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The transfer function of the SARS-CoV-2 spread in different countries with stability, observability, and controllability analysis

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Abstract

This paper presents the determination of the transfer function of the spreading pandemic caused by SARS-CoV-2 in different countries. The methodology of system identification, well known in control system theory, based on the number of infected was used. Appropriate hypotheses have been adopted to determine the transfer function of the system. Each country is viewed as a separate system, and comparisons of determined systems are given. The systems are also presented in the state space, the stability of the systems is analysed, and the matrices of controllability and observability are determined. After analysis, it is shown that the spread of the SARS-CoV-2, for each country, can be described with the same order of transfer function and differential equation.

Introduction

This paper is presented the mathematical model of the spreading of the virus SARS-CoV-2 based on the number of new cases per day. The mathematical model was identified as a linear mathematical model in the Laplace, complex S domain. The system identification algorithm from the theory of automatic control is applied.

In December 2019, China reported a cluster of cases of pneumonia in people in Wuhan, Hubei Province. The responsible pathogen is a novel coronavirus, named severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). The first cases in Europe of confirmed infection, named coronavirus disease 2019 (COVID-19), with the first patient diagnosed with the disease on 24/01/2020. COVID-19 is an infectious disease caused by the most recently discovered coronavirus. This new virus and disease were unknown before the outbreak began in Wuhan, China, in December 2019, [1],[2].

Thus, the only effective mechanisms currently available to control spread are social isolation and distancing, quarantine, travel restrictions, and changes in individual behaviour, e.g. usage of face masks, heightened preoccupation with hygiene [3] and vaccination

Predictive mathematical models for epidemics and pandemic can be used for a fundamental understanding of the course of the spread and to plan effective control strategies, [4].

Governments around the world are currently solving the spread of COVID-19 within their jurisdictions by developing, applying, and adjusting multiple variations on pandemic intervention strategies, [5]. Those strategies vary across nations, but fundamental approaches are adapted in most countries by national healthcare systems, [5].

Different mathematical models of the pandemic have been adopted, [4], [5], [6], [7], [8], [9] with two basic goals, namely pandemic prediction and control. Existing models [10], [11], [12], [13], [14], [15] which can be found in the literature are based on output functions. Currently, three basic output functions are monitored. Output function based on the number of new patients. All of these models are either statistical or stochastic or deterministic or graphical. These models all deal with the output functions of the model based on which pandemic predictions are made based on all three given functions.

The Control Systems theory approach considers a model described by the internal dynamics of systems from the aspect of system theory, system identification and control systems. The idea and motivation come from the theory of automatic control because it proved to be an excellent systemic approach to defining a mathematical model of the system when the initial equations and physical laws to define the system are unknown. This methodology applies to almost all technical systems where it is not possible to define a mathematical model in any other way. Very often this way of identification is also called black-box modelling because based on all theoretical considerations in the theory of automatic control, based on the input and output vectors, a mathematical model of the system can be defined. This methodology is also applied in other scientific disciplines, and it has certainly found its application in biomedical engineering.

In this paper, a methodology will be applied to determine the spread of the SARS-CoV-2 virus pandemic among the population in different countries, to determine an adequate mathematical model and determine stability, observability and controllability. Other mathematical models are mainly based on predictions, which is only one of the aspects that we can analyse with such a determined mathematical model, which is primarily given by its transfer function. The model determined by the transfer function, in addition to the prediction, also provides an insight into the internal dynamics of the system. In addition to determining the response, it is possible to determine the control algorithms when the stability conditions are met.

Results

Data Acquisition (DAQ)

The data used for the research in this paper were collected from the EU Open Data Portal. The European Union Open Data Portal (EU ODP) gives access to open data published by EU institutions and bodies, also there are data for other countries outside of the EU.

For this research, data collected daily were used, specific countries chosen by the authors will be considered in the differences in mathematical models. Data pre-processing involves consideration for different countries. In this research were considered data for Austria, Germany, Serbia, and Italy.

Before implementing the data, it is necessary to sort the data from the moment when the first case of SARS-Cov-2 virus infection was recorded in a chosen country. Period from the first recorded case on global level till first recorded case in the chosen country can be considered as time delay of the system.

day	month	year	cases	deaths	country	geoid	country	popDat	continent	Cumulative number for 14 days of
14			Sort A to Z		AT	AUT	8858775	Europe	472,4581	
13			Sort Z to A		AT	AUT	8858775	Europe	482,6401	
12			Sort by Color		AT	AUT	8858775	Europe	503,2637	
11			Sheet View		AT	AUT	8858775	Europe	523,4922	
10			Clear Filter From "countriesAndTerm..."		AT	AUT	8858775	Europe	557,1425	
9			Filter by Color		AT	AUT	8858775	Europe	590,8153	
8			Text Filters		AT	AUT	8858775	Europe	608,9217	
7			Search		AT	AUT	8858775	Europe	619,1827	
6					AT	AUT	8858775	Europe	647,0646	
5					AT	AUT	8858775	Europe	681,8211	
4					AT	AUT	8858775	Europe	715,4375	
3			☐ Anguilla		AT	AUT	8858775	Europe	749,8328	
2			☐ Antigua_and_Barbuda		AT	AUT	8858775	Europe	796,1823	
1			☐ Argentina		AT	AUT	8858775	Europe	821,0164	
30			☐ Armenia		AT	AUT	8858775	Europe	843,0963	
29			☐ Aruba		AT	AUT	8858775	Europe	858,5724	
28			☐ Australia		AT	AUT	8858775	Europe	885,969	
27			☒ Austria		AT	AUT	8858775	Europe	932,7475	
26			☐ Azerbaijan		AT	AUT	8858775	Europe	988,2969	
25			☐ Bahamas		AT	AUT	8858775	Europe	1030,594	
24			☐ Bahrain		AT	AUT	8858775	Europe	1027,253	
23			☐ Bangladesh		AT	AUT	8858775	Europe	1056,76	
22			☐ Barbados		AT	AUT	8858775	Europe	1065,689	
21			☐ Belarus		AT	AUT	8858775	Europe	1084,416	
20					AT	AUT	8858775	Europe	1081,165	
19					AT	AUT	8858775	Europe	1083,66	

Figure 1. Illustration of data pre-processing.

The models to be considered will include several periods, 90 days, the entire period for which daily data are available at the EU ODP, Figure 1. By comparative analysis of the obtained models, the differences of all obtained models will be considered. The data for the first 90 days from the beginning of the epidemic were analysed, the reason for this analysis is that in China, after 90 days in Wuhan, was declared the end of the epidemic. As is well known, the spread of the pandemic continued after 90 days in all other countries, therefore, mathematical models for longer time intervals are defined in accordance with the available data.

Data Analysis, Hypothesis and System Identification

Initial hypotheses within this research are that it is possible to define the input function of the system based on the output function, that is possible to obtain appropriate transfer functions based on output and input functions, the transfer functions, ie the differential equations of the behaviour of the model (pandemic spread) are linear and time-continuous, models can be defined by differential equations of n^{th} order where $n \geq 1$, that it is possible to perform model stabilization if the analysis shows that the models are unstable, it is possible to apply different engineering control algorithms and apply them to the obtained models, to control the spread of pandemic (epidemic), the sampling time is daily.

These hypotheses have already been partially considered and confirmed because the analogy of the system identification that has been applied is already considered on the other mathematical models of biological and biomedical systems.

One of the basic hypotheses of the research is that it is possible to define the input function and the linear transfer function, respecting all the stated assumptions that the model is linear and time continuous. Various possibilities of fitting data to obtain the appropriate transfer function of the model are considered. The output function is known, while the input function is assumed, in accordance with the hypotheses of this research. The system identification algorithm shown in Figure 2 was applied, for different variations of the transfer function parameters.

Following the research hypotheses, different transfer functions of the model were considered. The transfer functions having 1 pole and 0 zeros, then 2 poles and 0 zeros, then 2 poles and 1 zero, then 3 poles and 0 zeros, then 3 poles and 1 zero, then 3 poles and 2 zeros, then 3 poles and 3 zeros 4 poles and 0 zeros, 4 poles and 1 zero, 4 poles and 2 zeros, 4 poles and 3 zeros, etc. are assumed and mathematically modelled. It was concluded that the best match arises

when the spread of a pandemic is adopted as a third-order system. All these models are shown in Figure 2.

(a) (b)
Figure 2. Different transfer functions of the spreading pandemic SARS-CoV-2; period ~90 days (a); period ~300 days (b).

In the first iteration of obtaining a mathematical model, models were observed for 90 days through two different cases, the first from the day of data recording, ie from the first case of infection in the world, in the second case models were observed so that the first case is taken when registered in a particular country. For both cases, it was found that the mathematical model can be described as a system of the third - order, the system has 3 poles and 1 zero. In the time domain, the spread of a pandemic can be described by a 3rd-order differential equation.

In figure 5 are presented systems of a different order for one chosen country. Those models are considered as shown in the Austrian model, but for all considered countries a uniform conclusion can be made that the third-order system, namely the linear third-order system described by the transfer function in the S-Laplace domain with 3 poles and 1 zero, is acceptable as a pandemic spread model for all countries under consideration. In the time domain, this means that the spread of pandemic can be described by a third-order differential equation. It is common for technical systems to be described as at least second-order systems due to physical laws, although they can often be described as higher-order systems to obtain the most plausible mathematical model.

For the entire period, in this case for the period from the first registered case, until the last day when data were available on the EU open data portal, until 14.12.2020, the model, Figure 2b, was also adopted similarly. By analyzing the obtained models, it was decided to adopt a third-order model for a model that covers a longer period, in this case almost a year. For all considered countries, the system can be described by the transfer function of the third order, which corresponds to the differential equation of behaviour 3rd order.

Figure 3 shows the mathematical models, the transfer functions, of 4 different systems, representing 4 randomly selected countries on the European continent. These systems are characterized by different characteristics, in relation to the number of inhabitants, the habits of the population, the number of infected, etc.

Figure 3. Mathematical models, the transfer function for different countries, period ~90 days; Austria (a); Germany (b); Serbia (c); Italy (d).

The first period of 90 days was considered because after 90 days China declared the end of the epidemic, and based on the curve of the number of infected, a corresponding similarity can be observed in relation to the number of infected in Europe.

Identification of the mathematical model of the system, the model of the spread of the SARS-CoV-2 virus can be applied for the entire period of available data. Zero-day is taken as the day when the first case or cases when the infection occurred in a particular country occurred, while the last day is taken as the day until the data of daily data recording are available on the EU Open Data portal. Figure 4 represents mathematical models for different countries for the entire period.

Figure 4. Mathematical models, the transfer function for different countries, period ~300 days; Austria (a); Germany (b); Serbia (c); Italy (d).

For both observed periods of 90 and 300 days, respectively, a mathematical model can be identified, described by a transfer function, which has 3 poles and 1 zero. An illustration of the mathematical model of the system is given in Figure 5, where a block diagram of the identified mathematical model of the system is shown. It is represented by a corresponding transfer function.

Figure 5. Block diagram of the system, the spreading pandemic – SARS-CoV2

By applying the methodology presented in the paper, the transfer function of the system can be written as,

$$G(s) = \frac{b_1s + b_0}{a_3s^3 + a_2s^2 + a_1s + a_0} \quad (1)$$

Eq. (1) is defined under all initial conditions equal to zero, which is completely in accordance with the initial hypotheses of this paper. The coefficients b_1, b_0, a_3, a_2, a_1 and a_0 are known for all models. The coefficients are given in Table 1 and Table 2, for ~90 days and ~300 days, respectively. $G(s)$ in Eq. (1) is the transfer function shown in the block diagram, Figure 5.

Table 1. Coefficients of corresponding transfer functions for a period of ~ 90 days

	Coefficients					
	a_3	a_2	a_1	a_0	b_1	b_0
Austria	1	1430	559	474.8	624.8	436.2
Germany	1	250.7	495.1	239.1	-20.67	230.7
Serbia	1	12.93	44.96	42.99	-8.523	41.21
Italy	1	6.702	14.58	10.5	2.787	10.31

Table 2. Coefficients of corresponding transfer functions for a period of ~ 300 days

	Coefficients					
	a_3	a_2	a_1	a_0	b_1	b_0
Austria	1	305.6	847,8	531,5	-126,2	512,5
Germany	1	0.146	0,2039	1,665e-07	0,2079	2,961e-05
Serbia	1	5.191	7,559	2,412	0,9965	2,529
Italy	1	11.06	27,2	20,24	0,9806	20,5

It is interesting to note that the mathematical models for the whole period can be described by transfer functions of 3rd order, as in equation (3), although the coefficients differ to some extent. It is important to point out that all denominator coefficients are positive for all models. That is one of the necessary conditions for the stability of linear systems following the stability definitions and theorems.

Analyzing the coefficients of transfer functions, given in Tables 1 and 2, it is important to note that the coefficients in the highest degree of the Laplace variables are always equal to 1 and that the other coefficients a_2, a_1 and a_0 are positive, now the general model of the transfer function of a pandemic spread can be written as

$$G(s) = \frac{b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0} \quad (2)$$

If we apply the inverse Laplace transform, then the transfer function of the system (1) can be written as a differential equation of behaviour of 3rd- order, which is given by equation (3) below

$$a_3\ddot{Y}(t) + a_2\dot{Y}(t) + a_1Y(t) + a_0Y(t) = b_0U(t) + b_1\dot{U}(t) \quad (3)$$

Equation (4) represents the general dynamic equation of behaviour, obtained based on the analysis of parameters and equation (2).

$$\ddot{Y}(t) + a_2\dot{Y}(t) + a_1\dot{Y}(t) + a_0Y(t) = b_0U(t) + b_1\dot{U}(t) \quad (4)$$

Stability, Controllability and Observability of the spread SARS-CoV-2

The poles of the identified transfer functions are given in Table 3, for each country and period ~90 days and ~300 days. In addition to the poles of the transfer function, zeros of the transfer function were mentioned earlier, but they do not affect the stability of the system but show when the transfer function is equal to zero, for which values there will be no transfer from the input value to the output value. The system identification applied in this paper determines the transfer function for each system, which as previously shown by equations (2) has 3 poles and 1 zero.

Table 3. Poles and zero of the transfer functions of identified systems

	Poles		Zero	
	~90 days	~300 days	~90 days	~300 days
Austria	1430.1 -0.0002 + 0.0005i -0.0002 - 0.0005i	-302.8322 -1.8401 -0.9539	-0.6982	4.0609
Germany	-248.7584 -1.1527 -0.8336	-0.0730 + 0.4456i -0.0730 - 0.4456i -0.0000 + 0.0000i	11.1615	-1.4242e-04
Serbia	-7.9661 -3.3604 -1.6059	-2.7964 -1.9527 -0.4418	4.8359	-2.5376
Italy	-3.0491 + 0.0000i -1.8264 + 0.3263i -1.8264 - 0.3263i	-7.9688 -1.5470 + 0.3838i -1.5470 - 0.3838i	-3.6997	-20.4477poz

Figure 6. Pole – zero map of all transfer functions for period ~ 90 days (a), ~300 days (b).

Following the stated stability theorems and the analysis of the pole distribution of transfer functions, for all identified transfer functions, it is observed that they are stable in 90 days and the period of 300 days. Also, it can be seen that the poles distribution, Figure 6, for all countries is such that 1 real pole or 2 conjugate complex poles are very close to the stability limit and that the system can very easily become unstable, which certainly depends on the further spread of the pandemic.

The controllability and observability matrices were determined for all identified systems by transfer functions, which are represented by mathematical transformations in the state space, and then the controllability and observability matrices, as well as their rank, were determined.

Table 4. Rank matrix of controllability and observability

	$rank(C_o) \sim 90$	$rank(C_o) \sim 300$	$rank(O_b) \sim 90$	$rank(O_b) \sim 300$
Austria	3	3	3	3
Germany	3	3	3	3
Serbia	3	3	3	3
Italy	3	3	3	3

Analysing Table 4, it is clear that all identified system models are controllable and observable and that it is possible to determine appropriate control algorithms. Control algorithms can prevent the spread of the SARS-CoV-2 pandemic

Discussion

It is possible to determine the transfer function of the spread of the pandemic caused by the SARS-Cov2 virus. The identified systems, as transfer functions are of the third order, for each considered country the spread of the pandemic can be described by a third-order differential equation. All obtained transfer functions from the aspect of stability meet the stability criteria. All considered identified systems are stable. Mathematical transformations considered the identified mathematical models in the state space and determined the matrices of controllability and observability. The rank of controllability and observability matrices corresponds to the order of the system, $n = 3$, all mathematical models are controllable and observable.

Figure 7. Control in open loop with P – regulator.

It is possible to control a pandemic in an open-loop by applying only a proportional action of a regulator, which in practical terms means by applying various restrictions applied by governments around the world. The application of the lockdown on the engineering side is precisely the control of an open-loop pandemic, using proportional regulation, as shown in Figure 7.

The analysis of the poles of the transfer functions showed that some poles are close to the stability limit and that the systems can become unstable again by changing the parameters based on the daily number of newly infected. The systems can be controlled only by proportional action, the basic regulation of pandemic spread is a lockdown, which results in the application of the regulation effect, proportional. If it were possible to apply other actions in control algorithms, integrally or differentially as in classical control, the values of desired outputs would be reached faster. Good results are theoretically achieved by applying feedback control using P, PD, PID regulation, or MPC - model predictive control or Fuzzy control algorithms, Figure 8. Theoretically, the control of the pandemic, the spread of the virus SARS-CoV-2 in a closed loop can be realized by applying the aforementioned algorithms, while the realization of the system would be possible by realizing a simplified block diagram given in Figure 8, where it should be noted that it is realized with the negative feedback.

Figure 8. Control in closed loop with different types of regulators.

Methods

System Identification

System identification is methodology when obtained experimental or measurement data of output or input-output function are used to develop a mathematical model of the system [16]. The system identification of the system requires that input and output function data are available, and the structure of the system can be selected, [17]. There are different approaches of system identification linear or nonlinear, parametric or nonparametric. Methodology selection of the system identification is one of the most important decisions. This paper is considering different approaches to get the proper mathematical model of the system. Mathematical models can be described with a transfer function, state-space models, differential equations, and other well-known mathematical representation.

Figure 9. presents the general system identification approach, [18]. When there is a need for a mathematical model of a system, if this cannot be done by setting up appropriate physical and mathematical equations, then the determination of the mathematical model can be done experimentally. It is necessary to design an experiment in which we will consider which key parameters of the system we can measure.

After determined the values that can be measured, an experiment can be designed, and experimental data can be collected. Data set now can be filtered and all preprocessing analyses

should be done in this phase. The structure of the model is assumed, fitting data into the model is the next step in the flowchart, Figure 9. After data fitting model validation should be done.

Figure 9. represents an engineering approach in the system identification, [18]. This methodology also can be used in other non-technical systems as well as for all known systems and subsystems for example in biology, medicine, social sciences, etc

Figure 9. Flow chart of the key elements in the system identification cycle.

Identification of the pandemic spread

If we apply the system identification methodology to identify the mathematical model of pandemic spread, then it is certainly necessary to adjust the general system identification diagram to a certain extent as shown in Figure 10.

In this case, instead of the experiment, there is data acquisition. In this paper, acquired data will be used from official data available on EU Open Data Portal and WHO. When data for a particular time are collected data pre-processing should be done. All this data is then fitted into the appropriate, model, Figure 10. Once the results are obtained, it is necessary to perform model validation. If the model meets the conditions, the mathematical function of the model can be adopted, in the appropriate domain Laplace or time, and with a different representation of the model, as a transfer function, differential equation model, state-space model, etc. If we apply the analogy like in engineering systems stability, controllability and observability can be checked.

Figure 10. Flow chart of the key elements in the system identification of the spread of SARS-CoV-2.

Transfer function

The transfer function is one of the basic terms for linear stationary systems. Transfer function allows analysis of the dynamical characteristics of the systems in the complex domain. Transfer function as a mathematical model in many cases is the most efficient and the simplest way for the analysis of the dynamical properties of the system. The lowest-order differential equation describes the system completely dynamically, which allows determining the values of the

system output and all its derivatives on the interval $[t_0, \infty)$, for each initial point $t_0 \in \mathcal{R}$, and at known initial values (at time t_0) of all derivatives of the output and with known input values at the same interval $[t_0, \infty)$ is a differential equation behaviour of that system, or shorter equation of the behaviour of the system, [20]

The general form of the differential equation of the behaviour of a single transmission system can be represented by the following scalar inhomogeneous differential equation with constant coefficients, [20],

$$\begin{aligned} a_n Y^{(n)}(t) + a_{n-1} Y^{(n-1)}(t) + \dots + a_2 \ddot{Y}(t) + a_1 \dot{Y}(t) + a_0 Y(t) = \\ = b_0 U(t) + b_1 \dot{U}(t) + b_2 \ddot{U}(t) + \dots + a_{m-1} U^{(m-1)}(t) + b_m U^{(m)}(t) \end{aligned} \quad (5)$$

where n is the highest derivative of the output value, and m is the highest derivative of the input value and must be released $m \leq n$.

The transfer function, denoted by $G(s)$, is defined as the quotient of the left Laplace transforms of the output quantity $y(t)$ and the input quantity $u(t)$,

$$G(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}} = \frac{Y^-(s)}{U^-(s)} \quad (6)$$

at all initial conditions equal to zero.

The transfer function of a system describes the law, in the complex domain of the complex variables, by which the system effects are transferred from the input value to the output value.

The transfer function is one of the mathematical models of the system, given in the complex domain because as mentioned earlier it represents one of the simplest ways of analysis and the dynamic behaviour of the system. In this paper, the system identification algorithm will be applied, apropos the transfer function of the system as a mathematical description of the system.

It should be noted that the inverse Laplace transform represents the system in the time domain, in addition, the system will be displayed in the state space, to consider some important characteristics of the system, speaking of the spread of the SARS-CoV-2 virus pandemic among the population.

Stability, Controllability and Observability

Mathematical models of the system can be represented by a differential equation of behaviour, a transfer function, but they can also be represented in the state space. The representation of the model in the state space is important because of the analysis of the conditions of controllability and observability, but also because of the conditions of system stability, which will be explained first.

The general equation of a system in state space is given by state equation and output equation, eq. (7), eq. (8), respectively,

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (7)$$

$$y(t) = Cx(t) + Du(t) \quad (8)$$

Based on [20], the transfer function of the system can be written as,

$$G(s) = C(sI - A)^{-1}B + D \quad (9)$$

where A, B, C and D are the matrices given in the equation of state and the equation of output of the state space model representation, Eq.(7),(8), respectively.

The relationship between the poles of the transfer function $G(s)$ and the eigenvalues of matrix A is obtained from the relationship between the transfer matrix and the matrices that determine the equation of state and the equation of output.

$$G(s) = C \frac{\text{adj}(sI - A)}{\det(sI - A)} B + D = \frac{C \text{adj}(sI - A)B + D \det(sI - A)}{\det(sI - A)} \quad (10)$$

The differential equations of the behaviour of the system and the transfer functions of the system for each country are known, and that the systems have been successfully identified as linear with certain assumptions, stability analysis is possible and simple.

Since the eigenvalues of the matrix A are determined by solving the characteristic equations,

$$f(s) = \det(sI - A) = 0 \quad (11)$$

where $f(s)$ is a characteristic polynomial of the matrix A. Its roots represent eigenvalues matrix A.

The stability theorems [20] and the conditions of stability, is given below.

Theorem 1. For the zero-equilibrium state $x_r = 0_x$ of the linear system eq. (7-8) to be asymptotically stable, it is necessary and sufficient that the real parts of all eigenvalues of the matrix A are negative:

$$\text{Res}_i^*(A) < 0, \quad \forall i = 1, 2, \dots, \mu \quad (12)$$

Theorem 2. For a linear system eq. (7-8) to be border stable (at the stability limit), it is necessary and sufficient that (a) - (c):

the real parts of all eigenvalues of matrix A are non-positive,

$$\text{Res}_i^*(A) \leq 0, \quad \forall i = 1, 2, \dots, \mu \quad (13)$$

there is at least one eigenvalue of matrix A with the real part equal to zero

$$\exists k \in \{1, 2, \dots, \mu\} \Rightarrow \text{Res}_k^*(A) = 0 \quad (14)$$

all eigenvalues of matrix A with zero real part are single

$$\text{Res}_k^*(A) = 0 \Rightarrow \vartheta_k^* = 1 \quad (15)$$

Theorem 3. For a linear system eq. (7-8) to be unstable it is necessary and sufficient that, there is at least one eigenvalue of matrix A with a positive real part

$$\exists k \in \{1, 2, \dots, \mu\} \Rightarrow \text{Res}_k^*(A) > 0 \quad (16)$$

or

there is at least one eigenvalue of matrix A with zero real part of multiplicity greater than one

$$\exists k \in \{1, 2, \dots, \mu\} \Rightarrow \text{Res}_k^* = 0, \vartheta_k^* > 1 \quad (17)$$

or

to apply both (a) and (b) simultaneously.

An illustration of the stability theorems is given in Figure 11. Figure 11 shows the Laplace plane, where the x-axis is the real axis and the y axis of the coordinate system is the imaginary axis. The imaginary axis also represents the border of stability. The left half of the Laplace half-plane following the definitions and theorems is a stable region, while the right half-plane of the Laplace plane is the region of instability.

Figure 11. Pole map of continuous-time systems

By mathematical transformations, for each system, a controllability matrix and an observability matrix were obtained. By the definitions and theorems given in the literature, [20], it is stated that the system is given by Eq. (7), (8) is controllable if and only if

$$\text{rank}(C_o) = n \quad (17)$$

where C_o is controllability matrix defined as

$$C_o = [B \ AB \ A^2B \ \dots \ A^{n-1}B] \quad (18)$$

and n is the dimension of the matrix A, which corresponds to the order of the system defined by the transfer function of the system and/or the order of the differential equation of behaviour of the system.

The system is given by Eq. (7), (8) is observable if and only if

$$\text{rank}(O_b) = n \quad (19)$$

where O is observability matrix defined as

$$O = (C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \dots \quad (A^T)^{n-1} C^T) \quad (20)$$

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Author Contribution statement

Jasmina Lozanovic Sajic prepared the main manuscript text and figures. Sonja Langthaler participated in the work of making figures 5, 7 and 8. Sara Stoppacher participated in the making of figures 9 and 10. Christian Baumgartner and Jasmina Lozanovic Sajic adopted research hypotheses and applied systems management theory to the spread of the pandemic. All authors reviewed the manuscript.

Competing interests

The authors declare no competing interests. Authors declare that the authors have no competing interests as defined by Nature Research, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Figures

B	C	D	E	F	G	H	I	J	K	L	M	N	O
day	month	year	cases	deaths	country	geold	country	popDat	contine	Cumul	e_number_for_14_days_of		
14	1	2020	500	10	Austria	AT	AUT	8858775	Europe	472,4581			
13	1	2020	450	8	Austria	AT	AUT	8858775	Europe	482,6401			
12	1	2020	400	6	Austria	AT	AUT	8858775	Europe	503,2637			
11	1	2020	350	4	Austria	AT	AUT	8858775	Europe	523,4922			
10	1	2020	300	3	Austria	AT	AUT	8858775	Europe	557,1425			
9	1	2020	250	2	Austria	AT	AUT	8858775	Europe	590,8153			
8	1	2020	200	1	Austria	AT	AUT	8858775	Europe	608,9217			
7	1	2020	150	0	Austria	AT	AUT	8858775	Europe	619,1827			
6	1	2020	100	0	Austria	AT	AUT	8858775	Europe	647,0646			
5	1	2020	50	0	Austria	AT	AUT	8858775	Europe	681,8211			
4	1	2020	0	0	Austria	AT	AUT	8858775	Europe	715,4375			
3	1	2020	0	0	Austria	AT	AUT	8858775	Europe	749,8328			
2	1	2020	0	0	Austria	AT	AUT	8858775	Europe	796,1823			
1	1	2020	0	0	Austria	AT	AUT	8858775	Europe	821,0164			
30	1	2020	0	0	Austria	AT	AUT	8858775	Europe	843,0963			
29	1	2020	0	0	Austria	AT	AUT	8858775	Europe	858,5724			
28	1	2020	0	0	Austria	AT	AUT	8858775	Europe	885,969			
27	1	2020	0	0	Austria	AT	AUT	8858775	Europe	932,7475			
26	1	2020	0	0	Austria	AT	AUT	8858775	Europe	988,2969			
25	1	2020	0	0	Austria	AT	AUT	8858775	Europe	1030,594			
24	1	2020	0	0	Austria	AT	AUT	8858775	Europe	1027,253			
23	1	2020	0	0	Austria	AT	AUT	8858775	Europe	1056,76			
22	1	2020	0	0	Austria	AT	AUT	8858775	Europe	1065,689			
21	1	2020	0	0	Austria	AT	AUT	8858775	Europe	1084,416			
20	1	2020	0	0	Austria	AT	AUT	8858775	Europe	1081,165			
19	1	2020	0	0	Austria	AT	AUT	8858775	Europe	1083,66			

Figure 1

Illustration of data pre-processing.

Figure 2

Different transfer functions of the spreading pandemic SARS-CoV-2; period ~90 days (a); period ~300 days (b).

Figure 3

Mathematical models, the transfer function for different countries, period ~90 days; Austria (a); Germany (b); Serbia (c); Italy (d).

(a)

(b)

(c)

(d)

Figure 4

Mathematical models, the transfer function for different countries, period ~ 300 days; Austria (a); Germany (b); Serbia (c); Italy (d).

Figure 5

Block diagram of the system, the spreading pandemic – SARS-CoV2

Figure 6

Pole – zero map of all transfer functions for period ~ 90 days (a), ~300 days (b).

Figure 7

Control in open loop with P – regulator.

Figure 8

Control in closed loop with different types of regulators.

Flow chart of the key elements in the system identification cycle.

Figure 10

Flow chart of the key elements in the system identification of the spread of SARS-CoV-2.

Figure 11

Pole map of continuous-time systems