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On active disturbance rejection control for lower-triangular nonlinear uncertain systems with mismatched uncertainties and unknown control coefficients

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Abstract

The paper investigates the control problem for a class of lower-triangular nonlinear uncertain systems with mismatched uncertainties and unknown values of control coefficients. Based on the signs of control coefficients rather than the nominal values or the approximative mathematical expressions, a new active disturbance rejection control is proposed. The design procedure can be concluded by three steps: determining the equivalent integrators chain form, constructing the extended state observer to estimate the total disturbance, and designing a dynamical system to let the actual input track the ideal input. Then under a mild assumption for mismatched uncertainties and unknown control coefficients, the paper rigorously analyzes the bounds of tracking error, estimating error and the error between the actual and ideal inputs. The presented theoretical results reveal the strong robustness of the proposed method to mismatched uncertainties and uncertain control input coefficients. Moreover, the tuning law of observer parameter and the parameter of dynamical input design is theoretically shown.

Keywords: nonlinear uncertain system, active disturbance rejection control, mismatched uncertainty, unknown control coefficient.

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1. Introduction

Uncertainties, including unmodeled nonlinear dynamics, external disturbances and parametric perturbations, are ubiquitous in practice. In control science and technology, it is a central issue to ensure the normal operation of control systems despite various uncertainties [1]. Motivated by this important objective, numerous control methods have been developed, such as proportional-integral-derivative control [2], adaptive control [3] and disturbance rejection methods [4–7]. Among various disturbance rejection methods, active disturbance rejection control (ADRC) has drawn lots of attention from researches due to its simplicity in practical implementation and superior performance to handle uncertainties, which has been successfully applied to flight systems [8–10], motion control systems [11, 12] and process control systems [13–15], just to name a few.

In the framework of ADRC, the extended state observer (ESO) is novelly constructed to estimate the total effect of various uncertainties, named as “total disturbance” [16], and then the control input is composed of the compensation for total disturbance and the feedback of system states. In the last two decades, the theoretical foundation of ADRC has been substantially established. For the ESO which is the vital component in ADRC, the literatures [17] investigated the convergence of ESO and the bound of estimating error. The closed-loop stability of ADRC was firstly presented in [18], where the considered uncertainties and their derivatives with respect to the time are assumed to be bounded. For the linear time-invariant uncertain systems, the literature [19] theoretically illustrated the satisfied tracking performance of ADRC despite a large scope of parametric variations. In [20], by assuming the existence of the Lyapunov functions related with uncertainties, the capability of ADRC to deal with nonlinear uncertainties was proved. By assuming that the uncertainties and their partial derivatives are bounded if the states are in a bounded set, the references [21, 22] rigorously studied the convergence and the transient performance of ADRC. It is remarkable that the conditions in [21, 22] can describe a wide class of nonlinear uncertainties in practice. By utilizing the concept of total disturbance, the literatures [23, 24] illuminated the capability of ADRC to handle the mismatched nonlinearities.

ties in both dynamical systems and measurement model. Besides, several successful modifications of ADRC have been made for nonlinear uncertain systems with other complicated practical factors, such as time delay [25, 26] and stochastic uncertainties [27, 28].

Up to now, the theoretical results demonstrate the effectiveness of ADRC to tackle the time-varying disturbances and the nonlinear internal uncertainties dependent on system states. In the conventional ADRC design, it is remarkable that the information of the control coefficient is required to design the compensation term for total disturbance [7, 29, 30]. In practical systems, the true value of control coefficient is usually unknown [19], which promotes the development of the ADRC based on the nominal values or approximative mathematical expression of control coefficient. For the ADRC based on the nominal information of control coefficient, the literatures [19, 21, 23] quantitatively analyzed the stability range of the uncertain control coefficient. Unfortunately, compared with the nonlinear uncertainties dependent on system states, the capability of the conventional ADRC to deal with the uncertainties of control coefficient is limited [31]. Besides, in some practical systems, it is difficult to obtain the nominal values or approximative mathematical expression of control coefficient [13, 14, 25]. Hence it is significant to design a new ADRC which is featured with strong robustness to uncertainties and does not rely on the nominal value or the approximative mathematical expression of control coefficient.

The paper studies the control problem for a class of lower-triangular nonlinear uncertain systems with mismatched uncertainties. Based on the signs of control coefficients rather than the nominal values or the mathematical expressions, a new ADRC design is proposed to handle the uncertainties caused by the mismatched nonlinearities and the unknown values of control coefficients. The design procedure of the new ADRC can be separated into three parts: 1) determining the transformation from the original states to the states of an integrators chain system; 2) designing the ESO to estimate the total disturbance and the states of the integrators chain system; 3) forcing the actual input to track the ideal input by a dynamical system. By rigorously analyzing the properties of the closed-loop system, the bounds of tracking error, estimating error and the error between the actual and ideal inputs are explicitly shown as the functions

of control parameters. Based on the detailed expressions of error bounds, it is demonstrated that the satisfied closed-loop performance can be obtained by suitably enlarging the ESO's parameter despite a wide class of uncertainties. Moreover, by meticulously
 65 studying the relationship between the original states and integrators chain states, the paper removes the bounded assumption for integrators chain states [16, 23]. The main contributions of the paper are as follows.

- (1). Based on the signs of control coefficients rather than the nominal values or the mathematical expressions, a new ADRC is proposed.
- 70 (2). By considering a large scope of nonlinear uncertainties, the satisfied transient performance of the proposed ADRC is proved.
- (3). The tuning principle of the control parameters is provided. Especially, the relationship between the parameter of input dynamical system and the ESO's parameter is explicitly shown.

75 The rest of the paper has the following organization. In Section 2, the problem formulation is given. In Section 3, the new ADRC design based on the signs of control coefficients is proposed. In Section 4, the transient performance of the closed-loop system is presented. The simulation studies are shown in Section 5. The conclusion is presented in Section 6.

80 1.1. Notations

The following notations are used throughout the paper. $y^{(k)}(t)$ is the k -th order derivative of y respect to the variable t for $k \geq 1$ and $y^{(0)}(t) \triangleq y(t)$. The notations $|\cdot|$ and $\|\cdot\|$ are the absolute value of a scalar and the 2-norm of a vector or a matrix, respectively. The notation $\text{diag}(a_1, a_2, \dots, a_m)$ represents a diagonal matrix with the dimension $m \times m$, whose i -th diagonal element is a_i . For the real symmetric matrix A , the maximal eigenvalue of A is denoted as $\lambda_{\max}(A)$ and the minimal eigenvalue of A is

denoted as $\lambda_{\min}(A)$. The following useful matrices are introduced.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1}, \quad B_f = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(n+1) \times 1}, \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}, \\ A_e &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}_{(n+1) \times (n+1)}, \quad C_e = \begin{bmatrix} C \\ 0 \end{bmatrix}_{(n+1) \times 1}. \end{aligned} \quad (1)$$

The function $\text{sgn}(\cdot)$ represents the sign function, which satisfies

$$\text{sgn}(a) = \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{if } a = 0, \\ -1, & \text{if } a < 0. \end{cases} \quad (2)$$

2. Problem formulation

Consider the following class of lower-triangular nonlinear uncertain systems with unknown control coefficients.

$$\begin{cases} \dot{x}_i(t) = \theta_i(t)x_{i+1}(t) + \phi_i(x_1(t), \dots, x_i(t), t), & 1 \leq i \leq n-1, \\ \dot{x}_n(t) = \theta_n(t)u(t) + \phi_n(x_1(t), \dots, x_n(t), t), \\ \dot{z}(t) = g(z, x, t), \\ y(t) = x_1(t), & t \geq t_0, \end{cases} \quad (3)$$

where $x(t) = [x_1(t) \ \cdots \ x_n(t)]^T \in \mathbb{R}^n$ and $z(t) \in \mathbb{R}^m$ are the system states, $y(t) \in \mathbb{R}$ is the measured output to be controlled, $u(t) \in \mathbb{R}$ is the control input, $g(\cdot)$ represents the dynamics of the state $z(t)$, $\phi_i(\cdot)$ ($1 \leq i \leq n$) represent the uncertainties in various channels which might be mismatched, and $\theta_i(t)$ ($1 \leq i \leq n$) are the control coefficients. As shown in [32], the signs of the control coefficients $\theta_i(t)$, i.e., $\text{sgn}(\theta_i(t))$ ($1 \leq i \leq n$), represent the control directions of the system (3). The paper considers the situation that the control directions $\text{sgn}(\theta_i(t))$ ($1 \leq i \leq n$) are known. However, both the approximate mathematical expressions and the nominal values of the control coefficients $\theta_i(t)$ are unknown.

Based on the control directions $\text{sgn}(\theta_i(t))$ rather than the nominal values or mathematical expressions of control coefficients, the control objective of the system (3) is

to design the control input $u(t)$ such that the output $y(t)$ can track the reference signal $r(t)$ despite the multiple uncertainties $\phi_i(\cdot)$ ($1 \leq i \leq n$).

95 Before the detailed control design, the following assumptions for the reference signal $r(t)$, the multiple uncertainties $\phi_i(\cdot)$ ($1 \leq i \leq n$) and the dynamics of the state $z(t)$ are introduced.

Assumption 1. *There exists a positive constant M_r such that $\sup_{t \geq t_0} |r^{(i)}(t)| \leq M_r$ for $0 \leq i \leq n+1$.*

Assumption 2. *The functions $\phi_i(\cdot)$ and $\theta_i(\cdot)$ are $(n+1-i)$ -th order differentiable with respect to their variables for $1 \leq i \leq n$. There exist positive constants $\bar{M}_{\theta,i}$ and $\underline{M}_{\theta,i}$ and continuous functions $\psi_{\phi,i}(x_1, \dots, x_i)$ such that*

$$\sup_{t \geq t_0, 0 \leq j \leq n+1-i} |\theta_i^{(j)}(t)| \leq \bar{M}_{\theta,i}, \quad \inf_{t \geq t_0} |\theta_i(t)| \geq \underline{M}_{\theta,i} > 0, \quad (4)$$

$$\sup_{t \geq t_0} \left| \frac{\partial^{\Sigma_{k=1}^{i+1} j_k} \phi_i(x_1, \dots, x_i, t)}{\partial x_1^{j_1} \dots \partial x_i^{j_i} \partial t^{j_{i+1}}} \right| \leq \psi_{\phi,i}(x_1, \dots, x_i), \quad (5)$$

100 for $\Sigma_{k=1}^{i+1} j_k \leq n+i-1$, $1 \leq i \leq n$ and $j_p \geq 0$ ($1 \leq p \leq i+1$).

Assumption 3. *There exists a radially unbounded positive-definite function $V_z(z)$ such that*

$$\dot{V}_z(z(t)) = \frac{dV_z}{dz} g(z, x, t) \leq 0, \quad \forall V_z(z) \geq r_z(\|x\|), \quad t \geq t_0, \quad (6)$$

where $r_z(\cdot)$ is a nonnegative continuous increasing function.

Assumption 1 implies that the reference signal and its derivatives are bounded, which is rational in practice [29]. Assumption 2 assumes that the control coefficients and their derivatives are bounded. Moreover, (4) implies that the control coefficients are not equal to zero for $t \geq t_0$ which ensures the controllability of the system (3). From (5), the uncertainties $\phi_i(\cdot)$ are bounded by some continuous functions $\psi_{\phi,i}(\cdot)$ dependent on the system states, which allows a large scope of internal uncertainties and external disturbances in practice. By regarding $x(t)$ as the input of the z -subsystem, Assumption 3 implies that the z -subsystem is uniformly input-state-stable [21]. For the system (3) being a linear time invariant system, Assumption 3 is equivalent to that the system (3) is a minimum phase plant.

110

In the next section, an ADRC design based on the control directions $\text{sgn}(\theta_i(t))$ is proposed to achieve the tracking objective despite multiple uncertainties.

Remark 1. *As shown in the conventional ADRC design [7, 29], the control coefficient plays an important role in the design of ESO and feedback term. However, it is sometimes hard to acquire the nominal values or the mathematical expressions of control coefficients in practical systems [13, 14]. Moreover, due to the physical mechanism, the control directions, i.e., the signs of the control coefficients, can be determined. Hence the paper aims to develop a new ADRC design based on the control directions rather than the nominal values or the approximative mathematical expressions of control coefficients.*

3. ADRC design based on control directions

In this section, a new ADRC design based on control directions is proposed. The rest of this section consists of the following three parts.

1. The equivalent integrators chain form for the system (3) is investigated, which shows the essential relationship from the input to the output. Moreover, the output regulation task is transmitted to the state tracking task.
2. Based on the integrators chain system, an ESO is presented to estimate the derivatives of the output and the unknown term named as “total disturbance”.
3. Via the estimations from ESO, the control input is generated by a dynamical system, which forces the input to track the ideal input signal.

3.1. Analysis for the equivalent integrators chain form

In this subsection, the equivalent integrators chain form for the system (3) is investigated.

Denote the new state vector $\tilde{x}(t) = [\tilde{x}_1(t) \cdots \tilde{x}_n(t)]^T$ as follows.

$$\begin{cases} \tilde{x}_1(t) = x_1(t), \\ \tilde{x}_i(t) = \dot{\tilde{x}}_{i-1}(t) \\ \quad = \left(\prod_{j=1}^{i-1} \theta_j(t) \right) x_i(t) + \sum_{j=1}^{i-2} \frac{d^{j-1}}{dt^{j-1}} \left(\frac{d \left(\prod_{k=1}^{i-j-1} \theta_k(t) \right)}{dt} x_{i-j}(t) \right) \\ \quad + \sum_{j=1}^{i-1} \frac{d^{j-1}}{dt^{j-1}} \left(\left(\prod_{k=0}^{i-j-1} \theta_k(t) \right) \phi_{i-j}(x_1, \dots, x_{i-j}, t) \right), \quad 2 \leq i \leq n, \end{cases} \quad (7)$$

135 where $\theta_0(t) = 1$ for $t \geq t_0$.

The following lemma illustrates the relationship between the state vectors x and $\tilde{x}$.

Lemma 1. *Consider the transformation (7) with Assumption 2. Then, there exist two continuous mappings satisfying*

$$\gamma(x, t) \triangleq \begin{bmatrix} \gamma_1(x, t) \\ \vdots \\ \gamma_{n+1}(x, t) \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \\ t \end{bmatrix}, \quad \varphi(\tilde{x}, t) \triangleq \begin{bmatrix} \varphi_1(\tilde{x}, t) \\ \vdots \\ \varphi_{n+1}(\tilde{x}, t) \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ t \end{bmatrix} \quad (8)$$

and

$$\sup_{t \geq t_0, \|x\| \leq \rho_x} \left\{ |\gamma_i(x, t)|, \left\| \frac{\partial \gamma_i(x, t)}{\partial x} \right\|, \left| \frac{\partial \gamma_i(x, t)}{\partial t} \right| \right\} \leq \psi_\gamma(\rho_x), \quad \forall \rho_x \geq 0, \forall 1 \leq i \leq n, \quad (9)$$

$$\sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} \left\{ |\varphi_i(\tilde{x}, t)|, \left\| \frac{\partial \varphi_i(\tilde{x}, t)}{\partial \tilde{x}} \right\|, \left| \frac{\partial \varphi_i(\tilde{x}, t)}{\partial t} \right| \right\} \leq \psi_\varphi(\tilde{\rho}_x), \quad \forall \tilde{\rho}_x \geq 0, \forall 1 \leq i \leq n, \quad (10)$$

where $\psi_\gamma(\cdot)$ and $\psi_\varphi(\cdot)$ are continuous increasing functions dependent on $\bar{M}_{\theta,i}$, $\underline{M}_{\theta,i}$ and $\psi_{\phi,i}$ for $1 \leq i \leq n$.

The proof of Lemma 1 is given in Appendix.

140 **Remark 2.** *In the existing studies [16, 23], the bounds of the mappings $\gamma_i(\cdot)$ and $\varphi_i(\cdot)$ are provided as an additional assumption. In this paper, by rigorously analyzing the detailed expressions of the mappings $\gamma_i(\cdot)$ and $\varphi_i(\cdot)$, we prove that $\gamma_i(\cdot)$ and $\varphi_i(\cdot)$ and their partial derivatives are bounded, i.e., (9)–(10). Hence the assumptions for the bounds of $\gamma_i(\cdot)$ and $\varphi_i(\cdot)$ are removed in the paper.*

Based on the transformation (7) and Lemma 1, the system (3) can be rewritten as the following integrators chain system.

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{x}_{i+1}(t), & 1 \leq i \leq n-1, \\ \dot{\tilde{x}}_n(t) = b(t)u(t) + f(\tilde{x}(t), t), \\ \dot{z}(t) = g(z, \varphi_1(\tilde{x}, t), \dots, \varphi_n(\tilde{x}, t), t), \\ y(t) = \tilde{x}_1(t), & t \geq t_0, \end{cases} \quad (11)$$

with the initial condition $\tilde{x}(t_0) = [\gamma_1(x(t_0), t_0) \cdots \gamma_n(x(t_0), t_0)]^T$. The control coefficient $b(t)$ and the uncertainty $f(\tilde{x}(t), t)$ have the following form.

$$\begin{cases} b(t) = \Pi_{i=1}^n \theta_i(t), \\ f(\tilde{x}, t) = \sum_{j=1}^{n-1} \frac{d^{j-1}}{dt^{j-1}} \left(\frac{d(\Pi_{k=1}^{n-j} \theta_k(t))}{dt} \varphi_{n+1-j}(\tilde{x}, t) \right) \\ \quad + \sum_{j=1}^n \frac{d^{j-1}}{dt^{j-1}} \left((\Pi_{k=0}^{n-j} \theta_k(t)) \tilde{\varphi}_{n+1-j}(\tilde{x}, t) \right), \end{cases} \quad (12)$$

145 where $\tilde{\varphi}_i(\tilde{x}, t) = \phi_i(\varphi_1(\tilde{x}, t), \dots, \varphi_i(\tilde{x}, t), t)$ for $1 \leq i \leq n$.

If the nominal value of the control coefficient $b(t)$, denoted as $\bar{b}(t)$, can be obtained, the conventional ADRC design can be applied to the system (11) by regarding $f(\tilde{x}, t) + (b(t) - \bar{b}(t))u(t)$ as the total disturbance [23, 29]. However, only the sign of the control coefficient $b(t)$ is known in the paper, rather than the nominal value or the mathematical expression.

$$\text{sgn}(b(t)) = \Pi_{i=1}^n \text{sgn}(\theta_i(t)). \quad (13)$$

Next, based on the integrators chain form (11) and the sign of control coefficient $b(t)$, an ADRC design will be proposed.

3.2. ESO design

Since the nominal value of the control coefficient $b(t)$ is unknown, the total disturbance of the system (11) is denoted as

$$f_t(\tilde{x}, u, t) \triangleq b(t)u(t) + f(\tilde{x}, t). \quad (14)$$

Then the following ESO is presented to estimate $\tilde{x}$ and f_t .

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{f}}_t(t) \end{bmatrix} = A_e \begin{bmatrix} \hat{x}(t) \\ \hat{f}_t(t) \end{bmatrix} + L_e \left(y(t) - C_e^T \begin{bmatrix} \hat{x}(t) \\ \hat{f}_t(t) \end{bmatrix} \right), \quad (15)$$

where $\hat{\tilde{x}}(t) = [\hat{\tilde{x}}_1(t) \ \cdots \ \hat{\tilde{x}}_n(t)]^T \in R^n$ is the estimation for the system state vector $\tilde{x}(t)$ and $\hat{f}_t(t) \in R$ is the estimation for the total disturbance $f_t(\tilde{x}, u, t)$. In addition, the constant vector $L_e \in R^{(n+1) \times 1}$ is the tunable parameter vector of ESO such that the matrix $A_L \triangleq A_e - L_e C_e^T$ is Hurwitz. Owing to [33], the following concise tuning method of L_e is presented.

$$L_e = \begin{bmatrix} \varsigma_1 \omega_o & \varsigma_2 \omega_o^2 & \cdots & \varsigma_{n+1} \omega_o^{n+1} \end{bmatrix}^T, \quad \varsigma_i = \frac{(n+1)!}{(n+1-i)!i!}, \quad \omega_o \geq 0, \quad (16)$$

which ensures that all the eigenvalues of A_L are set at $-\omega_o$.

150 Next, a dynamical design of ADRC input based on the estimations from ESO will be presented.

3.3. Dynamical input design

Firstly, the ideal control input is introduced. For the system (11), the desired closed-loop system satisfies the following form.

$$\begin{cases} \dot{\tilde{x}}_i^*(t) = \tilde{x}_{i+1}^*(t), & 1 \leq i \leq n-1, \\ \dot{\tilde{x}}_n^*(t) = -K^T (\tilde{x}^*(t) - \bar{r}(t)) + r^{(n)}(t), \\ \dot{z}^*(t) = g(z^*, \varphi_1(\tilde{x}^*, t), \cdots, \varphi_n(\tilde{x}^*, t), t), \\ y^*(t) = \tilde{x}_1^*(t), \quad t \geq t_0, \quad \tilde{x}^*(t_0) = \tilde{x}(t_0), \end{cases} \quad (17)$$

155 where $x^*(t) = [x_1^*(t) \ \cdots \ x_n^*(t)] \in R^n$ and $z^*(t) \in R^m$ are the states of the desired system, $y^*(t) \in R$ is the desired output, $\bar{r}(t) = [r(t) \ \cdots \ r^{(n-1)}(t)]^T \in R^n$ and the constant vector $K \in R^{n \times 1}$ is feedback gain vector satisfying that the matrix $A_K \triangleq A - BK^T$ is Hurwitz.

Remark 3. For the desired system (17), the output $y^*(t)$ can exponentially converge to the reference signal $r(t)$ with the desired convergence rate by tuning the feedback gain vector K . Moreover, the system states $(\tilde{x}(t), z^*(t))$ are bounded for $t \geq t_0$.

By comparing the integrators chain system (11) with the desired system (17), the ideal control input can be obtained as follows.

$$u^*(t) = \frac{-f(\tilde{x}, t) - K^T (\tilde{x}(t) - \bar{r}(t)) + r^{(n)}(t)}{b(t)}. \quad (18)$$

Inspired by the approximative dynamic inversion method in [34, 35], the actual control input is generated by the following dynamics.

$$\dot{u}(t) = -\text{sgn}(b(t))\kappa(\omega_o)(\hat{f}_t(t) + K^T(\hat{x}(t) - \bar{r}(t)) - r^{(n)}(t)), \quad (19)$$

where $\kappa(\omega_o) > 0$ is a function to be designed.

160 The parameter $\kappa(\omega_o)$ should be carefully selected to ensure that the dynamics (19) is “slower” than the dynamics of the ESO (15) [35]. However, the explicit tuning law of $\kappa(\omega_o)$ has not been provided in the existing studies. To make the proposed method more friendly to practitioners, the tuning law of $\kappa(\omega_o)$ is investigated and is explicitly presented in the following assumption.

Assumption 4. *The increasing function $\kappa(\omega_o) > 0$ for $\omega_o > 0$ and*

$$\lim_{\omega_o \rightarrow \infty} \frac{\ln \omega_o}{\sqrt{\kappa(\omega_o)}} = 0, \quad \lim_{\omega_o \rightarrow \infty} \frac{\kappa(\omega_o)}{\omega_o} = 0. \quad (20)$$

165 Assumption 4 describes the growth rate of the increasing function $\kappa(\omega_o)$ which grows faster than the log function $f_1(\omega_o) = (\ln \omega_o)^2$ and slower than the linear function $f_2(\omega_o) = \omega_o$. It is remarkable that the function $\kappa(\omega_o) = \omega_o^k$ with $0 < k < 1$ can satisfy Assumption 4.

Remark 4. *The ideology of approximative dynamic inversion method is to design the dynamics of $u(t)$ which satisfies the following equation.*

$$\dot{u}(t) = -a(t)(u(t) - u^*(t)), \quad (21)$$

where $a(t) > 0$ is a function to be designed. By substituting (18) into (21), we have

$$\begin{aligned} \dot{u}(t) &= -\frac{a(t)}{b(t)}(b(t)u(t) + f(\tilde{x}, t) + K^T(\tilde{x} - \bar{r}(t)) - r^{(n)}(t)) \\ &= -\frac{a(t)}{b(t)}(f_t(\tilde{x}, u, t) + K^T(\tilde{x} - \bar{r}(t)) - r^{(n)}(t)). \end{aligned} \quad (22)$$

By designing $a(t) = \kappa(\omega_o)|b(t)|$ and substituting the estimations $(\hat{f}_t, \hat{x})$ into (22), then
170 (19) is obtained.

4. Performance analysis of the closed-loop system

The performance of the closed-loop system based on the ADRC design (15) and (19) will be investigated in this section.

The following theorem shows the satisfactory transient performance of the closed-loop system based on the proposed ADRC despite a wide class of uncertainties.

Theorem 1. Consider the system (3) with Assumptions 1–4. Let $u(t) = 0$ for $t \in [t_0, t_u)$ where

$$\begin{cases} t_u = t_0 + 2nc_{\zeta 2} \frac{\max\{\ln(\omega_o \rho_0), 0\}}{\sqrt{\kappa(\omega_o)}}, & \rho_0 = \max_{2 \leq i \leq n} |\tilde{x}_i(t_0) - \hat{x}_i(t_0)|^{\frac{1}{n}}, \\ c_{\zeta 2} = \lambda_{\max}(P_{\zeta}), \quad A_{\zeta}^T P_{\zeta} + P_{\zeta} A_{\zeta} = -I, \quad A_{\zeta} = \begin{bmatrix} -\zeta_1 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\zeta_n & 0 & \cdots & 0 & 1 \\ -\zeta_{n+1} & 0 & \cdots & \cdots & 0 \end{bmatrix}. \end{cases} \quad (23)$$

For $t \geq t_u$, $u(t)$ is designed according to (15) and (19). Then there exist positive constants η_i^* ($1 \leq i \leq 5$) and ω^* dependent on $(x(t_0), \hat{x}(t_0), \hat{f}_t(t_0), M_r, \underline{M}_{\theta, i}, \bar{M}_{\theta, i}, \psi_{\phi i}, K)$ such that

$$\sup_{t \geq t_0} |y(t) - y^*(t)| \leq \eta_1^* \max \left\{ \frac{\ln \omega_o}{\sqrt{\kappa(\omega_o)}}, \frac{\kappa(\omega_o)}{\omega_o}, \frac{1}{\kappa(\omega_o)} \right\}, \quad (24)$$

$$\left\| \begin{bmatrix} \tilde{x}(t) - \hat{x}(t) \\ f_t(\tilde{x}, u, t) - \hat{f}_t(t) \end{bmatrix} \right\| \leq \eta_2^* \left(\frac{\kappa(\omega_o)}{\omega_o} + e^{-\eta_3^* \omega_o (t - t_u)} \right), \quad \forall t \geq t_u, \quad (25)$$

$$|u(t) - u^*(t)| \leq \eta_4^* \left(\frac{\kappa(\omega_o)}{\omega_o} + \frac{1}{\kappa(\omega_o)} + e^{-\eta_5^* \kappa(\omega_o) (t - t_u)} \right), \quad \forall t \geq t_u, \quad (26)$$

for any $\omega_o \geq \omega^*$.

Theorem 1 demonstrates that the tracking error between the actual and ideal outputs, the estimating error of the ESO (15) and the error between the actual and ideal inputs are bounded. Furthermore, (24)–(26) explicitly show the bounds of the tracking error, estimating error and the error between the actual and ideal inputs. More importantly, as shown in (24), the tracking error between the actual and ideal outputs can be sufficiently small for $t \geq t_0$ by tuning ω_o to be suitably large, which ensures the satisfied transient performance despite various uncertainties. In addition, (25)–(26) implies that

both the estimating error and the error between the actual and ideal inputs can converge
 185 into a neighbourhood with a small boundary by suitably enlarging ω_o , which illustrates
 the tuning law of the parameter ω_o .

Remark 5. *It is remarkable that the satisfied closed-loop performance despite a large
 scope of uncertainties can be achieved by tuning one scale parameter, which is friendly
 to practical practitioner.*

190 **Remark 6.** *The design (23) is a common way to avoid the peaking phenomenon of
 ESO [21]. Moreover, since plenty of practical systems satisfy the initial condition that
 $\tilde{x} = [0 \ \cdots \ 0]^T$, it can be obtained by designing $\hat{\tilde{x}} = [0 \ \cdots \ 0]^T$ that $t_u = t_0$. In addition,
 for the systems satisfying $\rho \geq \frac{1}{\omega_o}$, it can be deduced from (23) and Assumption 4 that
 t_u can be closed to t_0 by suitably enlarging ω_o .*

195 To simplify the proof of Theorem 1, Lemmas 2–3 are introduced. Lemma 2 de-
 scribes the bounds of the uncertainty $f(\tilde{x}, t)$, the control coefficient $b(t)$ and their
 derivatives. Lemma 3 provides the closed-loop form and the bounds of the uncertain
 terms in the closed-loop system. The proofs of Lemmas 2–3 are given in Appendix.

Lemma 2. *Let Assumption 2 hold. For any given positive constant $\tilde{\rho}_x$, the functions
 $b(t)$ and $f(\tilde{x}, t)$ in (12) satisfy the following equations:*

$$\sup_{t \geq t_0} \{ |(b(t))^{-1}|, |b(t)|, |\dot{b}(t)| \} \leq \psi_b, \quad (27)$$

$$\sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} \left\{ |f(\tilde{x}, t)|, \left\| \frac{\partial f(\tilde{x}, t)}{\partial \tilde{x}} \right\|, \left| \frac{\partial f(\tilde{x}, t)}{\partial t} \right| \right\} \leq \psi_f(\tilde{\rho}_x), \quad (28)$$

where the positive constant ψ_b and the non-decreasing function $\psi_f(\cdot)$ are dependent
 200 on $\bar{M}_{\theta, i}$, $\underline{M}_{\theta, i}$ and $\psi_{\phi, i}$ for $1 \leq i \leq n$.

Denote the tracking error vector, estimating error vector and the error between the
 actual and ideal control inputs as follows.

$$e(t) = \tilde{x}(t) - \tilde{x}^*(t), \quad \zeta(t) = T_1^{-1} \begin{bmatrix} \tilde{x}(t) - \hat{\tilde{x}}(t) \\ f_t(\tilde{x}, u, t) - \hat{f}_t(t) \end{bmatrix}, \quad \delta_u(t) = u(t) - u^*(t), \quad (29)$$

where $T_1 = \text{diag}(\omega_o^{-n}, \dots, \omega_o^{-1}, 1)$. Then the following lemma presents the closed-loop system and further analyzes the properties of the uncertain terms in the closed-loop system.

Lemma 3. *Let Assumptions 1-3 hold. Design $u(t) = 0$ for $t \in [t_0, t_u)$, and design $u(t)$ by (15) and (19) for $t \in [t_u, \infty)$. Then the closed-loop system is shown as follow.*

$$\begin{cases} \dot{e}(t) = Ae(t) + B\Delta_{e0}(e, t), \\ \dot{z}(t) = g(z, \varphi_1(\tilde{x}^* + e, t), \dots, \varphi_n(\tilde{x}^* + e, t), t), \\ \dot{\zeta}(t) = \omega_o A_\zeta \zeta(t) + B_f \Delta_{\zeta 0}(e, t), \\ \dot{\delta}_u(t) = -|b(t)|\kappa \delta_u(t) + \Delta_{\delta_u 0}(e, \zeta, \omega_o, \kappa, t), \end{cases} \quad t \in [t_0, t_u), \quad (30)$$

$$\begin{cases} \dot{e}(t) = A_K e(t) + B\Delta_{e1}(\delta_u, t), \\ \dot{z}(t) = g(z, \varphi_1(\tilde{x}^* + e, t), \dots, \varphi_n(\tilde{x}^* + e, t), t), \\ \dot{\zeta}(t) = \omega_o A_\zeta \zeta(t) + B_f \Delta_{\zeta 1}(e, \zeta, \delta_u, \omega_o, \kappa, t), \\ \dot{\delta}_u(t) = -|b(t)|\kappa \delta_u(t) + \Delta_{\delta_u 1}(e, \zeta, \delta_u, \omega_o, \kappa, t), \end{cases} \quad t \in [t_u, \infty). \quad (31)$$

Moreover, the uncertain terms $(\Delta_{e0}, \Delta_{\zeta 0}, \Delta_{\delta_u 0}, \Delta_{e1}, \Delta_{\zeta 1}, \Delta_{\delta_u 1})$ have the following bounds.

$$\begin{cases} |\Delta_{e0}| \leq \pi_{e0}(\rho_e), \quad |\Delta_{e1}| \leq \psi_b |\delta_u|, \\ |\Delta_{\zeta 0}| \leq \pi_{\zeta 0}(\rho_e), \quad |\Delta_{\zeta 1}| \leq \pi_{\zeta 1}(\rho_e) + \pi_\omega(\omega_o^*)\kappa \|\zeta\| + (\kappa + 1)\pi_{\delta_u}(\rho_e, \rho_u), \\ |\Delta_{\delta_u 0}| \leq \pi_{\delta_u 0}(\rho_e) + \pi_\omega(\omega_o^*)\kappa \|\zeta\|, \quad |\Delta_{\delta_u 1}| \leq \pi_{\delta_u 1}(\rho_e, \rho_u) + \pi_\omega(\omega_o^*)\kappa \|\zeta\|, \end{cases} \quad (32)$$

for $e \in \{e \mid \|e\| \leq \rho_e\}$, $\zeta \in \{\zeta \mid \|\zeta\| \leq \rho_\zeta\}$, $\delta_u \in \{\delta_u \mid \|\delta_u\| \leq \rho_u\}$ and $\omega_o \in \{\omega_o \mid \omega_o \geq \omega_o^*\}$

205 with any given positives $\rho_e, \rho_\zeta, \rho_u$ and ω_o^* . The functions $\pi_{e0}(\cdot)$, $\pi_{\zeta 0}(\cdot)$, $\pi_{\zeta 1}(\cdot)$, $\pi_{\delta_u}(\cdot)$, $\pi_{\delta_u 0}(\cdot)$ and $\pi_{\delta_u 1}(\cdot)$ are non-decreasing and are dependent on K , M_r , $\bar{M}_{\theta, i}$, $\underline{M}_{\theta, i}$ and $\psi_{\phi, i}$ for $1 \leq i \leq n$. The function $\pi_\omega(\cdot)$ is non-increasing and is dependent on $\bar{M}_{\theta, i}$, $\underline{M}_{\theta, i}$ and K .

Based on Lemmas 2–3, the proof of Theorem 1 is presented as follows.

210 **Proof 1 (Proof of Theorem 1).** *With the mapping from (x, t) to $(\tilde{x}, t)$ presented in Lemma 1 and the discussions in Section 3.1, the plant (3) can be rewritten as the integrators chain system (11). Owing to Lemmas 2–3, the closed-loop system is formulated as (30)–(31).*

Since the matrices A_K and A_ζ are Hurwitz, there exist positive definite matrices P_K and P_ζ such that $A_K^T P_K + P_K A_K = -I$ and $A_\zeta^T P_\zeta + P_\zeta A_\zeta = -I$. Then the following Lyapunov functions are introduced.

$$V_K(t) = e^T(t) P_K e(t), \quad V_\zeta(t) = \zeta^T(t) P_\zeta \zeta(t), \quad V_u(t) = \frac{\delta_u^2(t)}{2}. \quad (33)$$

Let c_{k1} and c_{k2} be the minimal and maximal eigenvalues of P_K , and denote c_ζ and $c_{\zeta 2}$ as the minimal and maximal eigenvalues of P_ζ . Then the following inequalities hold.

$$c_{k1} \|e(t)\|^2 \leq V_K(t) \leq c_{k2} \|e(t)\|^2, \quad c_{\zeta 1} \|\zeta(t)\|^2 \leq V_\zeta(t) \leq c_{\zeta 2} \|\zeta(t)\|^2. \quad (34)$$

Next, we analyze the properties of the closed-loop system for $t \in [t_0, t_u)$ and $t \in [t_u, \infty)$.

215 Part 1: The analysis for the closed-loop system for $t \in [t_0, t_u)$.

Owing to (11) and (17), we get the initial condition $e(t_0) = 0$. The dynamics (30) implies the continuity of $e(t)$. Hence for a sufficiently small positive Δt satisfying $t_0 + \Delta t \leq t_u$, there exists a positive constant η_{e1} such that

$$\|e(t)\| \leq \eta_{e1}, \quad \forall t \in [t_0, t_0 + \Delta t]. \quad (35)$$

According to Assumption 4, it can be verified from that $\lim_{\omega_o \rightarrow \infty} \frac{\ln \omega_o}{\sqrt{\kappa(\omega_o)}} = 0$. In addition, it can be verified from (23) that $\lim_{\omega_o \rightarrow \infty} t_u = t_0$. Hence there exists a positive constant ω_1 such that $t_u - t_0 \leq \Delta t$ for $\omega_o \geq \omega_1$. Combined with the statement (35), it can be concluded that $\|e(t)\| \leq \eta_{e1}$ for $t \in [t_0, t_u)$ and $\omega_o \geq \omega_1$.

Due to Lemma 1, the bound of $x(t)$ for $t \in [t_0, t_u)$ satisfies that $\sup_{t_0 \leq t \leq t_u} \|x(t)\| \leq n\psi_\phi(\eta_{e1} + M_{x^*})$. Combined with Assumption 3, the following equation is satisfied.

$$\dot{V}_z(z(t)) \leq 0, \quad \forall V_z(z) \geq r_z(n\psi_\phi(\eta_{e1} + M_{x^*})), \quad t_0 \leq t \leq t_u, \quad (36)$$

which further implies that

$$\sup_{t_0 \leq t \leq t_u} V_z(z(t)) \leq \eta_{V_z 1} \triangleq r_z(n\psi_\phi(\eta_{e1} + M_{x^*})). \quad (37)$$

Based on the bound of $e(t)$ for $t \in [t_0, t_u)$, the dynamics (30) implies that

$$\begin{aligned} \sup_{t_0 \leq t \leq t_u} \|e(t)\| &\leq (t_u - t_0) \cdot \sup_{t_0 \leq t \leq t_u} \{\|A\| \|e(t)\| + \|B\| |\Delta_{e0}(e, t)|\} \\ &\leq 4nc_{\zeta 2} (\|A\| \eta_{e1} + \pi_{e0}(\eta_{e1})) \frac{\ln \omega_o}{\sqrt{\kappa}}. \end{aligned} \quad (38)$$

Due to (30) and (32), the following dynamics of $\sqrt{V_\zeta(t)}$ and $\sqrt{V_u(t)}$ for $t \in [t_0, t_u]$ hold for $\sqrt{V_\zeta(t)} \neq 0$ and $\sqrt{V_u(t)} \neq 0$.

$$\begin{cases} \frac{d\sqrt{V_\zeta(t)}}{dt} = \frac{-\omega_o \|\zeta\|^2 + 2\zeta^T P_\zeta B_f \Delta_{\zeta 0}}{2\sqrt{V_\zeta}} \leq -\frac{\omega_o}{2c_{\zeta 2}} \sqrt{V_\zeta(t)} + \frac{\|P_\zeta\| \pi_{\zeta 0}(\eta_{e1})}{\sqrt{c_{\zeta 1}}}, \\ \frac{d\sqrt{V_u(t)}}{dt} = \frac{-|b|\kappa|\delta_u|^2 + \delta_u \Delta_{\delta_u 0}}{2\sqrt{V_u}} \\ \leq -\psi_b^{-1} \kappa \sqrt{V_u(t)} + \frac{\sqrt{2}\pi_{\delta_u 0}(\eta_{e1}) + \sqrt{2}\pi_\omega(\omega_1)\kappa\|\zeta(t)\|}{2}, \end{cases} \quad (39)$$

for $\omega_o \geq \omega_1$. With the help of Gronwall lemma, it can be deduced from (39) that $\sqrt{V_\zeta(t)}$ has the following bound.

$$\sqrt{V_\zeta(t)} \leq \frac{2c_{\zeta 2}\|P_\zeta\|\pi_{\zeta 0}(\eta_{e1})}{\sqrt{c_{\zeta 1}}\omega_o} + \sqrt{V_\zeta(t_0)}e^{-\frac{\omega_o(t-t_0)}{2c_{\zeta 2}}}, \quad t \in [t_0, t_u]. \quad (40)$$

Based on Gronwall lemma and (39)–(40), the bound of $\sqrt{V_u(t)}$ is obtained as follows.

$$\begin{aligned} \sqrt{V_u(t)} &\leq \frac{\sqrt{2}\pi_{\delta_u 0}(\eta_{e1})}{2\psi_b^{-1}\kappa} + \sqrt{V_u(t_0)}e^{-\psi_b^{-1}\kappa(t-t_0)} + \int_{t_0}^t e^{-\psi_b^{-1}\kappa(t-s)} \sqrt{2}\pi_\omega \kappa \|\zeta(s)\| ds \\ &\leq \frac{\sqrt{2}\pi_{\delta_u 0}(\eta_{e1})}{2\psi_b^{-1}\kappa} + \frac{\sqrt{2}\pi_\omega c_{\zeta 2}\|P_\zeta\|\pi_{\zeta 0}(\eta_{e1})}{c_{\zeta 1}\psi_b^{-1}\omega_o} + \sqrt{V_u(t_0)}e^{-\psi_b^{-1}\kappa(t-t_0)} \\ &\quad + \frac{\sqrt{2}\pi_\omega \kappa \sqrt{V_\zeta(t_0)}}{\sqrt{c_{\zeta 1}}(\frac{\omega_o}{2c_{\zeta 2}} - \psi_b^{-1}\kappa)} e^{-\psi_b^{-1}\kappa(t-t_0)} (1 - e^{-(\frac{\omega_o}{2c_{\zeta 2}} - \psi_b^{-1}\kappa)(t-t_0)}). \end{aligned}$$

220 Next the bounds of $\sqrt{V_\zeta(t_u)}$ and $\sqrt{V_u(t_u)}$ will be analyzed.

Consider the initial condition satisfying $\rho_0 > \frac{1}{\omega_o}$. The initial value of $\sqrt{V_\zeta}$ satisfies that $\sqrt{V_\zeta(t_0)} \leq n\sqrt{c_{\zeta 2}}\omega_o^{n-1}\rho_0^{n-1}$. Owing to the definition of t_u (23), there is

$$\begin{cases} \sqrt{V_\zeta(t_0)}e^{-\frac{\omega_o(t_u-t_0)}{2c_{\zeta 2}}} = \sqrt{V_\zeta(t_0)}e^{-\frac{n\omega_o}{\sqrt{\kappa}}\ln(\omega_o\rho_0)} \leq \frac{n\sqrt{c_{\zeta 2}}}{\omega_o\rho_0}, \\ e^{-\psi_b^{-1}\kappa(t_u-t_0)} = e^{-2nc_{\zeta 2}\psi_b^{-1}\sqrt{\kappa}\ln(\omega_o\rho_0)} \leq \frac{1}{\rho_0^n\omega_o^n}, \end{cases} \quad (41)$$

for $\frac{\omega_o}{\kappa} \geq 1$ and $\kappa \geq \frac{1}{4\psi_b^{-2}c_{\zeta 2}^2}$. Owing to (40)–(41), we have

$$\begin{cases} \sqrt{V_\zeta(t_u)} \leq \eta_{V_\zeta 1}(\omega_2) \triangleq \frac{2c_{\zeta 2}\|P_\zeta\|\pi_{\zeta 0}(\eta_{e1})}{\sqrt{c_{\zeta 1}}\omega_2} + \frac{n\sqrt{c_{\zeta 2}}}{\rho_0\omega_2}, \\ \sqrt{V_u(t_u)} \leq \eta_{V_u 1}(\omega_2, \kappa_2) \triangleq \frac{\sqrt{2}\pi_{\delta_u 0}(\eta_{e1})}{2\psi_b^{-1}\kappa_2} + \frac{\sqrt{2}\pi_\omega(\omega_2)c_{\zeta 2}\|P_\zeta\|\pi_{\zeta 0}(\eta_{e1})}{c_{\zeta 1}\psi_b^{-1}\omega_2} \\ \quad + \frac{\sqrt{V_u(t_0)}}{\omega_2^n\rho_0^n} + \frac{\sqrt{2}\pi_\omega(\omega_2)n\sqrt{c_{\zeta 2}}}{\sqrt{c_{\zeta 1}}\psi_b^{-1}\rho_0\omega_2}, \end{cases}$$

for any $\omega_o \geq \omega_2$, $\kappa \geq \kappa_2$ and $\frac{\omega_o}{\kappa} \geq \tau_2$, where $\omega_2 = \max\{\kappa_2 \tau_2, \omega_1\}$, $\kappa_2 = \frac{1}{4\psi_b^{-2}c_{\zeta 2}^2}$ and $\tau_2 = \max\{1, 4\psi_b^{-1}c_{\zeta 2}\}$.

Considering the initial condition satisfying $\rho_0 \leq \frac{1}{\omega_o}$, (23) implies that $t_u = t_0$. It can be directly proved that $\|e(t_u)\|$, $V_z(z(t_u))$, $\sqrt{V_{\zeta}(t_u)}$ and $\sqrt{V_u(t_u)}$ are bounded for $\omega_o \geq \omega_2$. Without loss of generality, the bounds of $\|e(t_u)\|$, $V_z(z(t_u))$, $\sqrt{V_{\zeta}(t_u)}$ and $\sqrt{V_u(t_u)}$ can be also denoted as η_{e1} , $\eta_{V_{\zeta}1}$, $\eta_{V_{\zeta}1}$ and η_{V_u1} , respectively.

Part 2: The analysis for the closed-loop system for $t \in [t_u, \infty)$.

According to the control design (19) and the closed-loop system (30)–(31), the variables (e, ζ, z, u) are continuous at t_u . To simplify the mathematical expressions in this part, we introduce the following notations.

$$\begin{cases} \eta_{V_K2} \triangleq \sqrt{c_{k2}} \max\{\eta_{e1}, 2\|P_K\|\psi_b\eta_{\delta_u2}\}, & \eta_{V_{\zeta}2} \triangleq \eta_{V_{\zeta}1}, \\ \eta_{V_u2} \triangleq \max\{\eta_{V_u1}, 2\pi_{\omega}(\omega_2)\eta_{\zeta2}\psi_b\}, & \eta_{V_z2} = r_z(n\psi_{\phi}(\eta_{e2} + M_{x^*})), \\ \eta_{e2} \triangleq \frac{\eta_{V_K1}}{\sqrt{c_{k1}}}, & \eta_{\zeta2} \triangleq \frac{\eta_{V2}}{\sqrt{c_{\zeta1}}}, & \eta_{\delta_u2} \triangleq \frac{\eta_{V2}}{\sqrt{c_{21}}}. \end{cases} \quad (42)$$

Then we proceed to prove that there exist positive constants $(\omega_3, \kappa_3, \tau_3)$ such that $(e(t), z(t), \zeta(t), \delta_u(t))$ stay in a bounded set $\Omega_1 \triangleq \{(e, z, \zeta, \delta_u) \mid \sqrt{V_K} \leq \eta_{V_K2}, \sqrt{V_z} \leq \eta_{V_z2}, \sqrt{V_{\zeta}} \leq \eta_{V_{\zeta}2}, \sqrt{V_u} \leq \eta_{V_u2}\}$ for $\omega_o \geq \omega_3$, $\kappa \geq \kappa_3$, $\frac{\omega_o}{\kappa} \geq \tau_3$ and $t \geq t_u$. The proof consists of the following four steps:

Step 1. We assume that there exists $t^* \in [t_u, \infty)$ such that $\sqrt{V_{\zeta}(t^*)} = \eta_{V_{\zeta}2}$. Besides, $\sqrt{V_{\zeta}(t)} \leq \eta_{V_{\zeta}2}$, $\sqrt{V_K(t)} \leq \eta_{V_K2}$, $V_z(t) \leq \eta_{V_z2}$ and $\sqrt{V_{\delta_u}(t)} \leq \eta_{V_u2}$ for $t \in [t_u, t^*]$. Then it can be deduced that $\|e(t)\| \leq \eta_{e2}$, $\|\zeta(t)\| \leq \eta_{\zeta2}$ and $|\delta_u(t)| \leq \eta_{\delta_u2}$ for $t \in [t_u, t^*]$. Owing to the dynamics (31) and the bound of $\Delta_{\zeta 1}$ (32), the derivative of $\sqrt{V_{\zeta}(t^*)}$ satisfies

$$\frac{d\sqrt{V_{\zeta}(t^*)}}{dt} \leq -\frac{\omega_o\eta_{V_{\zeta}2}}{2c_{\zeta 2}} + \frac{\|P_{\zeta}\|(\pi_{\zeta 1}(\eta_{e2}) + (\kappa + 1)\pi_{\delta_u}(\eta_{e2}, \eta_{\delta_u2}) + \pi_{\omega}(\omega_o)\kappa\eta_{\zeta 2})}{\sqrt{c_{\zeta 1}}}. \quad (43)$$

By selecting $\omega_3 = \max\{\omega_2, (6c_{\zeta 2}\|P_{\zeta}\|(\pi_{\zeta 1}(\eta_{e2}) + \pi_{\delta_u}(\eta_{e2}, \eta_{\delta_u2}))/(\eta_{V_{\zeta}2}\sqrt{c_{\zeta 1}})\}$ and $\tau_3 = (6c_{\zeta 2}\|P_{\zeta}\|(\pi_{\omega}(\omega_2)\eta_{\zeta 2} + \pi_{\delta_u}(\eta_{e2}, \eta_{\delta_u2}))/(\eta_{V_{\zeta}2}\sqrt{c_{\zeta 1}}))$, (43) directly implies that $\frac{d\sqrt{V_{\zeta}(t^*)}}{dt} < 0$ for the (ω_o, κ) satisfying $\omega_o \geq \omega_3$ and $\frac{\omega_o}{\kappa} \geq \tau_3$.

Step 2. We assume that there exists $t^* \in [t_u, \infty)$ such that $V_z(t^*) = \eta_{V_z2}$. Besides, $\sqrt{V_u(t)} \leq \eta_{V_u2}$, $\sqrt{V_K(t)} \leq \eta_{V_K2}$, $V_z(t) \leq \eta_{V_z2}$ and $\sqrt{V_{\zeta}(t)} \leq \eta_{V_{\zeta}2}$ for $t \in [t_u, t^*]$. From

Lemma 1, the bound of $x(t)$ for $t \in [t_u, t^]$ satisfies that*

$$\sup_{t_u \leq t \leq t^*} \|x(t)\| \leq n\psi_\phi \left(\sup_{t_u \leq t \leq t^*} \|\tilde{x}(t)\| \right) \leq n\psi_\phi(\eta_{e2} + M_{x^*}).$$

235 *The definition of η_{V_z2} (42) implies that $\eta_{V_z2} \geq r_z(\sup_{t_u \leq t \leq t^*} \|x(t)\|) \geq r_z(\|x(t^*)\|)$. Due to Assumption 3, it can be verified that $\dot{V}_z(z(t^*)) \leq 0$.*

Step 3. We assume that there exists $t^* \in [t_u, \infty)$ such that $\sqrt{V_u(t^*)} = \eta_{V_u2}$. Besides, $\sqrt{V_u(t)} \leq \eta_{V_u2}$, $\sqrt{V_K(t)} \leq \eta_{V_K2}$, $V_z(t) \leq \eta_{V_z2}$ and $\sqrt{V_\zeta(t)} \leq \eta_{V_\zeta2}$ for $t \in [t_u, t^*]$. Hence $\|e(t)\| \leq \eta_{e2}$, $\|\zeta(t)\| \leq \eta_{\zeta2}$, $|\delta_u(t)| \leq \eta_{\delta_u2}$ and $|b(x, t)| \geq \eta_{b2}$ for $t \in [t_u, t^*]$. Based on (31) and (32), the derivative of $\sqrt{V_u(t^*)}$ satisfies

$$\frac{d\sqrt{V_u(t^*)}}{dt} \leq -\psi_b^{-1}\eta_{V_u2}\kappa + \frac{\sqrt{2}}{2}(\pi_{\delta_u1}(\eta_{e2}, \eta_{\delta_u2}) + \pi_\omega\kappa\eta_{\zeta2}). \quad (44)$$

Owing to the definition of η_{V_u2} (42), we have $-\frac{\psi_b^{-1}\eta_{V_u2}}{2} + \pi_\omega(\omega_2)\eta_{\zeta2} = -\pi_\omega(\omega_2)\eta_{\zeta2} \leq 0$ for any $\omega_o \geq \omega_3$. By choosing $\kappa_3 = \frac{2\sqrt{2}\pi_{\delta_u1}(\eta_{e2}, \eta_{\delta_u2})}{\eta_{b2}\eta_{V_u2}}$, there is

$$-\frac{\psi_b^{-1}\eta_{V_u2}}{2}\kappa + \frac{\sqrt{2}\pi_{\delta_u1}(\eta_{e2}, \eta_{\delta_u2})}{2} < 0$$

for $\kappa \geq \kappa_3$. Hence $\frac{d\sqrt{V_u(t^*)}}{dt} < 0$ for $\omega_o \geq \omega_3$ and $\kappa \geq \kappa_3$.

Step 4. We assume that there exists $t^* \in [t_u, \infty)$ such that $\sqrt{V_K(t^*)} = \eta_{V_K2}$. Besides, $\sqrt{V_K(t)} \leq \eta_{V_K2}$, $\sqrt{V_\zeta(t)} \leq \eta_{V_\zeta2}$, $V_z(t) \leq \eta_{V_z2}$ and $\sqrt{V_u(t)} \leq \eta_{V_u2}$ for $t \in [t_u, t^*]$. Hence $\|e(t)\| \leq \eta_{e2}$, $\|\zeta(t)\| \leq \eta_{\zeta2}$ and $|\delta_u(t)| \leq \eta_{\delta_u2}$ for $t \in [t_u, t^*]$. According to (31), (32) and (42), the derivative of $\sqrt{V_K(t^*)}$ satisfies

$$\frac{d\sqrt{V_K(t^*)}}{dt} \leq -\frac{\|e(t^*)\|}{2\sqrt{V_K(t^*)}}(\|e(t^*)\| - 2\|P_K\|\psi_b\eta_{\delta_u2}) < 0. \quad (45)$$

Due to Steps 1–4, it can be concluded that the variables $(e(t), z(t), \zeta(t), \delta_u(t))$ stay in Ω_1 for $t \in [t_u, \infty)$ if the parameters satisfy $\omega_o \geq \omega_3$, $\kappa \geq \kappa_3$ and $\frac{\omega_o}{\kappa} \geq \tau_3$. Next, we
240 will carefully analyze the bounds of $\sqrt{V_K(t)}$, $\sqrt{V_\zeta(t)}$ and $\sqrt{V_u(t)}$ for $t \geq t_u$.

The analysis of the bound of $\sqrt{V_\zeta(t)}$. Due to (31) and (32), for $t \geq t_u$, we have

$$\frac{d\sqrt{V_\zeta(t)}}{dt} \leq -\frac{\omega_o}{2c_{\zeta2}}\sqrt{V_\zeta(t)} + \frac{\|P_\zeta\|(\pi_{\zeta1}(\eta_{e2}) + (\kappa + 1)\pi_{\delta_u}(\eta_{e2}, \eta_{\delta_u2}) + \pi_\omega(\omega_o)\kappa\eta_{\zeta2})}{\sqrt{c_{\zeta1}}}. \quad (46)$$

Combined with Gronwall lemma, $\sqrt{V_\zeta(t)}$ has the following bound.

$$\sqrt{V_\zeta(t)} \leq \eta_{V_\zeta 2} e^{-\frac{\omega_o(t-t_u)}{2c_{\zeta 2}}} + \theta_{\zeta 1} \frac{1}{\omega_o} + \theta_{\zeta 2} \frac{\kappa}{\omega_o}, \quad \forall t \in [t_u, \infty), \quad (47)$$

for $\omega_o \geq \omega_3$, $\kappa \geq \kappa_3$ and $\frac{\omega_o}{\kappa} \geq \tau_3$, where $\theta_{\zeta 1} = (\|P_\zeta\|(\pi_{\zeta 1}(\eta_{e2}) + \pi_{\delta_u}(\eta_{e2}, \eta_{\delta_u 2}))/\sqrt{c_{\zeta 1}})$ and $\theta_{\zeta 2} = (\|P_\zeta\|(\pi_{\delta_u}(\eta_{e2}, \eta_{\delta_u 2}) + \pi_\omega(\omega_3)\eta_{\zeta 2}))/\sqrt{c_{\zeta 1}}$.

The analysis of the bound of $\sqrt{V_u(t)}$. Owing to (31), (32) and (47), for $t \geq t_u$, there is

$$\begin{aligned} \frac{d\sqrt{V_u(t)}}{dt} &\leq -\psi_b^{-1} \kappa \sqrt{V_u(t)} + \frac{\sqrt{2}}{2} (\pi_{\delta_{u1}}(\eta_{e2}, \eta_{\delta_{u2}}) + \pi_\omega \kappa \|\zeta(t)\|) \\ &\leq -\psi_b^{-1} \kappa \sqrt{V_u(t)} \\ &\quad + \frac{\sqrt{2}}{2} \left(\pi_{\delta_{u1}}(\eta_{e2}, \eta_{\delta_{u2}}) + \kappa \frac{\pi_\omega(\eta_{V_\zeta 2} e^{-\frac{\omega_o}{2c_{\zeta 2}}(t-t_u)} + \theta_{\zeta 1} \frac{1}{\omega_o} + \theta_{\zeta 2} \frac{\kappa}{\omega_o})}{\sqrt{c_{\zeta 1}}} \right). \end{aligned}$$

Combined with Gronwall lemma, the bound of $\sqrt{V_u(t)}$ is shown as follows.

$$\begin{aligned} \sqrt{V_u(t)} &\leq \eta_{V_u 2} e^{-\psi_b^{-1} \kappa(t-t_u)} + \frac{\theta_{u1}}{\omega_o} + \frac{\theta_{u2} \kappa}{\omega_o} + \frac{\theta_{u3}}{\kappa} + \theta_{u4} \int_{t_u}^t e^{-\psi_b^{-1} \kappa(t-s)} e^{-\frac{\omega_o}{2c_{\zeta 2}}(s-t_u)} ds \\ &\leq (\eta_{V_u 2} + \frac{\theta_{u4}}{\psi_b^{-1} \kappa_3}) e^{-\psi_b^{-1} \kappa(t-t_u)} + \theta_{u1} \frac{1}{\omega_o} + \theta_{u2} \frac{\kappa}{\omega_o} + \theta_{u3} \frac{1}{\kappa}, \quad \forall t \in [t_u, \infty), \end{aligned} \quad (48)$$

for $\omega_o \geq \omega_3$, $\kappa \geq \kappa_3$ and $\frac{\omega_o}{\kappa} \geq \tau_4 \triangleq \max\{\tau_3, 4c_{\zeta 2} \psi_b^{-1}\}$, where $\theta_{u1} = \frac{\sqrt{2}\pi_\omega(\omega_3)\theta_{\zeta 1}}{2\psi_b^{-1}\sqrt{c_{\zeta 1}}}$, $\theta_{u2} = \frac{\sqrt{2}\pi_\omega(\omega_3)\theta_{\zeta 2}}{2\psi_b^{-1}\sqrt{c_{\zeta 1}}}$, $\theta_{u3} = \frac{\sqrt{2}\pi_{\delta_{u1}}(\eta_{e2}, \eta_{\delta_{u2}})}{2\psi_b^{-1}}$ and $\theta_{u4} = \frac{\sqrt{2}\pi_\omega(\omega_3)\eta_{V_\zeta 2}}{2\psi_b^{-1}\sqrt{c_{\zeta 1}}}$.

The analysis of the bound of $\sqrt{V_K(t)}$. By denoting $\tilde{t}_u = t_u + \frac{\ln \omega_o}{\psi_b^{-1} \sqrt{\kappa}}$, it can be deduced from (31) and (38) that

$$\begin{cases} \sup_{t_u \leq t \leq \tilde{t}_u} \sqrt{V_K(t)} \leq \sqrt{c_{k2}} \left(\|e(t_u)\| + (\|A_K\| \eta_{e2} + \psi_b \eta_{\delta_{u2}}) \frac{\ln \omega_o}{\psi_b^{-1} \sqrt{\kappa}} \right) \leq \theta_e \frac{\ln \omega_o}{\sqrt{\kappa}}, \\ e^{-\psi_b^{-1} \kappa(\tilde{t}_u - t_u)} = \frac{1}{\omega_o \sqrt{\kappa}} \leq \frac{1}{\omega_o}, \end{cases} \quad (49)$$

for $\omega_o \geq \omega_3$, $\kappa \geq \kappa_4 \triangleq \max\{\kappa_3, 1\}$ and $\frac{\omega_o}{\kappa} \geq \tau_4$, where $\theta_e = 4\sqrt{c_{k2}}nc_{\zeta 2}(\|A\|\eta_{e1} + \pi_{e0}(\eta_{e1})) + \frac{\sqrt{c_{k2}}\|A_K\|\eta_{e2} + \sqrt{c_{k2}}\psi_b\eta_{\delta_{u2}}}{\psi_b^{-1}}$. Then the bound of $\sqrt{V_K(t)}$ for $t \geq \tilde{t}_u$ is analyzed.

According to (31), (32) and (48)–(49), the dynamics of $\sqrt{V_K(t)}$ satisfies the following

equation for $t \geq \tilde{t}_u$.

$$\begin{aligned} \frac{d\sqrt{V_K(t)}}{dt} &\leq -\frac{\sqrt{V_K(t)}}{2c_{k2}} + \frac{\|P_K\|\psi_b|\delta_u(t)|}{\sqrt{c_{k1}}} \\ &\leq -\frac{\sqrt{V_K(t)}}{2c_{k2}} + \frac{\sqrt{2}\|P_K\|\psi_b}{\sqrt{c_{k1}}} \left((\eta_{V_u2} + \frac{\theta_{u4}}{\psi_b^{-1}\kappa})e^{-\psi_b^{-1}\kappa(t-t_u)} + \frac{\theta_{u1}}{\omega_o} + \frac{\theta_{u2}\kappa}{\omega_o} + \frac{\theta_{u3}}{\kappa} \right) \\ &\leq -\frac{\sqrt{V_K(t)}}{2c_{k2}} + \frac{\sqrt{2}\|P_K\|\psi_b}{\sqrt{c_{k1}}} \left((\eta_{V_u2} + \frac{\theta_{u4}}{\psi_b^{-1}\kappa} + \theta_{u1})\frac{1}{\omega_o} + \frac{\theta_{u2}\kappa}{\omega_o} + \frac{\theta_{u3}}{\kappa} \right) \end{aligned}$$

for $\omega_o \geq \omega_3$, $\kappa \geq \kappa_4$ and $\frac{\omega_o}{\kappa} \geq \tau_4$. With the help of Gronwall lemma, we get the following bound of $\sqrt{V_K}$.

$$\sup_{t \geq \tilde{t}_u} \sqrt{V_K(t)} \leq \sqrt{V_K(\tilde{t}_u)} + \frac{\theta_{e1}}{\omega_o} + \frac{\theta_{e2}\kappa}{\omega_o} + \frac{\theta_{e3}}{\kappa} \leq \theta_e \frac{\ln \omega_o}{\sqrt{\kappa}} + \frac{\theta_{e1}}{\omega_o} + \frac{\theta_{e2}\kappa}{\omega_o} + \frac{\theta_{e3}}{\kappa}, \quad (50)$$

245 for $\omega_o \geq \omega_3$, $\kappa \geq \kappa_4$, and $\frac{\omega_o}{\kappa} \geq \tau_4$, where $\theta_{e1} = \frac{2\sqrt{2}c_{k2}\|P_K\|\psi_b}{\sqrt{c_{k1}}}(\eta_{V_u2} + \frac{\theta_{u4}}{\psi_b^{-1}\kappa} + \theta_{u1})$,
 $\theta_{e2} = \frac{2\sqrt{2}c_{k2}\|P_K\|\psi_b}{\sqrt{c_{k1}}}\theta_{u2}$ and $\theta_{e3} = \frac{2\sqrt{2}c_{k2}\|P_K\|\psi_b}{\sqrt{c_{k1}}}\theta_{u3}$.

According to Assumption 4, there exists a positive constant $\omega_4 \geq \omega_3$ such that

$$\kappa(\omega_o) \geq \kappa_4, \quad \frac{\omega_o}{\kappa(\omega_o)} \geq \tau_4, \quad (51)$$

for any $\omega_o \geq \omega_4$. Notice that $\frac{1}{\omega_o} \leq \frac{\kappa}{\omega_o}$ for $\kappa \geq \kappa_4 \geq 1$. With the combination of the bounds of $\sup_{t_0 \leq t \leq t_u} \|e(t)\|$, $\sqrt{V_K(t)}$, $\sqrt{V_\varsigma(t)}$ and $\sqrt{V_u(t)}$, i.e., (38), (47), (48), (49) and (50), we prove (24)–(26).

250 5. Simulation

In this section, the simulation for an application example, Chua's circuit, is presented.

Figure 1 describes Chua's circuit, which is featured with strong nonlinearity and can generate chaotic response [36]. The mathematical model of Chua's circuit is presented as follows [32, 36].

$$\begin{cases} \dot{x}_1(t) = -\frac{1}{C_1 R}x_1(t) + \frac{1}{C_1 R}x_2(t) - \frac{1}{C_1}f(x_1), \\ \dot{x}_2(t) = -\frac{1}{C_2 R}x_1(t) + \frac{1}{C_2 R}x_2(t) + \frac{1}{C_2}x_3(t), \\ \dot{x}_3(t) = \frac{1}{L}u(t) - \frac{1}{L}x_2(t) - \frac{R_0}{L}x_3(t), \end{cases} \quad (52)$$

Figure 1: Chua's circuit.

where x_1 and x_2 represent the voltages across the capacitor C_1 and C_2 , x_3 is the current through the inductor L , u is the input voltage, and $f(x_1)$ represents the nonlinear current caused by the nonlinear resistor D .

The control objective is to design the input voltage $u(t)$ such that the system states (x_1, x_2, x_3) can track $(0, 0, 0)$.

In the simulation, the measurement of x_1 can be obtained, while (x_2, x_3) cannot be measured. Besides, the detailed values of the system parameters (C_1, C_2, L, R, R_0) are unknown for the control design, whereas the signs of (C_1, C_2, L, R, R_0) can be directly verified by physical mechanism:

$$C_1 > 0, \quad C_2 > 0, \quad L > 0, \quad R > 0, \quad R_0 \geq 0. \quad (53)$$

According to [32, 36], the nonlinearity $f(x_1)$ satisfies that $f(0) = 0$.

Remark 7. The presented control objective is a classical stabilization problem [32, 36]. However, the methods proposed in [32, 36] require the measurements for all states (x_1, x_2, x_3) . In this paper, only the measurement of x_1 is utilized for the stabilization problem. Moreover, the nominal values of system parameters (C_1, C_2, L, R, R_0) are unknown.

Figure 2: The response curves of the state x_1 . Figure 3: The response curves of the state x_2 .

Next, the proposed ADRC is applied to the system (52). Firstly, denote the new

Figure 4: The response curves of the state x_3 .

states

$$\begin{cases} \tilde{x}_1 = x_1, \\ \tilde{x}_2 = -\frac{1}{C_1 R} x_1(t) + \frac{1}{C_1 R} x_2(t) - \frac{1}{C_1} f(x_1), \\ \tilde{x}_3 = \left(-\frac{1}{C_1 R} - \frac{1}{C_1} \frac{df(x_1)}{dx_1} \right) \left(-\frac{x_1(t)}{C_1 R} + \frac{x_2(t)}{C_1 R} - \frac{f(x_1)}{C_1} \right) \\ \quad + \frac{1}{C_1 R} \left(-\frac{x_1(t)}{C_2 R} + \frac{x_2(t)}{C_2 R} + \frac{x_3(t)}{C_2} \right). \end{cases} \quad (54)$$

Then the integrators chain form is obtained as follows.

$$\begin{cases} \dot{\tilde{x}}_1(t) = \tilde{x}_2(t), \\ \dot{\tilde{x}}_2(t) = \tilde{x}_3(t), \\ \dot{\tilde{x}}_3(t) = bu(t) + f_i, \end{cases} \quad (55)$$

where b and f satisfy the following equation.

$$\begin{cases} b = \frac{1}{C_1 C_2 R L}, \\ f_t = -\frac{1}{C_1} \frac{d^2 f(x_1)}{dx_1^2} \left(-\frac{x_1(t)}{C_1 R} + \frac{x_2(t)}{C_1 R} - \frac{f(x_1)}{C_1} \right)^2 - \frac{1}{C_1 C_2 R^2} \left(-\frac{x_1(t)}{C_1 R} + \frac{x_2(t)}{C_1 R} - \frac{f(x_1)}{C_1} \right) \\ \quad + \frac{1}{C_1 C_2 R^2} \left(-\frac{x_1(t)}{C_2 R} + \frac{x_2(t)}{C_2 R} + \frac{x_3(t)}{C_2} \right) - \frac{1}{C_1 C_2 R} \left(\frac{x_2(t)}{L} + \frac{R_0 x_3(t)}{L} \right) \\ \quad + \left(-\frac{1}{C_1 R} - \frac{1}{C_1} \frac{df(x_1)}{dx_1} \right) \left(-\frac{1}{C_1 R} - \frac{1}{C_1} \frac{df(x_1)}{dx_1} \right) \left(-\frac{x_1(t)}{C_1 R} + \frac{x_2(t)}{C_1 R} - \frac{f(x_1)}{C_1} \right) \\ \quad + \frac{1}{C_1 R} \left(-\frac{1}{C_1 R} - \frac{1}{C_1} \frac{df(x_1)}{dx_1} \right) \left(-\frac{x_1(t)}{C_2 R} + \frac{x_2(t)}{C_2 R} + \frac{x_3(t)}{C_2} \right). \end{cases}$$

Due to the transformation (54), it can be verified that $[x_1 \ x_2 \ x_3] = [0 \ 0 \ 0]$ is equivalent to $[\tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3] = [0 \ 0 \ 0]$. Hence the stabilization problem of the system (52) can be reformulated as the stabilization problem of the system (55).

Although the nominal value of b is unknown due to the unknown system parameters (C_1, C_2, L, R, R_0) , it can be verified by (53) that $\text{sgn}(b) > 0$. Based on the sign of b , the proposed ADRC (15) and (19) with the following controller parameters is utilized.

$$\omega_o = 300, \quad K = [8 \ 12 \ 6]^T, \quad \kappa(\omega_o) = \sqrt{\omega_o}. \quad (56)$$

To investigate the capability of disturbance rejection, the following cases of uncertainties are considered, including the cubic function in [32] (Case 1).

$$\text{Case 1: } f(x_1) = -x_1 + x_1^3,$$

$$\text{Case 2: } f(x_1) = -1.3x_1 + x_1^3,$$

$$\text{Case 3: } f(x_1) = -x_1 + x_1^3 - \sin(x_1).$$

We consider the following system parameters and initial condition, which are provided in [32].

$$C_1 = C_2 = 1, \ R_0 = 0, \ R = 1, \ L = 2, \ x_1(0) = 0.01, \ x_2(0) = 0.2, \ x_3(0) = 0.5. \quad (57)$$

The simulation results of the proposed ADRC and the following backstepping

based funnel control [32] are presented in Figures 2–4.

$$\begin{cases} u = 7r_3 \cos(\pi r_3) z_3, \\ z_1 = x_1, \\ z_2 = x_2 - 10r_1 \cos(\pi r_1) z_1, \\ z_3 = x_3 - 7r_2 \cos(\pi r_2) z_2, \\ r_i = \frac{1}{1 - (e^t - 1)^2 z_i^2}, \quad i = 1, 2, 3. \end{cases} \quad (58)$$

270 For Case 1, the satisfied closed-loop performance of the proposed ADRC and the backstepping based funnel control are shown, while the smaller tracking error and the better transient performance of the proposed ADRC are illustrated by Figure 2. From Figures 3–4, the closed-loop performance of the backstepping based funnel control becomes poor for Cases 2–3. Moreover, despite various uncertainties, the performance of the proposed ADRC is highly consistent with each other, which illuminates the strong
275 capability of the proposed ADRC to handle uncertainties.

6. Conclusion

For a class of lower-triangular nonlinear uncertain systems, the paper proposes a new ADRC based on the control directions rather than the nominal values or mathematical expressions of control coefficients. The design ideology can be summarized
280 as the following three parts: 1) By transforming the original states into the states of an integrators chain system, the effects from the control input and uncertainties to the controlled output are clearly shown; 2) Based on the integrators chain form, the ESO is presented to estimate the total disturbance and the integrators chain states; 3) Inspired
285 by the approximative dynamic inversion method, a dynamical system is designed to force the actual control input to track the ideal input. Moreover, by associating the parameter of the dynamical input design with the ESO's parameter, the tuning method of the dynamical input design is explicitly provided. With the consideration of a large scope of mismatched nonlinear uncertainties, the transient performance of the proposed
290 ADRC is theoretically investigated. Based on the presented theoretical results, the sat-

isfied tracking and estimating performance can be ensured by suitably enlarging the ESO's parameter.

7. Appendix

7.1. Proof of Lemma 1

Part I: The analysis of the mapping φ . With the combination of the dynamics (3) and the explicit form of $\tilde{x}_i$ ($1 \leq i \leq n$) (7), it can be directly verified that $\tilde{x}_i = \dot{\tilde{x}}_{i-1}$ for $2 \leq i \leq n$. Then, for the transformation (7), we will prove that the term

$$\tilde{\tau}_i \triangleq \sum_{j=1}^{i-2} \frac{d^{j-1}}{dt^{j-1}} \left(\frac{d(\Pi_{k=1}^{i-j-1} \theta_k(t))}{dt} x_{i-j} \right) + \sum_{j=1}^{i-1} \frac{d^{j-1}}{dt^{j-1}} \left((\Pi_{k=0}^{i-j-1} \theta_k(t)) \phi_{i-j} \right) \quad (59)$$

is a function dependent on $(x_1, \dots, x_{i-1}, t)$ for $2 \leq i \leq n$ by mathematical induction in the following three steps.

Step 1 (consider the case that $i = 2$). It can be obtained that

$$\tilde{\tau}_2 = \theta_0(t) \phi_1(x_1, t) = \phi_1(x_1, t). \quad (60)$$

Hence $\tilde{\tau}_2$ is a function dependent on (x_1, t) .

Step 2 (consider the case that $i = k$ ($2 \leq k \leq n-1$)). Suppose that $\tilde{\tau}_k$ ($2 \leq k \leq n-1$) is a function dependent on $(x_1, \dots, x_{k-1}, t)$.

Step 3 (consider the case that $i = k+1$ ($2 \leq k \leq n-1$)). Since $\tilde{\tau}_k$ is a function dependent on $(x_1, \dots, x_{k-1}, t)$, the state $\tilde{x}_k$ has the following form:

$$\tilde{x}_k = (\Pi_{j=1}^{k-1} \theta_j(t)) x_i(t) + \tilde{\tau}_k(x_1, \dots, x_{k-1}, t). \quad (61)$$

By taking the derivative of $\tilde{x}_k$ and utilizing the dynamics (3), there is

$$\begin{aligned} \dot{\tilde{x}}_k = \tilde{x}_{k+1} = & (\Pi_{j=1}^{k-1} \theta_j(t)) (\theta_k(t) x_{k+1}(t) + \phi_k(x_1, \dots, x_k, t)) \\ & + \frac{d(\Pi_{j=1}^{k-1} \theta_j(t))}{dt} x_k(t) + \frac{\partial \tilde{\tau}_k(x_1, \dots, x_{k-1}, t)}{\partial t} \\ & + \sum_{j=1}^{k-1} \frac{\partial \tilde{\tau}_k(x_1, \dots, x_{k-1}, t)}{\partial x_j} (\theta_j x_{j+1} + \phi_j(x_1, \dots, x_j, t)). \end{aligned} \quad (62)$$

By comparing the form (62) with (7), it can be obtained that

$$\begin{aligned} \tilde{\tau}_{k+1} = & (\Pi_{j=1}^{k-1} \theta_j(t)) \phi_k(x_1, \dots, x_k, t) + \frac{d(\Pi_{j=1}^{k-1} \theta_j(t))}{dt} x_k(t) + \frac{\partial \tilde{\tau}_k}{\partial t} \\ & + \sum_{j=1}^{k-1} \frac{\partial \tilde{\tau}_k}{\partial x_j} (\theta_j x_{j+1} + \phi_j(x_1, \dots, x_j, t)). \end{aligned} \quad (63)$$

Due to the supposition in Step 2, (63) illustrates that $\tilde{\tau}_{k+1}$ is a function dependent on $(x_1, \dots, x_k, t)$.

Based on the fact that $\tilde{\tau}_i$ is a function dependent on $(x_1, \dots, x_{i-1}, t)$ for $2 \leq i \leq n$, (7) can be rewritten in the following form.

$$\begin{cases} \tilde{x}_1 = x_1, \\ \tilde{x}_i = (\prod_{j=1}^{i-1} \theta_j(t))x_i + \tilde{\tau}_i(x_1, \dots, x_{i-1}, t), \quad 2 \leq i \leq n. \end{cases} \quad (64)$$

According to Assumption 2, the term $\prod_{j=1}^{i-1} \theta_j(t)$ is nonzero for $2 \leq i \leq n$. Then the mapping from $(\tilde{x}, t)$ to (x, t) can be determined by the following recursive way.

From (64), the functions φ_1 and φ_2 can be obtained as follows.

$$\varphi_1(\tilde{x}, t) = \tilde{x}_1, \quad \varphi_2(\tilde{x}, t) = \frac{\tilde{x}_2 - \tilde{\tau}_2(x_1, t)}{\theta_1(t)} = \frac{\tilde{x}_2 - \tilde{\tau}_2(\varphi_1(\tilde{x}, t), t)}{\theta_1(t)}. \quad (65)$$

Suppose that $\varphi_i(\tilde{x}, t)$ exists for $2 \leq i \leq k \leq n-1$ being an integer. Owing to (64), we can acquire the function $\varphi_{k+1}(\tilde{x}, t)$ as follows.

$$\varphi_{k+1}(\tilde{x}, t) = \frac{\tilde{x}_{k+1} - \tilde{\tau}_{k+1}(\varphi_1(\tilde{x}, t), \dots, \varphi_k(\tilde{x}, t), t)}{\prod_{j=1}^k \theta_j(t)}, \quad 2 \leq k \leq n-1. \quad (66)$$

Additionally, it is obvious that $\varphi_{n+1}(\tilde{x}, t) = t$. Moreover, the inverse of the mapping φ can be directly obtained by (64).

Then the bounds of $\varphi_i(\tilde{x}, t)$ ($1 \leq i \leq n+1$) are analyzed. Owing to the dynamics (3), the expression of $\tilde{\tau}_i$ (59) implies that $\tilde{\tau}_i$ is composed of the finite sums and products of x_j ($1 \leq j \leq i-1$), ϕ_j ($1 \leq j \leq i-1$), the partial derivatives of ϕ_j ($1 \leq j \leq i-1$) up to $(i-j-1)$ -th order and $\theta_j^{(p)}$ ($1 \leq j \leq i-1, 0 \leq p \leq i-j-1$). According to Assumption 2, there exist continuous functions $\tilde{\Psi}_{\tau,i}(x_1, \dots, x_{i-1})$ ($2 \leq i \leq n$) such that

$$\sup_{t \geq t_0} |\tilde{\tau}_i(x_1, \dots, x_{i-1}, t)| \leq \tilde{\Psi}_{\tau,i}(x_1, \dots, x_{i-1}), \quad \forall [x_1 \dots x_{i-1}] \in R^{i-1}, \quad 2 \leq i \leq n. \quad (67)$$

By introducing the non-decreasing continuous function

$$\psi_{\tau,i}(\rho_x) \triangleq \sup_{\|[x_1 \dots x_{i-1}]\| \leq \rho_x} \tilde{\Psi}_{\tau,i}(x_1, \dots, x_{i-1})$$

for $\rho_x \geq 0$, (67) implies that

$$\sup_{t \geq t_0, \|[x_1 \dots x_{i-1}]\| \leq \rho_x} |\tilde{\tau}_i(x_1, \dots, x_{i-1}, t)| \leq \psi_{\tau,i}(\rho_x), \quad \forall \rho_x \geq 0, \quad 2 \leq i \leq n. \quad (68)$$

Next, the bounds of φ_i are presented by recursive method. Denote $\tilde{\rho}_x$ as a non-negative constant. Based on (65), the bounds of φ_1 and φ_2 are shown as follows.

$$\left\{ \begin{array}{l} \sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |\varphi_1(\tilde{x}, t)| \leq \sup_{\|\tilde{x}\| \leq \tilde{\rho}_x} |\tilde{x}_1| \leq M_{\varphi,1}(\tilde{\rho}_x) \triangleq \tilde{\rho}_x, \\ \sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |\varphi_2(\tilde{x}, t)| \leq \sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} \frac{|\tilde{x}_2| + |\tilde{\tau}_2(\varphi_1(\tilde{x}, t), t)|}{|\theta_1(t)|} \\ \leq M_{\varphi,2}(\tilde{\rho}_x) \triangleq \frac{\tilde{\rho}_x + \psi_{\tau,2}(M_{\varphi,1}(\tilde{\rho}_x))}{\underline{M}_{\theta,1}}. \end{array} \right. \quad (69)$$

Notice that $M_{\varphi,1}(\cdot)$ and $M_{\varphi,2}(\cdot)$ are increasing functions. Suppose that $\sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |\varphi_i(\tilde{x}, t)| \leq M_{\varphi,i}(\tilde{\rho}_x)$ where $M_{\varphi,i}$ is a continuous increasing functions for $2 \leq i \leq k$, $2 \leq k \leq n-1$ and $\tilde{\rho}_x \geq 0$. Then, the bound of $\varphi_{k+1}(\tilde{x}, t)$ can be obtained as follows.

$$\begin{aligned} \sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |\varphi_{k+1}(\tilde{x}, t)| &\leq \sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} \frac{|\tilde{x}_{k+1}| + |\tilde{\tau}_{k+1}(\varphi_1(\tilde{x}, t), \dots, \varphi_k(\tilde{x}, t), t)|}{|\Pi_{j=1}^k \theta_j(t)|} \\ &\leq M_{\varphi,k+1}(\tilde{\rho}_x) \triangleq \frac{\tilde{\rho}_x + \psi_{\tau,k+1} \left(\sqrt{\sum_{j=1}^k (M_{\varphi,j}(\tilde{\rho}_x))^2} \right)}{\Pi_{j=1}^k \underline{M}_{\theta,j}}. \end{aligned} \quad (70)$$

Hence it can be concluded that there exist continuous increasing functions $M_{\varphi,i}$ ($1 \leq i \leq n$) such that $\sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |\varphi_i(\tilde{x}, t)| \leq M_{\varphi,i}(\tilde{\rho}_x)$ for $\tilde{\rho}_x \geq 0$ and $1 \leq i \leq n$.

By taking the partial derivatives of φ_i along the dynamics (3), it can be deduced with the similar analysis (65)–(70) that there exist continuous functions $M_{\frac{\partial \varphi}{\partial \tilde{x}},i}$ and $M_{\frac{\partial \varphi}{\partial t},i}$ such that

$$\sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} \left\| \frac{\partial \varphi_i(\tilde{x}, t)}{\partial \tilde{x}} \right\| \leq M_{\frac{\partial \varphi}{\partial \tilde{x}},i}(\tilde{\rho}_x), \quad \sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} \left| \frac{\partial \varphi_i(\tilde{x}, t)}{\partial t} \right| \leq M_{\frac{\partial \varphi}{\partial t},i}(\tilde{\rho}_x), \quad 1 \leq i \leq n. \quad (71)$$

for any $\tilde{\rho}_x \geq 0$. Due to (69)–(71), (10) is proved by defining

$$\psi_{\varphi}(\tilde{\rho}_x) \triangleq \max_{1 \leq i \leq n} \left\{ M_{\varphi,i}(\tilde{\rho}_x), M_{\frac{\partial \varphi}{\partial \tilde{x}},i}(\tilde{\rho}_x) M_{\frac{\partial \varphi}{\partial t},i}(\tilde{\rho}_x) \right\}.$$

Part II: The analysis of the mapping γ . Based on the analysis of $\tilde{\tau}_i$, the mapping γ can be directly determined by (64). Combined with the bounds of $\tilde{\tau}_i$ (67)–(68), (64)

further implies that

$$\begin{cases} \sup_{t \geq t_0, \|x\| \leq \rho_x} |\gamma_1(x, t)| \leq |x_1| \leq M_{\gamma,1}(\rho_x) \triangleq \rho_x, \\ \sup_{t \geq t_0, \|x\| \leq \rho_x} |\gamma_i(x, t)| \leq \sup_{t \geq t_0, \|x\| \leq \rho_x} \left(|\Pi_{j=1}^{i-1} \theta_j(t)| |x_i| + |\tilde{\tau}_i| \right) \\ \leq M_{\gamma,i}(\rho_x) \triangleq \rho_x \Pi_{j=1} \bar{M}_{\theta,i} + \psi_{\tau,i}(\rho_x), \quad 2 \leq i \leq n, \end{cases} \quad (72)$$

for any given $\rho_x \geq 0$. Notice that $M_{\gamma,i}(\rho_x)$ ($1 \leq i \leq n$) are continuous increasing functions.

Similarly, by taking the partial derivatives of γ_i along the dynamics (3), it can be deduced by Assumption 2 that there exist continuous functions $M_{\frac{\partial \gamma}{\partial x},i}$ and $M_{\frac{\partial \gamma}{\partial t},i}$ such that

$$\sup_{t \geq t_0, \|x\| \leq \rho_x} \left\| \frac{\partial \gamma_i(x, t)}{\partial x} \right\| \leq M_{\frac{\partial \gamma}{\partial x},i}(\rho_x), \quad \sup_{t \geq t_0, \|x\| \leq \rho_x} \left| \frac{\partial \gamma_i(x, t)}{\partial t} \right| \leq M_{\frac{\partial \gamma}{\partial t},i}(\rho_x), \quad \forall 1 \leq i \leq n, \quad (73)$$

for any $\rho_x \geq 0$. Based on (72)–(73) and the notation

$$\psi_\gamma(\rho_x) \triangleq \max_{1 \leq i \leq n} \left\{ M_{\gamma,i}(\rho_x), M_{\frac{\partial \gamma}{\partial x},i}(\rho_x) M_{\frac{\partial \gamma}{\partial t},i}(\rho_x) \right\},$$

310 then (9) is proved.

7.2. Proof of Lemma 2

Firstly, the bounds of $b(t)$, $b^{-1}(t)$ and $\dot{b}(t)$ are analyzed. According to the bounds of θ_i and $\dot{\theta}_i$ in Assumption 2, the expression of $b(t)$ (12) implies that

$$\begin{cases} |b(t)| \leq \Pi_{i=1}^n \bar{M}_{\theta,i}, \\ |b^{-1}(t)| = |\Pi_{i=1}^n \theta_i^{-1}(t)| \leq \Pi_{i=1}^n \underline{M}_{\theta,i}^{-1}, \\ |\dot{b}(t)| = \Sigma_{j=1}^n |\dot{\theta}_j(t)| \Pi_{1 \leq i \leq n, i \neq j} |\theta_i(t)| \leq \Pi_{i=1}^n \bar{M}_{\theta,i}. \end{cases} \quad (74)$$

By denoting $\psi_b = \max\{\Pi_{i=1}^n \bar{M}_{\theta,i}, \Pi_{i=1}^n \underline{M}_{\theta,i}^{-1}\}$, (27) is proved.

Next the bound of $f(\tilde{x}, t)$ is analyzed. Lemma 1 illustrates that $\varphi_{n+1-j}(\tilde{x}, t) = x_{n+1-j}$ for $1 \leq j \leq n-1$. Additionally, $\tilde{\phi}_i(\tilde{x}, t) = \phi_i(x_1, \dots, x_i, t)$ for $1 \leq i \leq n$. From
315 the expression of $f(\tilde{x}, t)$ (12), $f(\tilde{x}, t)$ can be regarded as a function of (x, t) , denoted as $\tilde{f}(x, t)$.

Owing to the dynamics (3) and the notation $\tilde{f}(x, t)$, it can be verified that $\tilde{f}(x, t)$ is composed of the finite sums and products of x_j ($1 \leq j \leq n$), ϕ_j ($1 \leq j \leq n$), the partial derivatives of ϕ_j ($1 \leq j \leq n$) up to $(n-j)$ -th order and $\theta_j^{(p)}$ ($1 \leq j \leq n-1, 0 \leq p \leq n-j-1$). Due to Assumption 2, there exists a continuous function $\psi_{f,1}(x)$ such that

$$\sup_{t \geq t_0} |\tilde{f}(x, t)| \leq \psi_{f,1}(x), \quad (75)$$

which further implies that

$$\sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |f(\tilde{x}, t)| = \sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |\tilde{f}(x, t)| \leq \sup_{\|x\| \leq \tilde{\rho}_x} \psi_{f,1}(x), \quad \forall \tilde{\rho}_x \geq 0. \quad (76)$$

Recalling the continuous increasing function $\psi_\gamma(\cdot)$ in Lemma 1, owing to inverse function theorem, the inverse function of $\psi_\gamma(\cdot)$ can be denoted as $\psi_\gamma^{-1}(\cdot)$ which is a continuous function in $[0, \infty)$. According to (8)–(9) in Lemma 1, the following equation holds.

$$\begin{aligned} \sup_{\|x\| \leq \psi_\gamma^{-1}(\tilde{\rho}_x/\sqrt{n})} \|\tilde{x}\| &= \sup_{\|x\| \leq \psi_\gamma^{-1}(\tilde{\rho}_x/\sqrt{n})} \left\| \begin{bmatrix} \gamma_1(x, t) & \cdots & \gamma_n(x, t) \end{bmatrix} \right\| \\ &\leq \sqrt{n} \psi_\gamma(\psi_\gamma^{-1}(\tilde{\rho}_x/\sqrt{n})) = \tilde{\rho}_x. \end{aligned} \quad (77)$$

Based on (77), it can be verified that

$$\{\tilde{x} \mid \|\tilde{x}\| \leq \tilde{\rho}_x\} \subset \{\tilde{x} \mid \|x\| \leq \psi_\gamma^{-1}(\tilde{\rho}_x/\sqrt{n})\}, \quad \forall \tilde{\rho}_x \geq 0. \quad (78)$$

With the combination of (76) and (78), it can be obtained that

$$\sup_{t \geq t_0, \|\tilde{x}\| \leq \tilde{\rho}_x} |f(\tilde{x}, t)| \leq \sup_{\|\tilde{x}\| \leq \tilde{\rho}_x} \psi_{f,1}(x) \leq \tilde{\psi}_{f,1}(\tilde{\rho}_x) \triangleq \sup_{\|x\| \leq \psi_\gamma^{-1}(\tilde{\rho}_x/\sqrt{n})} \psi_{f,1}(x), \quad \forall \tilde{\rho}_x \geq 0. \quad (79)$$

Due to the continuity of $\psi_\gamma^{-1}(\cdot)$ and $\psi_{f,1}(\cdot)$, it can be deduced that $\tilde{\psi}_{f,1}(\tilde{\rho}_x)$ is a continuous function with respect to the variable $\tilde{\rho}_x$.

From Lemma 1, $\varphi_{n+1}(\tilde{x}, t) = t$ and

$$x_i = \varphi_i(\tilde{x}, t), \quad 1 \leq i \leq n. \quad (80)$$

Then the partial derivatives of $f(\tilde{x}, t)$ can be expressed as follows.

$$\begin{cases} \frac{\partial f(\tilde{x}, t)}{\partial \tilde{x}_i} = \frac{\partial \bar{f}(\varphi_1(\tilde{x}, t), \dots, \varphi_n(\tilde{x}, t), t)}{\partial \tilde{x}_i} \\ \quad = \sum_{j=1}^n \frac{\partial \bar{f}(x_1, \dots, x_n, t)}{\partial x_j} \frac{\partial \varphi_j(\tilde{x}, t)}{\partial \tilde{x}_i}, \quad 1 \leq i \leq n, \\ \frac{\partial f(\tilde{x}, t)}{\partial t} = \frac{\partial \bar{f}(x_1, \dots, x_n, t)}{\partial t}. \end{cases} \quad (81)$$

Due to Assumption 2, with the similar derivation as (75), there exist continuous functions $\psi_{f x, j}$ and $\psi_{f t}$ such that

$$\sup_{t \geq t_0} \left| \frac{\partial \bar{f}(x_1, \dots, x_n, t)}{\partial t} \right| \leq \psi_{f t}(x), \quad \sup_{t \geq t_0} \left| \frac{\partial \bar{f}(x_1, \dots, x_n, t)}{\partial x_j} \right| \leq \psi_{f x, j}(x), \quad 1 \leq j \leq n. \quad (82)$$

Based on the bounds of $\frac{\partial \varphi_j}{\partial x_i}$ (10) and with the similar procedure as (75)–(79), there exist continuous functions $\bar{\psi}_{f, 2}(\cdot)$ and $\bar{\psi}_{f, 3}(\cdot)$ such that

$$\sup_{t \geq t_0, \|\tilde{x}\| \leq \bar{\rho}_x} \left\| \frac{\partial f(\tilde{x}, t)}{\partial \tilde{x}} \right\| \leq \bar{\psi}_{f, 2}(\bar{\rho}_x), \quad \sup_{t \geq t_0, \|\tilde{x}\| \leq \bar{\rho}_x} \left| \frac{\partial f(\tilde{x}, t)}{\partial t} \right| \leq \bar{\psi}_{f, 3}(\bar{\rho}_x), \quad \forall \bar{\rho}_x \geq 0. \quad (83)$$

By denoting $\psi_f(\bar{\rho}_x) \triangleq \max\{\bar{\psi}_{f, 1}(\bar{\rho}_x), \bar{\psi}_{f, 2}(\bar{\rho}_x), \bar{\psi}_{f, 3}(\bar{\rho}_x)\}$, (79) and (83) imply (28).

320 7.3. Proof of Lemma 3

For $t \in [t_0, t_u)$, the control input satisfies that $u(t) = 0$. Hence the dynamics of $\tilde{x}$ in (11) can be rewritten as follows.

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bf(\tilde{x}, t). \quad (84)$$

Combined with the dynamics of $\tilde{x}^*$ (17), the dynamics of $e(t)$ for $t \in [t_0, t_u)$ is shown as follows.

$$\dot{e}(t) = Ae(t) + B\Delta_{e0}(e, t), \quad t \in [t_0, t_u), \quad (85)$$

where

$$\Delta_{e0}(e, t) = f(\tilde{x}^* + e, t) + K^T(\tilde{x}^*(t) - \bar{r}(t)) - r^{(n)}(t). \quad (86)$$

Then, consider the case that $t \in [t_u, \infty)$. With the notation (1) and the ideal input (18), the dynamics of $\tilde{x}$ in (11) can be reformulated as follows.

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B(b(t)u(t) + f(\tilde{x}, t) - b(xt)u^*(t) + b(t)u^*(t)) \\ &= A\tilde{x}(t) - BK^T(\tilde{x}(t) - \bar{r}(t)) + Br^{(n)}(t) + Bb(t)\delta_u(t) \\ &= A_K e(t) + Bb(t)\delta_u(t) + \dot{\tilde{x}}^*(t). \end{aligned} \quad (87)$$

Owing to the dynamics of $\tilde{x}^*$ (17), the dynamics of $e(t)$ for $t \in [t_u, \infty)$ is obtained as follows.

$$\dot{e}(t) = A_K e(t) + Bb(t)\delta_u(t), \quad t \in [t_u, \infty). \quad (88)$$

Before analyzing the closed-loop form of ζ and δ_u , the derivative of $u^*(t)$ is calculated due to (18).

$$\left\{ \begin{array}{l} \dot{u}^*(t) = -\frac{\dot{b}(t)(-f(\tilde{x}, t) - K^T(\tilde{x} - \bar{r}) + r^{(n)})}{b^2(t)} \\ \quad + \frac{-\frac{\partial f}{\partial \tilde{x}}(A(e + \tilde{x}^*) + Bf(e + \tilde{x}^*, t)) - \frac{\partial f}{\partial t}}{b(t)} \\ \quad + \frac{-K^T(A(e + \tilde{x}^*) + Bf(e + \tilde{x}^*, t) - \dot{r}) + r^{(n+1)}}{b(t)}, \quad t \in (t_0, t_u), \\ \dot{u}^*(t) = -\frac{\dot{b}(t)(-f(\tilde{x}, t) - K^T(\tilde{x} - \bar{r}) + r^{(n)})}{b^2(t)} \\ \quad + \frac{-\frac{\partial f}{\partial \tilde{x}}(A_K e + Bb\delta_u + \dot{\tilde{x}}^*) - \frac{\partial f}{\partial t}}{b(t)} \\ \quad + \frac{-K^T(A_K e + Bb\delta_u + \dot{\tilde{x}}^* - \dot{r}) + r^{(n+1)}}{b(t)}, \quad t \in (t_u, \infty). \end{array} \right. \quad (89)$$

With the help of Remark 4, the derivative of $u(t)$ is shown as follows.

$$\begin{aligned} \dot{u}(t) &= -\text{sgn}(b(t))\kappa(\omega_o)(\hat{f}_t(t) + K^T(\hat{\tilde{x}}(t) - \bar{r}(t)) - r^{(n)}(t)) \\ &= -\kappa|b(t)| \frac{f_t(\tilde{x}, u, t) + K^T(\tilde{x}(t) - \bar{r}(t)) - r^{(n)}(t) - K_e^T T_1 \zeta}{b(t)} \\ &= -\kappa|b(t)|\delta_u + \text{sgn}(b(t))\kappa K_e^T T_1 \zeta, \end{aligned} \quad (90)$$

where $K_e = [K^T \ 1]^T$.

Since $T_1^{-1}A_L T_1 = \omega_o A_\zeta$, the dynamics of ζ for $t \in [t_0, t_u)$ can be obtained by (11) and (15):

$$\dot{\zeta}(t) = \omega_o A_\zeta \zeta(t) + B_f \Delta_{\zeta 0}(e, t), \quad t \in [t_0, t_u), \quad (91)$$

where

$$\Delta_{\zeta 0}(e, t) = \frac{df(\tilde{x}, t)}{dt} \Big|_{\text{along (84)}} = \frac{\partial f}{\partial \tilde{x}}(A(e + \tilde{x}^*) + Bf(e + \tilde{x}^*, t)) + \frac{\partial f}{\partial t}. \quad (92)$$

Owing to (90), the dynamics of ζ for $t \in [t_u, \infty)$ is presented as follows.

$$\dot{\zeta}(t) = \omega_o A_\zeta \zeta(t) + B_f \Delta_{\zeta 1}(e, \zeta, \delta_u, \omega_o, \kappa, t), \quad t \in [t_0, t_u), \quad (93)$$

where

$$\begin{aligned}\Delta_{\zeta 1}(e, \zeta, \delta_u, \omega_o, \kappa, t) &= \frac{d(b(t)u(t) + f(\tilde{x}, t))}{dt} \Big|_{\text{along (87)}} \\ &= \dot{b}\delta_u + \dot{b}(-f(e + \tilde{x}^*, t) - K^T(e + \tilde{x}^* - \bar{r}) + r^{(n)})/b \\ &\quad + b(-\kappa|b|\delta_u + \text{sgn}(b)\kappa K_e^T T_1 \zeta) \\ &\quad + \frac{\partial f}{\partial \tilde{x}}(A_K e + Bb\delta_u + \dot{\tilde{x}}^*) + \frac{\partial f}{\partial t}.\end{aligned}\quad (94)$$

With the combination of (89) and (90), the derivative of δ_u is shown as follows.

$$\begin{cases} \dot{\delta}_u(t) = -|b(t)|\kappa\delta_u(t) + \Delta_{\delta_u 0}(e, \zeta, \omega_o, \kappa, t), & t \in [t_0, t_u], \\ \dot{\delta}_u(t) = -|b(t)|\kappa\delta_u(t) + \Delta_{\delta_u 1}(e, \zeta, \delta_u, \omega_o, \kappa, t), & t \in [t_u, \infty), \end{cases}\quad (95)$$

where

$$\begin{cases} \Delta_{\delta_u 0}(e, \zeta, \omega_o, \kappa, t) = \text{sgn}(b(t))\kappa K_e^T T_1 \zeta + \frac{\dot{b}(t)(-f(\tilde{x}, t) - K^T(e + \tilde{x}^* - \bar{r}) + r^{(n)})}{b^2(t)} \\ \quad - \frac{-\frac{\partial f}{\partial \tilde{x}}(A(e + \tilde{x}^*) + Bf(e + \tilde{x}^*, t)) - \frac{\partial f}{\partial t} - K^T(A(e + \tilde{x}^*) + Bf(e + \tilde{x}^*, t) - \dot{\bar{r}}) + r^{(n+1)}}{b(t)}, \\ \Delta_{\delta_u 1}(e, \zeta, \delta_u, \omega_o, \kappa, t) = \text{sgn}(b(t))\kappa K_e^T T_1 \zeta + \frac{\dot{b}(t)(-f(\tilde{x}, t) - K^T(e + \tilde{x}^* - \bar{r}) + r^{(n)})}{b^2(t)} \\ \quad - \frac{-\frac{\partial f}{\partial \tilde{x}}(A_K e + Bb\delta_u + \dot{\tilde{x}}^*) - \frac{\partial f}{\partial t} - K^T(A_K e + Bb\delta_u + \dot{\tilde{x}}^* - \dot{\bar{r}}) + r^{(n+1)}}{b(t)}. \end{cases}\quad (96)$$

As for the dynamics of z , it can be directly obtained from (11) that

$$\dot{z}(t) = g(z, \varphi_1(\tilde{x}, t), \dots, \varphi_n(\tilde{x}, t), t), \quad t \geq t_0. \quad (97)$$

Based on the dynamics of (e, ζ, δ_u, z) , i.e., (85), (88), (91), (93), (95) and (97), the closed-loop form can be written as (30)–(31).

325 Next we will prove that the bounds of $(\Delta_{e0}, \Delta_{\zeta 0}, \Delta_{\delta_u 0}, \Delta_{e1}, \Delta_{\zeta 1}, \Delta_{\delta_u 1})$ satisfy (32) for $e \in \{e \mid \|e\| \leq \rho_e\}$, $\zeta \in \{\zeta \mid \|\zeta\| \leq \rho_\zeta\}$, $\delta_u \in \{\delta_u \mid |\delta_u| \leq \rho_u\}$ and $\omega_o \in \{\omega_o \mid \omega_o \geq \omega_o^*\}$ with any given positives $\rho_e, \rho_\zeta, \rho_u$ and ω_o^* . Hence we only need to present the detailed expressions of the functions $\pi_{e0}(\cdot)$, $\pi_{\zeta 0}(\cdot)$, $\pi_{\zeta 1}(\cdot)$, $\pi_{\delta_u}(\cdot)$, $\pi_{\delta_u 0}(\cdot)$ and $\pi_{\delta_u 1}(\cdot)$.

Due to Assumption 1, there is

$$\sup_{t \geq t_0} \|\bar{r}(t)\| \leq nM_r, \quad \sup_{t \geq t_0} \{|r^{(n)}(t)|, |r^{(n+1)}(t)|\} \leq M_r. \quad (98)$$

Combined with the dynamics of $\tilde{x}^*$ (17), there exists a positive constant M_{x^*} such that

$$\sup_{t \geq t_0} \{ \|\tilde{x}^*(t)\|, \|\dot{\tilde{x}}^*(t)\| \} \leq M_{x^*}. \quad (99)$$

According to the bounds of $\left(b, \dot{b}, b^{-1}, f, \frac{\partial f}{\partial \tilde{x}}, \frac{\partial f}{\partial t}\right)$ shown in Lemma 2, (86) implies that

$$\sup_{\|e\| \leq \rho_e} |\Delta_{e0}(e, t)| \leq \pi_{e0}(\rho_e) \triangleq \psi_f(\rho_e + M_{x^*}) + \|K\|(M_{x^*} + nM_r) + M_r, \quad \forall \rho_e \geq 0. \quad (100)$$

Due to (88), it can be directly obtained that

$$|\Delta_{e1}(\delta_u, t)| = |b(t)\delta_u(t)| \leq \psi_b|\delta_u|. \quad (101)$$

Owing to (92) and (94), the bounds of $\Delta_{\zeta 0}$ and $\Delta_{\zeta 1}$ are provided as follows.

$$\sup_{\|e\| \leq \rho_e} |\Delta_{\zeta 0}(e, t)| \leq \pi_{\zeta 0}(\rho_e) \triangleq \psi_f(\rho_e + M_{x^*}) (1 + \|A\|(\rho_e + M_{x^*}) + \psi_f(\rho_e + M_{x^*})),$$

$$\sup_{\|e\| \leq \rho_e, |\delta_u| \leq \rho_u, \omega_o \geq \omega_o^*} |\Delta_{\zeta 1}(e, \zeta, \delta_u, \omega_o, \kappa, t)| \leq \pi_{\zeta 1}(\rho_e) + \pi_{\omega}(\omega_o^*) \kappa \|\zeta\| + (\kappa + 1) \pi_{\delta_u}(\rho_e, \rho_u),$$

for any $\rho_e \geq 0$, $\rho_u \geq 0$ and $\omega_o^* \geq 0$, where $\pi_{\zeta 1}(\rho_e) = \psi_b^2(\psi_f(\rho_e + M_{x^*}) + \|K\|(\rho_e + M_{x^*} + nM_r) + M_r) + \psi_f(\rho_e + M_{x^*})(1 + \|A_K\|\rho_e + M_{x^*})$, $\pi_{\omega}(\omega_o^*) = \max\{1, \psi_b\} \|K_e\| \|T_1(\omega_o^*)\|$ and $\pi_{\delta_u}(\rho_e, \rho_u) = \max\{\psi_b^2\rho_u, \psi_b\rho_u(1 + \psi_f(\rho_e + M_{x^*}))\}$. With the help of Lemma 2, the following bounds of $\Delta_{\delta_{u1}}$ and $\Delta_{\delta_{u2}}$ are obtained based on (96).

$$\begin{cases} \sup_{\|e\| \leq \rho_e} |\Delta_{\delta_{u0}}| \leq \pi_{\delta_{u0}}(\rho_e) + \pi_{\omega}(\omega_o^*) \kappa \|\zeta\|, \\ \sup_{\|e\| \leq \rho_e, |\delta_u| \leq \rho_u} |\Delta_{\delta_{u1}}| \leq \pi_{\delta_{u1}}(\rho_e, \rho_u) + \pi_{\omega}(\omega_o^*) \kappa \|\zeta\|, \end{cases} \quad \forall \rho_e \geq 0, \rho_u \geq 0, \quad (102)$$

where $\pi_{\delta_{u0}}(\rho_e) = \max\{\psi_b, \psi_b^3\}(\psi_f(\rho_e + M_{x^*})(2 + \|A\|(\rho_e + M_{x^*}) + \psi_f(\rho_e + M_{x^*})) + \|K\|(1 + \|A\|)(\rho_e + M_{x^*} + nM_r) + \psi_f(\rho_e + M_{x^*})) + 2M_r$ and $\pi_{\delta_{u0}}(\rho_e, \rho_u) = \pi_{\delta_{u0}}(\rho_e) + \psi_b(\psi_f(\rho_e + M_{x^*}) + \|K\|)(\|A_K\|\rho_e + \psi_b\rho_u + M_{x^*} + nM_r)$.

Based on (100)–(102), (32) is proved.

Data availability statements

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

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