \dots + I_{P_n}} \geq \frac{1}{1 + n}.$$

On the other hand $\mu_k(x, y) \geq \sigma(x) \wedge \sigma(y)$, for any inverse fuzzy edge $((x, y), \mu_k(x, y))$ in ψ^I . Since $\alpha = \min\{\sigma(x) : x \in V\}$, we get $\mu_k(x, y) \geq \alpha$, and so

$$I_{P_i} \geq \frac{\alpha + \alpha}{2} = \alpha.$$

For $i = \{1, 2, \dots, n\}$, we have $I_{P_1} + I_{P_2} + \dots + I_{P_n} \geq n\alpha$. Thus

$$P' = \frac{1}{1 + I_{P_1} + I_{P_2} + \dots + I_{P_n}} \leq \frac{1}{1 + n\alpha}.$$

□

Theorem 4.5. *For any inverse fuzzy multigraph ψ^I with geometric representation which has n intersecting points and which is edge stable:*

$$P' = \frac{1}{1 + n\gamma},$$

where $\gamma = \mu_k(x, y)$, for any inverse fuzzy edge $((x, y), \mu_k(x, y))$ in ψ^I .

Proof. According to Definition 4.1:

$$I_P = \frac{\mu_k(x, y) + \mu_l(z, t)}{2}.$$

Since ψ^I is edge stable, we get $\mu_k(x, y) = \mu_l(z, t) = \gamma$, for any inverse fuzzy edges $((x, y), \mu_k(x, y))$ and $((z, t), \mu_l(z, t))$ in ψ^I , and so

$$I_{P_i} = \frac{\gamma + \gamma}{2} = \gamma.$$

For $i = \{1, 2, \dots, n\}$, we obtain

$$P' = \frac{1}{1 + I_{P_1} + I_{P_2} + \dots + I_{P_n}} = \frac{1}{1 + n\gamma}.$$

□

Theorem 4.6. *For any inverse fuzzy multigraph ψ^I with geometric representation which has n intersecting points, all of them between non-effective inverse fuzzy edges:*

$$P' > \frac{1}{1 + n\beta},$$

where $\beta = \max\{\sigma(x) : x \in V\}$.

Proof. According to Definition 4.1:

$$I_P = \frac{\mu_k(x, y) + \mu_l(z, t)}{2}.$$

Since all n intersecting points in ψ^I are between non-effective inverse fuzzy edges, we get

$$\mu_k(x, y) < \max\{\sigma(x), \sigma(y)\} \text{ and } \mu_l(z, t) < \max\{\sigma(z), \sigma(t)\}.$$

Since $\beta = \max\{\sigma(x) : x \in V\}$, so

$$I_{P_i} = \frac{\mu_k(x, y) + \mu_l(z, t)}{2} < \frac{\beta + \beta}{2} = \beta,$$

for $i = \{1, 2, \dots, n\}$, so $I_{p_1} + I_{p_2} + \dots + I_{p_n} < n\beta$. Thus,

$$P' = \frac{1}{1 + I_{P_1} + I_{P_2} + \dots + I_{P_n}} > \frac{1}{1 + n\beta}.$$

□

Definition 4.7. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph with a certain geometric representation which has n intersecting points $P_1, P_2, \dots, P_n$, such that m of these intersecting points satisfy that $I_{P_i} \geq 0.5$, for $i = 1, 2, \dots, m$. Then we call this geometric representation of ψ^I *inverse fuzzy strong planar multigraph* if

$$P' > 0.5 \text{ and } m < \frac{n}{2}.$$

It means that a geometric representation of ψ^I is an inverse fuzzy strong planar multigraph if the inverse fuzzy planarity value is greater than 0.5 and the intersecting points which satisfy $I_P \geq 0.5$ are less than half of the total number of the intersecting points n . Otherwise, we call this geometric representation of ψ^I as *inverse fuzzy weak planar multigraph*.

Example 4.8. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph represented in a geometric representation showed in Figure 3.

Figure 3: The inverse fuzzy multigraph ψ^I

Then this geometric representation of ψ^I has 2 intersecting points, P_1 and P_2 . The intersecting value at the point P_1 is

$$I_{P_1} = \frac{0.4 + 0.2}{2} = 0.3,$$

and the intersecting value at the point P_2 is

$$I_{P_2} = \frac{0.3 + 0.1}{2} = 0.2.$$

So the inverse fuzzy planarity value of this geometric representation of ψ^I is

$$P' = \frac{1}{1 + 0.3 + 0.2} \approx 0.67.$$

Figure 4: Another geometric representation of ψ^I

then this geometric representation of ψ^I has one intersecting point P, that the intersecting value at the point P is

$$I_P = \frac{1+1}{2} = 1.$$

So the inverse fuzzy planarity value of this geometric representation of ψ^I is

$$P' = \frac{1}{1+1} = 0.5.$$

Hence this geometric representation of ψ^I is weak planar.

Note 4.9. In the crisp graph, the planarity is very important and usually we try to find a geometric representation of the graph without any intersecting points. If it is not possible we try to find a geometric representation of the graph with the lowest possible number of intersecting points because this kind will help us to solve some problems.

But in the inverse fuzzy graph the situation is different, because according to the application sometimes we need a specific number of intersecting points to solve our problem. So in the inverse fuzzy graph we do not care about finding a geometric representation without any intersecting points instead of that according to our problem we determine the wanted number of intersecting points and we try to find the geometric representation with this specific number and with lowest dangerous in the intersecting points (lowest $\{I_{P_i}\}$) and the greatest inverse planarity value(P'), as we will see in the application at the end of this paper.

Example 4.10. In the geometric representation of ψ^I which represented in Figure 2, we have 2 intersecting points such that $I_{P_1} = 0.3, I_{P_2} = 0.6$ and $P' \approx 0.53$. Even if $P' > 0.5$, then this geometric representation of ψ^I is weak planar because the condition $m < \frac{n}{2}$ is not achieved.

Theorem 4.11. If ψ^I is an inverse fuzzy multigraph with geometric representation which has n intersecting points satisfy that $I_{P_i} < 0.5$, for $i = 1, 2, \dots, n$, then

$$P' > \frac{2}{2+n}.$$

Proof. Since $I_{P_i} < 0.5$, for $i = 1, 2, \dots, n$, we get $I_{P_1} + I_{P_2} + \dots + I_{P_n} < (0.5)n$, and

$$P' = \frac{1}{1 + I_{P_1} + I_{P_2} + \dots + I_{P_n}} > \frac{1}{1 + (0.5)n} = \frac{2}{2 + n}.$$

□

Corollary 4.12. *If ψ^I is an inverse fuzzy multigraph with geometric representation which has 1 or 2 intersecting points satisfy that $I_P < 0.5$, for every intersecting point, then this geometric representation of ψ^I is an inverse fuzzy strong planar multigraph.*

Proof. According to Theorem 4.11, $P' > \frac{2}{2+n}$. For $n = 1$, we have $P' > \frac{2}{2+1} = \frac{2}{3} > 0.5$, and for $n = 2$, we have $P' > \frac{2}{2+2} = \frac{2}{4} = 0.5$. So this geometric representation of ψ^I is an inverse fuzzy strong planar multigraph. □

Example 4.13. Let $\psi^I = (V, \sigma, E)$ be an inverse fuzzy multigraph represented in a geometric representation showed in Figure 5.

Figure 5: The inverse fuzzy multigraph ψ^I

Then this geometric representation of ψ^I has 3 intersecting points $\{P_1, P_2, P_3\}$, satisfies that $I_{P_1} = 0.35 < 0.5$, $I_{P_2} = 0.45 < 0.5$, and $I_{P_3} = 0.45 < 0.5$, but $P' = \frac{1}{1+0.35+0.45+0.45} \approx 0.47 < 0.5$. Hence this geometric representation of ψ^I is weak planar, even if $I_{P_i} < 0.5$, for $i = 1, 2, 3$.

Theorem 4.14. *If ψ^I is an inverse fuzzy multigraph with geometric representation which has $n \geq 2$ intersecting points and which is edge stable satisfies that $\mu_k(x, y) = \gamma < \frac{1}{n}$, then this geometric representation of ψ^I is strong planar.*

Proof. Since ψ^I is an edge stable, we get $I_{P_i} = \frac{\gamma+\gamma}{2} = \gamma < \frac{1}{n}$, for $i = 1, 2, \dots, n$. From $I_{P_i} < \frac{1}{n}$ and $n \geq 2$, we have $I_{P_i} < \frac{1}{2}$, for $i = 1, 2, \dots, n$. According to Theorem 4.5, $P' = \frac{1}{1+n\gamma}$. Moreover $\gamma < \frac{1}{n}$, then $P' > \frac{1}{1+n \cdot \frac{1}{n}} = 0.5$. Hence this geometric representation of ψ^I is strong planar. $\square$

Note 4.15. If ψ^I is an inverse fuzzy multigraph with geometric representation which has only 1 intersecting point and which is edge stable satisfies that $\mu_k(x, y) = \gamma < 0.5$, then this geometric representation of ψ^I is strong planar, because $I_{P_i} = \frac{\gamma+\gamma}{2} = \gamma < 0.5$ and $P' = \frac{1}{1+n\gamma} > \frac{1}{1+0.5} \approx 0.67$.

Theorem 4.16. If ψ^I is an inverse fuzzy multigraph with geometric representation which has $n > 2$ intersecting points, all of them between non-effective inverse fuzzy edges and satisfy that $\beta = \max\{\sigma(x) : x \in V\} \leq \frac{1}{n}$, then this geometric representation of ψ^I is strong planar.

Proof. Since all of intersecting points of ψ^I are between non-effective inverse fuzzy edges, then

$$\mu_k(x, y) < \max\{\sigma(x), \sigma(y)\} \leq \beta,$$

for every inverse fuzzy edge $((x, y), \mu_k(x, y))$ in the intersecting. Hence,

$$I_{P_i} = \frac{\mu_k(x, y) + \mu_l(z, t)}{2} < \frac{\beta + \beta}{2} = \beta \leq \frac{1}{n}.$$

Since $n > 2$, we get $I_{P_i} < \frac{1}{n} < 0.5$. By Theorem 4.6, $P' > \frac{1}{1+n\beta}$. Since $\beta \leq \frac{1}{n}$, we have $n\beta \leq 1$ and so $1 + n\beta \leq 1 + 1 = 2$. Hence $\frac{1}{1+n\beta} \geq \frac{1}{2}$ and then $P' > \frac{1}{1+n\beta} \geq 0.5$. Therefore, this geometric representation of ψ^I is strong planar. $\square$

5 Application about planarity of inverse fuzzy multigraphs in decision-making how to reduce the cost of travel tours

For any tourism company, it is important to organize their tours such that they cover the most famous and lovely places in their destination with the lowest possible cost. We can use the concept of planarity in inverse fuzzy multigraph to help us to make a decision about the lowest cost plan for the tour.

For example, if we want to organize a tour for a group of tourists to visit 4 famous places in one day and we want to setup a program for the group to visit these 4 places in one day such that they visit two places together in the morning and another two places in the afternoon with the lowest possible cost. In this case we can use the concept of planarity in inverse fuzzy multigraph to help us with determining the better program for the tour.

First we consider every place of the 4 places to be a vertex in our graph. Then we want to give a membership degree for every vertex and because the cost is very important thing in our case so we give every vertex a membership degree which represent the cost of visiting this place. Of course here we have to normalize this membership degree by dividing every expected cost for every place into the total cost which we need to finish all the tour.

Hence if the cost (which including tickets price and transportation costs) for visiting the 4 places $\{x_1, x_2, x_3, x_4\}$ is $\{1500, 2000, 2500, 4000\}$, respectively. So the total cost is $1500 + 2000 + 2500 + 4000 = 10000$. Then we can give the vertex x_1 the membership degree $\sigma(x_1) = \frac{1500}{10000} = 0.15$. By the similar way we find the membership degrees for all vertices, $\sigma(x_2) = 0.2, \sigma(x_3) = 0.25$ and $\sigma(x_4) = 0.4$.

We want to setup the program such that we visit two places in one period (morning or afternoon). To discuss the choices we form an inverse fuzzy graph for every choice, such that we draw an edge between two vertices which we want to visit together in the morning and another edge between two vertices which we want to visit together in the afternoon and we draw these two edges such

that they intersect in one point P which represents the same day. Of course, if we want to organize a tour for n days we will have n intersecting points, every one of them represents one day of our tour, where the vertices of the intersecting edges represent the places which we want to visit in the same day. It means that for every choice we will have a graph with 4 vertices (every one represents a place), 2 edges (every one represents a period morning or afternoon) and 1 intersecting point (represents the same day). This graph will be inverse fuzzy graph if we give a membership degree for every edge to be the required cost for visiting both of places in these two vertices in the same period, and it is necessary to normalize this value by dividing it into the total cost. It is clear that our graph will be inverse fuzzy graph such that the membership degree of any edge (the required cost to visit two places together in the same period) for sure is greater than the cost of visiting every place alone, and it is better when we visit two places which they are near to each other because visiting them together economizes transportation costs.

So to determine which program is better it is enough to study planarity of every possible inverse fuzzy graph and choose the one which is strong planar and has the greatest inverse fuzzy planarity value P' . If all the choices are weak planar, then it is enough to choose the one with the greatest inverse fuzzy planarity value P' . In our example we have 3 possible choices (programs).

First choice, to visit places x_1 and x_2 in the morning and places x_3 and x_4 in the afternoon. We represent this choice by the inverse fuzzy graph ψ_1^I showed in Figure 6.

Figure 6: The inverse fuzzy graph ψ_1^I

Second choice, to visit places x_1 and x_3 in the morning and places x_2 and x_4 in the afternoon. We represent this choice by the inverse fuzzy graph ψ_2^I showed in Figure 7.

Figure 7: The inverse fuzzy graph ψ_2^I

Third choice, to visit places x_1 and x_4 in the morning and places x_2 and x_3 in the afternoon. We represent this choice by the inverse fuzzy graph ψ_3^I showed in Figure 8.

Figure 8: The inverse fuzzy graph ψ_3^I

To determine which choice is better we have to calculate the inverse fuzzy planarity value in every choice.

In ψ_1^I , we get $I_P = \frac{0.3+0.6}{2} = 0.45$ and so $P'_1 = \frac{1}{1+0.45} \approx 0.68$, that means ψ_1^I is strong planar.

In ψ_2^I , we get $I_P = \frac{0.4+0.6}{2} = 0.5$ and so $P'_2 = \frac{1}{1+0.5} \approx 0.67$, that means ψ_2^I is weak planar.

In ψ_3^I , we get $I_P = \frac{0.5+0.3}{2} = 0.4$ and so $P'_3 = \frac{1}{1+0.4} \approx 0.71$, that means ψ_3^I is strong planar.

Hence the inverse fuzzy strong planar graph with the greatest inverse fuzzy planarity value will be the better choice since the cost will be the lowest. It means that the third choice is the better one in our example.

6 Conclusion

In the present article, concept of inverse fuzzy multigraph was defined. Then the notation of planarity of an inverse fuzzy multigraph was presented by using the intersecting value and inverse fuzzy planarity value. Also some theorems give bounds for inverse fuzzy planarity value in general and in special cases is showed. After that strong (weak) planarity is defined and showed some related theorems and application. In the future, it is recommended to study more about planarity and the concept of faces and dual in inverse fuzzy multigraphs.

7 Declarations

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Conflict of interest: The authors declare that there is no conflict of interest.

Human and animal rights: This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent: Informed consent was obtained from all individual participants included in the study.

8 Author contributions

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by all. The first draft of the manuscript was co-written and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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Figures

Figure 1

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Figure 2

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Figure 3

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Figure 4

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Figure 5

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Figure 6

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Figure 7

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Figure 8

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