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Constitutive modeling of an electro-magneto-rheological fluid: A continuum mechanics approach

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ABSTRACT

The present article deals with a continuum mechanics-based method to model an electro-magneto-rheological (EMR) fluid deformation under an electromagnetic field. The proposed method follows the fundamental laws of physics, including the principles of thermodynamics. We start with the general balance laws for mass, linear momentum, angular momentum, energy, and the second law of thermodynamics in the form of Clausius-Duhem inequality with Maxwell's equations. Then, we derive the generalized constitutive relation for EMR fluids following the representation theorem. To validate of the same, the developed constitutive relation is applied to an electro-rheological fluid valve system. The analytical predictions of the considered system are consistent with the experimentation. At last, we simulate different velocity profiles from the developed constitutive relation in the case of the parallel plate configuration. As a result, we succeed in providing more physics-based analytical findings than the existing studies in the literature.

1 Introduction

In the current scenario, electro-magneto-rheology established a new research direction with enough potential to provide fruitful ideas for modern engineering and medical field applications. In line with that, new hybrid electro-magneto-rheological (EMR) fluid systems that couple electro-magneto-mechanical properties of smart fluids[1–3] are under development in laboratories worldwide. There is a specific need to understand the deformation of such smart fluids due to their usage finding in various applications like clutches and brakes in cars, vibration-dampers, and absorbers, lubricating fluids in bearings, smart transducers for medical purposes, etc. In engineering applications, EMR fluids are generally used in hydraulic systems, and these systems are inherently nonlinear. Therefore, their accurate model prediction is so essential that it will directly affect the system performance.

This work is motivated by our previous work[4] in case of deformable solids. We herein focus on the constitutive modeling of EMR fluids specifically used in smart hydraulic systems. These EMR fluids are characterized by their ability to undergo significant variation in their mechanical properties with an electromagnetic field. EMR fluids are smart, synthetic fluids that change their viscosity from liquid to semi-solid state within milliseconds if a sufficiently strong electric or magnetic field is applied[5,6]. These fluids usually consist of micron-sized polarizable and magnetizable solid particles dissolved in a non-conducting liquid like mineral or silicone oil. EMR fluids also offer an innovative potential for quick and adaptively controllable electro-magneto-mechanical interfaces in control units when used in suitable devices[7,8]. A coupled electro-magneto-rheological theory is needed to model such EMR fluids within the framework of the continuum mechanics-based approach. Also, the coupled theory must describe the electromagnetic field interaction with the moving deformable media[4,9–12] such as EMR fluids.

To mention some earlier works on electro-rheological (ER) and magneto-rheological (MR) fluids, Winslow[13] made an initial attempt to understand the behavior of ER fluids under external gradient. He[13] investigated an electrically induced fibration of small particles in fluid liquid suspension. Nevertheless, unfortunately, their applicability to understanding the deformation modeling of such rheological fluids was not fully identified for many years. After that, the first standardized approach to model these rheological behaviours under external gradient were presented by various researchers, namely, Atkin and Bullogh[14], Abu-Jdayil and Brunn[15–17], Rajagopal[5] along-with Yalamanchili[18] and Wineman[6]. Herein, they[5,6,18] adopted the classical continuum mechanics-based approach to model the rheological behaviours of ER fluids under external gradient. Further, some researchers, namely, Conard et al.[19], and Jordan et al.[20] also proposed different one-dimensional models for ER fluids.

Furthermore, Rajagopal and Wineman[5] developed three-dimensional models satisfying the appropriate invariance requirements. However, in these models[5], the electric field was treated as a constant field. In addition to the deformation modeling of ER fluid under external gradient, similar studies on the deformation modeling of MR fluid were also focused by the researchers[7,21,22] followed by similar approaches.

To mention some earlier works on EMR fluids, Fujita et al.[1] investigated a combined electromagnetic field control effect on the rheological properties of a fluid. However, they[1] have not fully utilized the magnetic field for fine control of the rheological properties. Next, Minagawal et al.[2] developed a new technique to measure the rheological properties of fluids under electric and magnetic fields. Further, Koyama et al.[3] compared an EMR effect observed in fluids with the parallel and cross-field systems. Moreover, they[3] suggested that the parallel-field effect was more significant than the crossed one.

From the literature, we find that many works on the modeling of ER and MR fluids have been devoted. However, to the best of our knowledge, a complete analysis dedicated to the deformation mechanics of an EMR fluid continua under an electromagnetic field is still not been reported. In line with that, the present article aims to generalize the deformation concept of fluid continua to electro-magneto-rheology in contrast to the existing studies on ER and MR fluids. For that, a generalized constitutive relation is developed for an incompressible isotropic non-Newtonian EMR fluid under an electromagnetic field. To develop the same, a thermodynamically consistent classical continuum mechanics-based approach[9,10] is adopted. Additionally, the developed constitutive relation is also experimentally validated in contrast to the existing works[5,6,18] aimed at the pure theoretical models for smart fluids.

The further part of the paper is organized as follows. In Section 2, a brief review on electro-magnetorheology related to an isotropic EMR fluid continua is summarized. In Section 3, a generalized constitutive relation is developed for an incompressible isotropic non-Newtonian EMR fluid under an electromagnetic field. Next, in Section 4, the developed constitutive relation is applied to an ER fluid valve system to validate the same. Further, in Section 5, different velocity profiles are predicted from the developed constitutive equation for different forms of shear viscosity in the parallel plate configuration. At last, Section 6 explains about some concluding remarks.

2 Fundamentals of electro-magneto-rheology

In this section, a brief review on electro-magneto-rheology related to the deformation of EMR fluid continua is presented, followed by the fundamental laws of physics. These fundamental laws provide initial steps to model the exact electro-magneto-rheological behavior of EMR fluids under an electromagnetic field application.

2.1 Kinematics

Consider an EMR fluid occupying the material space β_0 in a stress-free configuration. The material point in reference configuration β_0 is represented by the position vector $\mathbf{X}$ with respect to an origin. With an application of electromagnetic field, the material deforms and a point $\mathbf{X}$ in the material occupies a new position $\mathbf{x} = \kappa(\mathbf{X},t)$ in the current configuration β. Wherein κ represents an one-to-one deformation mapping. Therefore, the corresponding velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of that material point in three-dimensional Euclidean space, at the instant of time t are defined as

$$\mathbf{v} = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}, \quad \mathbf{a} = \frac{\partial^2 \mathbf{x}(\mathbf{X},t)}{\partial t^2}. \tag{1}$$

The velocity gradient tensor $\mathbf{L}(\mathbf{x},t)$ is represented as

$$\mathbf{L} = \nabla \mathbf{v}, \tag{2}$$

wherein $\nabla \mathbf{v}$ represents the gradient operator on velocity field. This velocity gradient tensor $\mathbf{L}(\mathbf{x},t)$ may also be represented in the sum of symmetric part $\mathbf{d}$ and anti-symmetric part $\mathbf{W}$ as

$$\mathbf{L} = \mathbf{d} + \mathbf{W} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) + \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right). \tag{3}$$

2.2 Electromagnetic field equations

At reference configuration β_0, the fluid continua is assumed as a free of current and electric charge with the time-independent properties. Additionally, we consider the electric field vector $\mathbf{E}$, the electric displacement vector $\mathbf{D}$ and the polarization density $\mathbf{P}$ as the electric field variables. Similarly, the magnetic field vector $\mathbf{H}$, the magnetic displacement vector $\mathbf{B}$ and the magnetization density vector $\mathbf{M}$ are considered as the magnetic field variables in the current configuration β. In condensed matter, these field variables are related as

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0[\mathbf{H} + \mathbf{M}], \tag{4}$$

wherein ε_0 and μ_0 represent the electric permittivity and magnetic permeability of free space, respectively. The electric field variables and the magnetic field variables both satisfy the given Maxwell's equations[23] under our constant electric field as well as constant magnetic field application assumption as

$$\nabla \times \mathbf{E} = 0, \quad \nabla.\mathbf{D} = 0, \quad \nabla \times \mathbf{H} = 0, \quad \nabla.\mathbf{B} = 0, \tag{5}$$

wherein $\nabla \times \mathbf{E}$ and $\nabla.\mathbf{D}$ denote the curl and divergence operators on $\mathbf{E}$ and $\mathbf{D}$, respectively.

2.3 Balance laws

The mechanical balance laws for fluids are expressed in the local form as

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0. \tag{6}$$

Conservation of linear momentum

$$\nabla.\mathbf{S} - \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{f}_{em} = 0, \tag{7}$$

wherein $\mathbf{S}$ and $\mathbf{f}_{em}$ denotes the Cauchy stress tensor and the electromagnetic body force (per unit volume), respectively. The generalized electromagnetic body force[23] $\mathbf{f}_{em}$ in terms of electromagnetic field variables is given as

$$\mathbf{f}_{em} = q\mathbf{E} + \mu_0^{-1} \nabla \mathbf{B}.\mathbf{M} + \nabla \mathbf{E}.\mathbf{P} + \frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B}) + \nabla.[\mathbf{v} \otimes (\mathbf{P} \times \mathbf{B})]. \tag{8}$$

Conservation of angular momentum

$$\varepsilon : \mathbf{S} + [\mu_0^{-1}\mathbf{M} + \mathbf{v} \times \mathbf{P}] \times \mathbf{B} + \mathbf{P} \times \mathbf{E} = \mathbf{0}, \tag{9}$$

wherein $[\varepsilon : \mathbf{S}]$ represents $\varepsilon_{ijk}\mathbf{S}_{jk}$ and ε_{ijk} is a permutation tensor. For the detailed discussion on electromagnetic interaction in deformable continua, we refer to the literature[24–26] and references therein.

Conservation of energy (First law of thermodynamics)

$$\frac{d}{dt}\left(\rho U + \frac{1}{2}\rho \mathbf{v}.\mathbf{v}\right) = \nabla.(\mathbf{S}.\mathbf{v}) + w_{em}, \tag{10}$$

wherein U is the internal energy (per unit mass) and w_{em} is the electromagnetic power (per unit volume) in the absence of heat. The general expression of the electromagnetic power w_{em} is given as[23]

$$w_{em} = \mathbf{f}_{em}.\mathbf{v} - \mathbf{M}_e.\dot{\mathbf{B}} + \rho \frac{d}{dt}\left(\frac{1}{\rho}\mathbf{P}\right).\mathbf{E}_e, \tag{11}$$

wherein the superposed dot $\dot{()}$ in the corresponding variable represents the material time derivative. From the Lorentz's theory of electrons, we have $\mathbf{M}_e = \mu_0{}^{-1}\mathbf{M} + \mathbf{v} \times \mathbf{P}$ and $\mathbf{E}_e = \mathbf{E} + \mathbf{v} \times \mathbf{B}$. Herein, the corresponding terms $\mathbf{M}_e$ and $\mathbf{E}_e$ represent the effective magnetization and the effective electric field, respectively. Now, from equations (6) and (7), the above equation (10) may be rewritten as

$$\rho \dot{U} = \mathbf{S} : \mathbf{L} + \mathbf{M}_e.\dot{\mathbf{B}} + \frac{\dot{\rho}}{\rho}(\mathbf{E}_e.\mathbf{P} + \mathbf{M}_e.\mathbf{B}) + \dot{\mathbf{P}}.\mathbf{E}_e. \tag{12}$$

Clausius–Duhem inequality (Second law of thermodynamics)

$$\mathbf{S} : \mathbf{L} + \rho \dot{\varphi} \geqslant 0, \tag{13}$$

wherein $\dot{\varphi}$ term represents the material time derivative of Helmholtz free energy function for a given fluid system. The above representation (13) is a systematic way of expressing the second law of thermodynamics used in the classical continuum mechanics in the absence of heat. This inequality is particularly useful in determining whether the constitutive relation of material is thermodynamically consistent or not.

3 Constitutive modeling

In this section, the concepts of an electro-magneto-rheological deformation of fluid continua presented in previous section 2 are utilized to derive the constitutive relation for an EMR fluid under an electromagnetic field.

In order to define our EMR fluid system under an electromagnetic field, we first have to define a Helmholtz free energy function φ as[27]

$$\varphi = U - \frac{1}{\rho}\left(\mathbf{E}_e.\mathbf{P} + \mathbf{M}_e.\mathbf{B}\right), \tag{14}$$

through which a thermodynamic pressure P_{th} is defined for the given fluid system. By substituting (14) and (10) into (13), we obtain an expression of the dissipation inequality given as

$$-\rho\dot{\varphi} + \mathbf{S}:\mathbf{L} - \mathbf{M}_e.\dot{\mathbf{B}} - \dot{\mathbf{E}}_e.\mathbf{P} \geq 0. \tag{15}$$

Following the principle of Euclidean invariance, we may assume that all relativistic effects as well as the effect of Earth's spin are neglected and there exist a generalized space vector $\mathbf{k}$ defined as

$$\mathbf{k} = \hat{\mathbf{k}}(\rho, \mathbf{d}, \mathbf{E}_e, \mathbf{B}). \tag{16}$$

Based on the above space vector (16), we obtain an expression of $\dot{\varphi}$ through the total derivative given as

$$\dot{\varphi} = \left(\frac{\partial\varphi}{\partial\rho}\right)\left(\frac{\partial\rho}{\partial t}\right) + \left(\frac{\partial\varphi}{\partial\mathbf{d}}\right):\left(\frac{\partial\mathbf{d}}{\partial t}\right) + \left(\frac{\partial\varphi}{\partial\mathbf{E}_e}\right).\left(\frac{\partial\mathbf{E}_e}{\partial t}\right) + \left(\frac{\partial\varphi}{\partial\mathbf{B}}\right).\left(\frac{\partial\mathbf{B}}{\partial t}\right). \tag{17}$$

In this regard, it is assumed that an objective part only of these time derivatives contributes to the rate of change of the Helmholtz free energy. By replacing the above $\dot{\varphi}$ term in equation (15), we have

$$\left(\mathbf{S}:\mathbf{L} - \rho\frac{\partial\varphi}{\partial\rho}\dot{\rho}\right) - \left(\rho\frac{\partial\varphi}{\partial\mathbf{d}}\right):\dot{\mathbf{d}} - \left(\rho\frac{\partial\varphi}{\partial\mathbf{E}_e} + \mathbf{P}\right).\dot{\mathbf{E}}_e - \left(\rho\frac{\partial\varphi}{\partial\mathbf{B}} + \mathbf{M}_e\right).\dot{\mathbf{B}} \geq 0. \tag{18}$$

Using equation (6), we may rewrite the above dissipation inequality (18) as

$$\left(\mathbf{S} + \rho^2\frac{\partial\varphi}{\partial\rho}\mathbf{I}\right):\mathbf{L} - \rho\frac{\partial\varphi}{\partial\mathbf{d}}:\dot{\mathbf{d}} - \left(\rho\frac{\partial\varphi}{\partial\mathbf{E}_e} + \mathbf{P}\right).\dot{\mathbf{E}}_e - \left(\rho\frac{\partial\varphi}{\partial\mathbf{B}} + \mathbf{M}_e\right).\dot{\mathbf{B}} \geq 0. \tag{19}$$

In general, the above inequality (19) is expected to hold for real materials at all times and at every fixed point in space for a certain class of admissible thermodynamic processes, i.e., processes compatible with the balance laws and the constitutive response functions. By considering the quantities $\dot{\mathbf{d}}, \dot{\mathbf{E}}_e$ and $\dot{\mathbf{B}}$ as independent quantities in the above inequality (19), we obtain the different constitutive laws for an EMR fluid system as

$$\frac{\partial\varphi}{\partial\mathbf{d}} = \mathbf{0}, \quad \mathbf{P} = -\rho\frac{\partial\varphi}{\partial\mathbf{E}_e}, \quad \mathbf{M}_e = -\rho\frac{\partial\varphi}{\partial\mathbf{B}}. \tag{20}$$

Now, the reduced dissipation inequality may be re-written as

$$\left(\mathbf{S} + \rho^2\frac{\partial\varphi}{\partial\rho}\mathbf{I}\right):\mathbf{L} \geq 0. \tag{21}$$

The above inequality (21) represents the Cauchy stress $\mathbf{S}$ as an isotropic in nature. In this regard, we may define a thermodynamic pressure acting on a homogeneous EMR fluid with an application of electromagnetic field as $P_{th} = \rho^2\frac{\partial\varphi}{\partial\rho}$. For a given EMR fluid that is not capable of dissipating, i.e., there is no entropy generation, a constitutive law for Cauchy stress tensor in terms of thermodynamic pressure is given as $\mathbf{S} = -P_{th}\mathbf{I}$. Further, we would like to point out that some of our assumptions in the present work are rather based on general observations than on careful experimental evidence. Accordingly, we assume that the considered EMR fluid is non-Newtonian and non-conducting in nature. From the invariance requirements, it follows that $\mathbf{S}:\mathbf{L} = \mathbf{S}:\mathbf{d}$. In addition, the Cauchy stress tensor $\mathbf{S}$ is an isotropic function of its arguments and has various mini-subspaces. This mini-subspaces span can be utilized for the span required to map other tensors that functionally depend on the tensor $\mathbf{S}$. Now, from the representation theorem[28], we consider that the tensors $\mathbf{d}, \mathbf{E}\otimes\mathbf{E}$ and $\mathbf{H}\otimes\mathbf{H}$ belong to the subspace spanned

by the mini-subspaces of tensor **S**. We may here now link our previous physical assumption (isotropic Cauchy stress) that is verified from the relation (21) as well. Hence, the corresponding representation theorem spencer1971part yields

$$\begin{aligned}\mathbf{S}=\mathbf{S}(\rho,\mathbf{d},\mathbf{E},\mathbf{H})=\alpha_1\mathbf{I}+\alpha_2\mathbf{d}+\alpha_3\mathbf{d}^2+\alpha_4\mathbf{E}\otimes\mathbf{E}+\alpha_5(\mathbf{dE}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{dE})+\alpha_6(\mathbf{d}^2\mathbf{E}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{d}^2\mathbf{E})+\alpha_7\mathbf{H}\otimes\mathbf{H}\\ +\alpha_8(\mathbf{dH}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{dH})+\alpha_9(\mathbf{d}^2\mathbf{H}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{d}^2\mathbf{H}),\end{aligned} \tag{22}$$

wherein $\alpha_{1,2,3...9}=f(I_1,I_2,I_3,....I_9)$ are functions of the following invariants that may be represented as

$$\begin{aligned}I_1=\mathrm{tr}\mathbf{d},\quad I_2=\mathrm{tr}\mathbf{d}^2,\quad I_3=\mathrm{tr}\mathbf{d}^3,\quad I_4=\mathrm{tr}[\mathbf{E}\otimes\mathbf{E}],\quad I_5=\mathrm{tr}[\mathbf{dE}\otimes\mathbf{E}],\quad I_6=\mathrm{tr}[\mathbf{d}^2\mathbf{E}\otimes\mathbf{E}],\quad I_7=\mathrm{tr}[\mathbf{H}\otimes\mathbf{H}],\\ I_8=\mathrm{tr}[\mathbf{dH}\otimes\mathbf{H}],\quad I_9=\mathrm{tr}[\mathbf{d}^2\mathbf{H}\otimes\mathbf{H}].\end{aligned} \tag{23}$$

Finally, the above relation (22) combined with (23) represents a physics-based thermodynamically consistent generalized constitutive relation for an EMR fluid under an applied electromagnetic field.

4 Application to few standard fluid flow situations

In this section, the developed constitutive relation (22) for an EMR fluid is directly applied to few common fluid flow situations (e.g., shear flows) in order to formulate the associated non-zero components of the stress tensor for each of the considered cases.

In line with the shear flow applications, we consider three different standard cases as shown in Figures 1 (**Case-I**), 2 (**Case-II**) and 3 (**Case-III**). Wherein an EMR fluid flows in the X_1-X_2 plane of a Cartesian system. For the given configurations, an EMR fluid is assumed to be isotropic, incompressible, and isothermal. In the first **Case-I**, the applied electric and magnetic fields are mutually parallel and normal to the EMR fluid flow direction. Next, in **Case-II**, the applied electric field is mutually perpendicular to the magnetic field and parallel to the EMR fluid flow direction. At last, in **Case-III**, the applied electric and magnetic fields are mutually perpendicular and normal to the EMR fluid flow direction. In addition, we herein assume that the fluid velocity varies only in one direction significantly, and the other directional variations are negligible.

Case-I

For the defined **Case-I** as shown in Figure 1, the forms of the velocity field vector **v**, electric field vector **E** and the magnetic field vector **H** are obtained as

$$\mathbf{v}=\begin{bmatrix}\gamma(t)X_2\\0\\0\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}0\\E_0\\0\end{bmatrix},\quad \mathbf{H}=\begin{bmatrix}0\\-H_0\\0\end{bmatrix}, \tag{24}$$

wherein $\gamma(t)$ is the instantaneous shear rate and E_0, H_0 are the corresponding electric and magnetic field components of an applied electromagnetic field. Now, the corresponding symmetric part of the velocity gradient tensor **d** from the definitions (2)

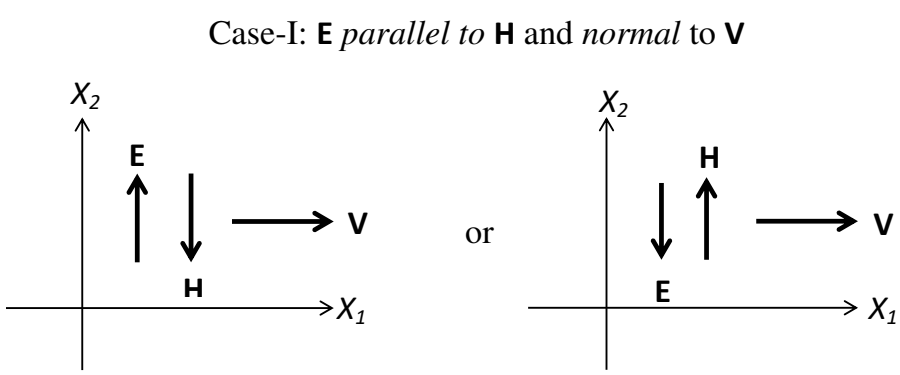

Figure 1. Case-I: Applied electric and magnetic fields are mutually parallel and normal to the EMR fluid flow direction.

and (3) is given as

$$\mathbf{d}=\begin{bmatrix}0&\frac{\gamma}{2}&0\\\frac{\gamma}{2}&0&0\\0&0&0\end{bmatrix}. \tag{25}$$

Further, the non-zero resulting components of the stress tensor from the generalized constitutive relation (22) are

$$S_{11} = \alpha_1 + \alpha_3 \frac{\gamma^2}{4}, \quad S_{22} = \alpha_1 + \alpha_3 \frac{\gamma^2}{4} + \alpha_4 {E_0}^2 + \alpha_6 \frac{\gamma^2 E_0}{2} + \alpha_7 {H_0}^2 + \alpha_9 \frac{\gamma^2 H_0}{2}, \quad S_{33} = \alpha_1,$$
$$S_{12} = \alpha_2 \frac{\gamma}{2} + \alpha_5 \gamma {E_0}^2 + \alpha_8 \gamma {H_0}^2, \quad S_{21} = \alpha_2 \frac{\gamma}{2}, \quad S_{13} = S_{31} = S_{23} = S_{32} = 0. \tag{26}$$

The corresponding invariants for the defined field vectors (24) that help to express $\alpha_i = f(I_1, I_2, I_3, ...I_9)$ where $i = 1, 2, 3...9$ from their definitions (23) are given as

$$I_1 = 0, \quad I_2 = \frac{\gamma^2}{2}, \quad I_3 = 0, \quad I_4 = {E_0}^2, \quad I_5 = 0, \quad I_6 = \frac{\gamma^2 {E_0}^2}{4}, \quad I_7 = {H_0}^2, \quad I_8 = 0, \quad I_9 = \frac{\gamma^2 {H_0}^2}{4}. \tag{27}$$

Case-II

For the defined **Case-II** as shown in Figure 2, the forms of the velocity field vector **v**, electric field vector **E** and the magnetic field vector **H** are obtained as

$$\mathbf{v} = \begin{bmatrix} \gamma(t) X_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} -E_0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ H_0 \\ 0 \end{bmatrix}. \tag{28}$$

Now, the obtained symmetric part of the velocity gradient tensor **d** from (25) may be reused with the above defined field vectors

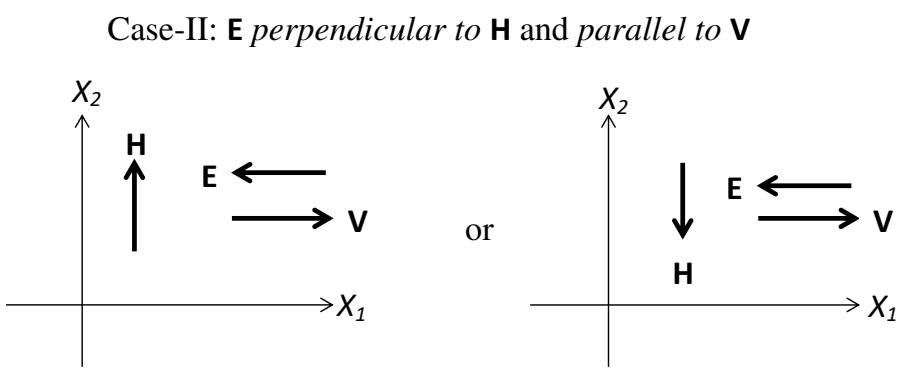

Figure 2. Case-II: Applied electric field is mutually perpendicular to magnetic filed and parallel to the EMR fluid flow direction.

(28) to formulate the non-zero resulting components of the stress tensor from the generalized constitutive relation (22) as

$$S_{11} = \alpha_1 + \alpha_3 \frac{\gamma^2}{4} + \alpha_4 {E_0}^2 + \alpha_6 \frac{\gamma^2 {E_0}^2}{2}, \quad S_{22} = \alpha_1 + \alpha_3 \frac{\gamma^2}{4} + \alpha_7 {H_0}^2 + \alpha_9 \frac{\gamma^2 H_0}{2}, \quad S_{33} = \alpha_1,$$
$$S_{12} = \alpha_2 \frac{\gamma}{2} + \alpha_8 \gamma {H_0}^2, \quad S_{21} = \alpha_2 \frac{\gamma}{2} + \alpha_5 \gamma {E_0}^2, \quad S_{13} = S_{31} = S_{23} = S_{32} = 0. \tag{29}$$

The corresponding invariants for the defined field vectors (28) that help to express $\alpha_i = f(I_1, I_2, I_3, ...I_9)$ where $i = 1, 2, 3...9$ from their definitions (23) are

$$I_1 = 0, \quad I_2 = \frac{\gamma^2}{2}, \quad I_3 = 0, \quad I_4 = {E_0}^2, \quad I_5 = 0, \quad I_6 = \frac{\gamma^2 {E_0}^2}{4}, \quad I_7 = {H_0}^2, \quad I_8 = 0, \quad I_9 = \frac{\gamma^2 {H_0}^2}{4}. \tag{30}$$

Case-III

For the defined **Case-III** as shown in Figure 3, the forms of the velocity field vector **v**, electric field vector **E** and the magnetic field vector **H** are obtained as

$$\mathbf{v} = \begin{bmatrix} \gamma(t) X_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 \\ E_0 \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ 0 \\ -H_0 \end{bmatrix}. \tag{31}$$

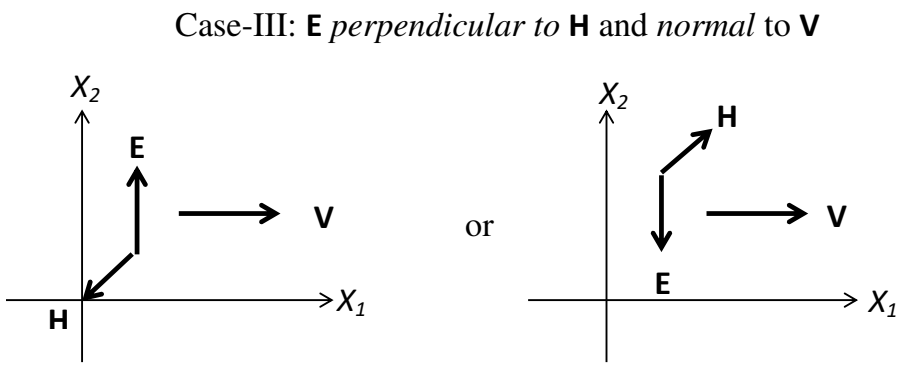

Figure 3. Case-III: Applied electric and magnetic fields are mutually perpendicular and normal to the EMR fluid flow direction.

Next, the obtained symmetric part of the velocity gradient tensor **d** from (25) is reused with the above defined field vectors (31) to formulate the non-zero resulting components of the stress tensor from the generalized constitutive relation (22) as

$$S_{11} = \alpha_1 + \alpha_3 \frac{\gamma^2}{4}, \quad S_{22} = \alpha_1 + \alpha_3 \frac{\gamma^2}{4} + \alpha_4 {E_0}^2 + \alpha_6 \frac{\gamma^2 E_0}{2}, \quad S_{33} = \alpha_1, \quad S_{12} = \alpha_2 \frac{\gamma}{2} + \alpha_5 \gamma {E_0}^2, \\ S_{21} = \alpha_2 \frac{\gamma}{2}, \quad S_{13} = S_{31} = S_{23} = S_{32} = 0. \tag{32}$$

The corresponding invariants for the defined field vectors (31) that help to express $\alpha_i = f(I_1, I_2, I_3, ...I_9)$ where $i = 1, 2, 3...9$ from their definitions (23) are

$$I_1 = 0, \quad I_2 = \frac{\gamma^2}{2}, \quad I_3 = 0, \quad I_4 = {E_0}^2, \quad I_5 = 0, \quad I_6 = \frac{\gamma^2 {E_0}^2}{4}, \quad I_7 = {H_0}^2, \quad I_8 = 0, \quad I_9 = 0. \tag{33}$$

5 Results and discussion

In this section, an analytical model for an ER fluid valve system configuration consistent with[29] is modeled through the developed constitutive relation (22) in order to validate the same. In addition, the different velocity profiles are also predicted from the same constitutive equation (22) for different forms of shear viscosity in the parallel plate configuration.

5.1 Experimental validation of the constitutive relation

To check the validity of constitutive relation (22) derived in section 3, we compare our theoretical result obtained from (22) at zero magnetic field with the existing experimental work performed by Kamelreiter et al.[29]. Kamelreiter et al.[29] designed an ER valve geometry to measure the field-dependent yield stress of a typical ER fluid. This ER valve consists of a flat (or annular) channel of height h formed by two electrodes as shown in Figure 4. The electrodes have the dimensions as length L and width B. The ER fluid in the channel is assumed to be isotropic, incompressible, and isothermal. Herein, the volume flow was driven by the pressure difference between inlet and outlet pressures. With an application of electrical voltage U at a single electrode and by earthening the other electrode, an electric field was generated that changes the rheological properties of the ER fluid.

We begin our theoretical analysis of the same ER valve geometry used by Kamelreiter et al.[29] as shown in Figure 4. To obtain an analytical expression of the field-dependent shear stress model S_{12}, we first have to choose the appropriate functional forms of α_2 and α_5 based on the definitions of invariants (27) through $(26)_4$. In line with that, we may choose the corresponding appropriate functional dependency of α_2 and α_5 based on the simplest algebraic functions for the considered ER valve configuration as

$$\alpha_2 = \alpha_2(I_1, I_2, ...I_6) = K_1 \gamma^2 {E_0}^2, \quad \alpha_5 = \alpha_5(I_1, I_2, ...I_6) = K_2 \gamma^2 {E_0}^2. \tag{34}$$

Now, for the above simplest functional forms of α_2 and α_5, the simplified component of the Cauchy stress tensor (22) through $(26)_4$ at $H_0 = 0$ is represented as

$$S_{12} = K_1 \frac{\gamma^3}{2} {E_0}^2 + K_2 \gamma^3 {E_0}^4. \tag{35}$$

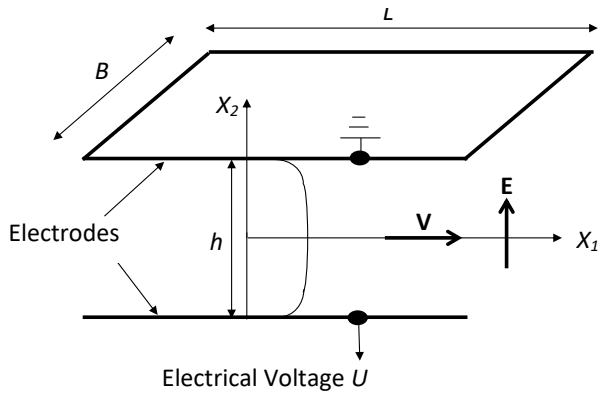

Figure 4. System configuration of flat channel ER valve.

The above equation (35) represents a physics-based thermodynamically consistent analytical expression of the field dependent shear stress model derived from (22) for an ER fluid under an applied electric field. In addition, Kamelreiter et al.[29] had developed an analytical model similar to our (35) for the same ER valve geometry as shown in Figure 4 as

$$S_{12} = a_1 E_0 + a_2 {E_0}^2 + a_3 {E_0}^3, \tag{36}$$

wherein a_1, a_2 and a_3 are the constant parameters, which were obtained from the measurements for the characteristic behaviorof a typical ER fluid.

To compare the proposed model (35) derived from (22) with the model (36) as well as the experimental data of a typical ER fluid from[29], we plot the same on Figure 5 for $\gamma = 1$, $K_1 = 1$ and $K_2 = 119.68$ based on the geometrical dimensions of the system configuration. We here note that the theoretical predictions of the considered ER valve system through the model (35) are consistent with the experimental data[29]. In addition, the success of the proposed model (35) in characterizing the considered system is evident in two aspects. Firstly, it may produce a simple closed-form analytical solution to the problem. Secondly, a comparison of the proposed model (35) with the experimental data[29] requires the determination of only two material constants: K_1 and K_2 even for an extensive range of the applied field. Whereas, another model (36) that compares favorably with the experimental data do so by fitting three parameters: a_1, a_2 and a_2 that lack, in most cases, any physical interpretation at all and also demand considerable computation.

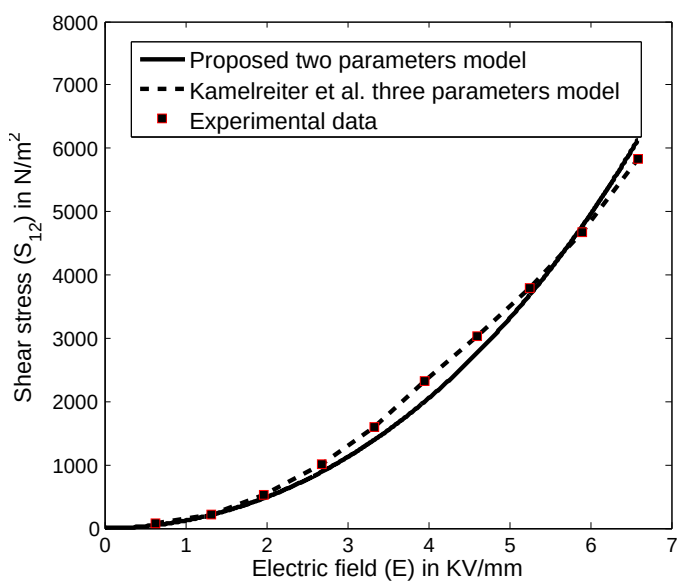

Figure 5. Comparison of the proposed model (35) derived from (22) with the kamelreiter et al. model (36) as well as the experimental data[29] on the field dependent shear (yield) stress of a typical ER fluid.

5.2 Possible simulational results

In current discussion, consider the effect of the electric and magnetic field components E_0 and H_0 in each of the considered cases. Herein, the in-plane stress components S_{11}, S_{22}, S_{12}, and S_{21} depend on E_0 and H_0 through the corresponding functions of $\alpha_i = f(I_1, I_2, I_3, ...I_9)$ where $i = 1, 2, 3...9$. Additionally, it may be seen from the relations (26), (29) and (32) with (23) that $\alpha_{1,2,3...9} = f(I_1, I_2, I_3, ...I_9)$ can be regarded as functions of ${E_0}^2, {H_0}^2$, and γ^2. Now for each of the considered cases, we represent the shear rate equation in the generalized shear form as

$$S_{12} = \mu({E_0}^2, {H_0}^2, \gamma^2)\gamma(t), \tag{37}$$

wherein $\mu({E_0}^2, {H_0}^2, \gamma^2)$ represents the shear viscosity function. Herein, $\gamma = \dfrac{dv}{dX_2}$ and some of the α_i terms represent the non-Newtonian nature of EMR fluid. Some of the α terms arise from the interaction of the shear flow and the component of the electric field in the direction normal to the flow direction. From the standard observations[5,6], we have $S_{12} = -PX_2$. Wherein, P is the pressure gradient, and it helps to represent some of the α_i terms, correspondingly. Next, by substituting $S_{12} = -PX_2$ in the above expressions of S_{12} for each of the considered cases, we may easily obtain the differential equations for the velocity $v(X_2)$ in terms of the parameters P, E^2, H^2, and γ^2, etc.

Now, consider an EMR fluid compassed between two parallel plates having a distance $2h$ apart as shown in Figure 6. In the given configuration, a constant electric field $\mathbf{E} = E_0\mathbf{e}_2$ and a constant magnetic field $\mathbf{H} = -H_0\mathbf{e}_2$ is applied perpendicular to the plates. The EMR fluid flow is assumed to be rectilinear in the direction parallel to the plates. For the given fluid flow similar to **Case-I** in section 4 situation and the corresponding shear stress expression of S_{12} from $(26)_4$ with the condition $S_{12} = -PX_2$, we may easily obtain the velocity profiles of EMR fluid. Some of the corresponding velocity profiles are predicted successfully for different forms of shear viscosity like $\mu = constant$, $\mu = \mu(E_0, \gamma)$ and $\mu = \mu(H_0, \gamma)$ as shown in Figure 6. Additionally, the simulational results shown in Figure 6 are also consistent with the existing results[5,6] at zero magnetic field condition.

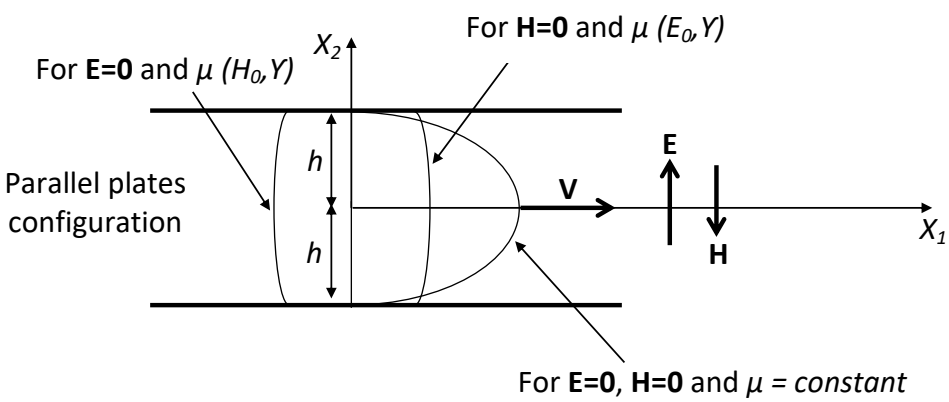

Figure 6. Different velocity profiles for flow between two parallel plates configuration.

In summary, we may control the EMR fluid flow situations and play with the corresponding theoretical parameters for the given situation through a specific choice of the relationships between S_{12}, E^2, H^2, and γ^2 from the standard experimental evidence. At last, these mathematical arguments will directly help us in the design and development of the different EMR fluid actuators for various engineering and medical field applications.

5.3 An additional comment

From (26), (29) and (32) for each of the considered case, we note that the shear response in **Case-III** is independent of an additional magnetic field application. However, for the remaining **Case-I** and **Case-II**, this additional magnetic field application significantly affects the EMR fluid deformation. Based on this observation, we may comment that an electric or magnetic field normal to fluid flow direction with different plane does not affect the shear flow deformation in EMR fluids.

6 Concluding remarks

In the present paper, we develop a thermodynamically consistent generalized constitutive relation (22) for an incompressible isotropic non-Newtonian electro-magneto-rheological (EMR) fluid under an electromagnetic field. The developed constitutive

relation (22) relates the Cauchy stress tensor **S** with the stretch rate tensor **d**, electric field vector **E** and magnetic field vector **H**. Next, the relation (22) is applied to study the most common fluid flow situations in modern system configurations and validated with an ER fluid valve system. Later on, the different velocity profiles are also predicted from the same for different forms of shear viscosity in the parallel plate configuration.

The major contribution of the present study is an attempt to generalize the deformation concept of fluid continua to electro-magneto-rheology in contrast to the existing works on ER and MR fluids. Moreover, the previous works are also obtainable from the same as a special case. In addition, an analytical model (35) proposed from (22) is succeed in two ways. Firstly, it produces a simple closed-form solution to the problem. Secondly, it requires a lesser number of material parameters that have a certain physical basis for defining the same phenomenon compared to other existing models (36), that lack, in most cases, like for a large range of the applied field. At last, the proposed study will enrich the physical understanding of the EMR fluids like other systems of interest, which are under development in laboratories worldwide.

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