Technology factor as society's information stock: An elegant proof

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Technology factor as society's information stock: An elegant proof

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Abstract: It has been empirically observed that the income structure of the vast majority of populations in market-economy countries follows an exponential distribution. The empirical evidence has covered more than 66 countries, ranging from Europe to Latin America, North America, and Asia. Here, to further support exponential income distribution as a signature of the well-functioning market economy, we empirically show how the income structure of China evolved towards an exponential distribution after the market-oriented economic reformation. In particular, we strictly prove that, if the income structure of an economy obeys an exponential distribution, the income summation over all households leads to a neoclassical aggregate production function, in which the technology factor is exactly equal to society's information stock. This finding provides an insight into understanding the underlying implication of technological progress.

Keywords: Generalized Pareto distribution; Exponential income distribution; Technology factor; Information stock.

JEL Classification: O33; D31; D83; C46

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1. Introduction

The technology factor in aggregate production function is a mysterious variable. Distinguishing from labor and capital, Solow (1956) has used his famous model to show that the technology factor is the most powerful driver of economic growth. However, in his model, the technology factor is exogenously given. As a result, for the given technology, Solow's model predicts that, due to the diminishing marginal return of capital, the growth rate of per-capita income will decline to zero. This implies that, in the economic world described by Solow's model, the growth of per-capita income is unsustainable. To overcome the dilemma of Solow's model, motivated by Arrow's learning by doing (Arrow 1962), Lucas (1988), Romer (1986, 1990), and Barro (1990) endogenized the technology factor. Following their models, the technology factor will persistently rise with knowledge accumulation so that the growth rate of per-capita income is always higher than zero. These models are called the "endogenous growth models", in which the technology factor has been proposed as "knowledge" or "information stock". In the 1990s, the conception of technology factor as "knowledge" or "information stock" was further used to explain the success of China's gradual reform of economic development[1] (Murrell, 1992). Although the technology factor has been commonly interpreted as "knowledge" or "information stock", to date there is no formal proof for the validity of this interpretation. Our paper fills this gap. Here, we strictly prove that if the income structure of an economy obeys an exponential distribution, the technology factor in aggregate production function is exactly equal to society's information stock.

In the economic literature, it has been observed empirically that, for market-economy countries, the low- and middle-income class (about 95% of populations) follows an exponential distribution (Nirei and Souma, 2007; Newby et al., 2011;

[1] Recently, Lin (2011) attempted to explain the success of China's gradual reform of economic development from the perspective of matching technological choice and factor endowment. This is in accordance with the conception of technology factor as "knowledge" or "information stock". In this regard, Murrell (1992) has pointed out that a society's stock of personal knowledge is acquired through a long historical process and is shaped by the institutions and organizations of that particular society that has the given factor endowment.

Clementi et al., 2012; Prinz, 2016; Irwin and Irwin, 2017; Rosser, 2019; Tao et al., 2019; Ma and Ruzic, 2020) and the top income class (less than 5% of populations) follows a Pareto distribution (Mandelbrot, 1960; Nirei and Souma, 2007; Atkinson et al., 2011; Aoki and Nirei, 2017; Tao et al., 2019). In this regard, Blanchet et al. (2017, 2018) have used the generalized Pareto distribution to describe the income structure of the total population. The generalized Pareto distribution is a fairly general family that includes the Pareto distribution and the exponential distribution as two special cases (Singh and Maddala, 1976; Cowell, 2000; Jenkins, 2016). In this paper, we further demonstrate that when the equal opportunity is guaranteed, the generalized Pareto distribution yields an exponential distribution, and when the equal opportunity is undermined, the generalized Pareto distribution turns to a Pareto distribution. This finding implies that the exponential distribution is expected to describe the income structure of an equal-opportunity market economy, just as described by a peer-to-peer economy (Tao, 2016), which indicates a well-functioning market economy. Using a large sample, Tao et al. (2019) analyzed datasets of household income from 66 countries, ranging from Europe to Latin America, North America, and Asia, and found that, for all the countries, the income structure for the low and middle classes (about 95% of populations) uniformly follows an exponential distribution. Because the empirical evidence has shown that exponential income distribution dominates the vast majority of the populations, we assume that the income structure of the total population in a market-economy country is approximately described by an exponential distribution. Using this assumption, we strictly prove that the income summation over all households leads to a neoclassical aggregate production function, in which the technological factor is exactly equal to society's information stock.

The remainder of the paper is organized as follows. In section 2, we empirically show the validity of the exponential income distribution in describing the income structure of the well-functioning market economy. In section 3, we show that if the income structure of an economy obeys the exponential distribution, then the income summation over all households leads to a neoclassical aggregate production function with Hicks-neutral-like technical progress. In section 4, we further prove that the technology factor in this aggregate production function is exactly equal to society's information stock. Section 5 concludes the paper.

2. Exponential income distribution

For market-economy countries, it has been empirically observed that the income structure of top income class (less than 5% of populations) obeys a Pareto distribution (Mandelbrot, 1960; Nirei and Souma, 2007; Atkinson et al., 2011; Aoki and Nirei, 2017; Tao et al., 2019), while the low- and middle-income class (about 95% of populations) obeys an exponential distribution (Nirei and Souma, 2007; Newby et al., 2011; Clementi et al., 2012; Prinz, 2016; Irwin and Irwin, 2017; Rosser, 2019; Tao et al., 2019; Ma and Ruzic, 2020). Theoretically, it has been known that the generalized Pareto distribution is a fairly general family that includes the Pareto distribution and the exponential distribution as two special cases (Balkema and Haan, 1974; Pickands, 1975; Singh and Maddala, 1976; Cowell, 2000). In this regard, Blanchet et al. (2017, 2018) used the generalized Pareto distribution to describe the income structure of the total population, where the generalized Pareto distribution is defined as follows:

$$F_{\xi}(t \geq x) = \left(1 + \xi \frac{x-\mu}{\theta}\right)^{-1/\xi}, \tag{1}$$

where $\xi > 0$, x denotes the income level, and $F_{\xi}(t \geq x)$ denotes the fraction of population with the income being higher than x.

It can be seen that equation (1) yields the Pareto distribution $F_{\xi}(t \geq x) = (x/\mu)^{-1/\xi}$ when $\mu = \theta/\xi$ (Jenkins, 2016; Blanchet et al., 2017), where the Pareto exponent is denoted by $1/\xi$, which measures the degree of income inequality, that is, a larger Pareto exponent is associated with lower income inequality (Jones and Kim, 2018). Intuitively, a lower income inequality should correspond to a more equal-opportunity society. Thus, we anticipate that, as the Pareto exponent $1/\xi \to \infty$ (or $\xi \to 0$), equation (1) yields a distribution being close to equal opportunity; that is,

$$F_0(t \geq x) = \lim_{\xi \to 0} \left(1 + \xi \frac{x-\mu}{\theta}\right)^{-1/\xi} = e^{-\frac{x-\mu}{\theta}}. \tag{2}$$

Equation (2) is an exponential distribution, which is expected to describe the income structure of an equal-opportunity economy. In the economic literature, a peer-to-peer economy without an intermediary third-party is recognized as an equal-opportunity market economy (Einav et al., 2016; Davidson et al., 2018; Filippi et al., 2020). In this regard, Tao (2016) has theoretically shown that the exponential income distribution emerges spontaneously in a peer-to-peer economic network described by an Arrow-Debreu economy[2]:

$$\begin{cases} a_j = e^{-\frac{\varepsilon_j - \mu}{\theta}} \\ j = 1, \dots, n \end{cases}, \tag{3}$$

where $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$, and a_j denotes that there are a_j households, each of which obtains ε_j units of income. We will clarify the economic implication of μ and θ in section 3.

Because exponential income distributions (2) and (3) are derived by assuming an equal-opportunity market economy, we propose to identify the exponential income distribution as a signature of the well-functioning market economy. Next, we employ the empirical evidence to support this proposal.

Before doing this, we first demonstrate the relationship between equations (2) and (3). Equation (2) is the cumulative distribution of income, where the income level takes continuous value. By contrast, equation (3) is the density distribution of income, where the income level takes discrete values to keep up with reality. Tao et al (2019) have shown that the cumulative form of equation (3) is equivalent to equation (2) when the income level takes continuous value. From this sense, the discrete form (3) has included the continuous form (2) as a special example. In this paper, we will use the discrete form (3) of exponential income distribution to perform a theoretical analysis. By contrast, it is more convenient to carry out empirical investigation by using the continuous form (2).

Because there has been a large body of empirical literature to support that the exponential distribution (2) dominates an extremely large proportion of populations (Nirei and Souma, 2007; Newby et al., 2011; Clementi et al., 2012; Prinz, 2016; Irwin

[2] Equation (3) has been known as a Boltzmann-like distribution (Cerreia-Vioglio et al., 2020).

and Irwin, 2017; Rosser, 2019; Tao et al., 2019; Ma and Ruzic, 2020), here we simply use the household income data from two representative market-economy countries to support the validity of the exponential distribution (2) in describing the income structure of the well-functioning market economy. The two countries include one typical developed economy (the United Kingdom) and one typical developing economy (China). Figure 1 shows that the low- and middle-income classes (about 97% of populations) in China and the United Kingdom uniformly obey the exponential distribution (2).

[**Insert Figure 1 here**]

In particular, because China is a special sample that has undergone the transition from a planned economy to a market economy, it is more important to examine if there is a transition towards an exponential income distribution after the market-oriented economic reformation. Figure 2 empirically shows that the transition of China's income structure evolving to an exponential income distribution occurred after the market-oriented economic reformation (i.e., 1978). The transition of China's income structure is compelling evidence for identifying the exponential income distribution as a signature of the well-functioning market economy.

[**Insert Figure 2 here**]

Because the empirical evidence has shown that the exponential distribution (2) dominates an extremely large proportion of populations, we assume that the income structure of the total population in a market economy is approximately described by an exponential distribution. Based on this assumption, we next use the discrete form (3) of exponential income distribution to verify that the technology factor in aggregate production function is exactly equal to society's information stock.

3. Aggregate production function

According to the economic meaning of exponential income distribution (3), the total number of households, N, and the GDP, Y, of the economy can be written as below:

$$N = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} e^{-\frac{\varepsilon_j-\mu}{\theta}}, \tag{4}$$

$$Y = \sum_{j=1}^{n} a_j\, \varepsilon_j = \sum_{j=1}^{n} \varepsilon_j e^{-\frac{\varepsilon_j-\mu}{\theta}}. \tag{5}$$

Here, we order:

$$\alpha = -\frac{\mu}{\theta}, \tag{6}$$

$$\beta = \frac{1}{\theta}. \tag{7}$$

By using equations (6) and (7), equations (4) and (5) can be rewritten as

$$N = \sum_{j=1}^{n} \frac{1}{e^{\alpha+\beta\varepsilon_j}}, \tag{8}$$

$$Y = \sum_{j=1}^{n} \frac{\varepsilon_j}{e^{\alpha+\beta\varepsilon_j}}. \tag{9}$$

Equations (4) and (5) indicate that N and Y are related via two parameters α and β. This means that the aggregate production function (or GDP) Y should be a function of total number of agents, N. By using equations (8) and (9), we can obtain

$$dY = -\frac{\alpha}{\beta} dN + \frac{1}{\beta} d\left(N - \alpha\frac{\partial N}{\partial \alpha} - \beta\frac{\partial N}{\partial \beta}\right). \tag{10}$$

The derivation for equation (10) can be found in Appendix A.

Let us order:

$$T = N - \alpha\frac{\partial N}{\partial \alpha} - \beta\frac{\partial N}{\partial \beta}. \tag{11}$$

The following proposition 1 guarantees that if equation (12) is solvable, then N is independent of T (Tao, 2019).

Proposition 1: *Assume that N agents obey the exponential income distribution (3) so that equation (10) holds. Then N is independent of T if and only if the following partial differential equation (12) holds and is solvable:*

$$N\frac{\partial Y(N,T)}{\partial N} + (T-N)\frac{\partial Y(N,T)}{\partial T} = Y(N,T). \tag{12}$$

Proof. The proof can be found in Appendix B. □

It is easy to verify that equation (12) is solvable (Tao, 2019). Henceforth, we always assume that equation (12) holds. Following this assumption, N is independent of T. Therefore, in the context of exponential income distribution (3), we have the aggregate production function as below:

$$Y = Y(N, T). \tag{13}$$

Substituting equations (6), (7), and (11) into equation (10) we obtain

$$dY(N, T) = \mu \, dN + \theta \, dT. \tag{14}$$

Since N is independent of T, by equation (14) we have

$$\mu = \frac{\partial Y(N,T)}{\partial N}, \tag{15}$$

$$\theta = \frac{\partial Y(N,T)}{\partial T}. \tag{16}$$

To identify the economic implication of μ and θ, we denote T by the technology factor. Here, we further assume that households are the owners of labor and capital. This assumption is natural. If a household has no labor or capital, it cannot survive in a real society. Therefore, the total number of households, N, can be regarded as the function of the labor L and the capital stock K; that is, we have

$$N = N(L, K). \tag{17}$$

In the context of neoclassical economics, the aggregate production function can be written as

$$Y = Y(L, K, T). \tag{18}$$

Because equation (12) holds, N is independent of T. Therefore, by equations (17) and (18), the aggregate production function (13) can be written in the form:

$$Y = Y(N(L, K), T), \tag{19}$$

which implies a Hicks-neutral-like technical progress.

By equations (15) and (17), μ denotes the marginal labor-capital return, which denotes the increment of GDP when a new household enters markets, while technology factor T is held constant. Similarly, θ denotes the marginal technology return. The main purpose of this paper is to prove that if equation (12) holds, then the technology factor T is exactly equal to society's information stock.

4. Main results

To prove that T stands for society's information stock, we need to make three assumptions.

Assumption 1: *The GDP, Y, is a function of collective strategies (or collective decisions) of N agents (or households[3]), $\boldsymbol{s} = (s_1, s_2, \dots, s_N)$; that is, $Y = Y(\boldsymbol{s})$, where s_i denotes the economic strategy[4] of the i-th agent and $i = 1, \dots, N$.*

The economic meaning of Assumption 1 is that the GDP of a country is determined by the economic strategy (or economic decision) of each agent. This assumption is natural. Indeed, the GDP consists of all agents' incomes and an agent's income is determined by her economic strategy. For example, a good strategy may increase income and a bad strategy may reduce income. Furthermore, $\boldsymbol{s} = (s_1, s_2, \dots, s_N)$ indicates that the number of agents is discrete (or countable). However, as with Aumann's research of a continuum of traders (Aumann, 1966), Assumption 1 can be applied to the case that the number of agents is continuous (or uncountable). For a continuum of agents, we order $\boldsymbol{s} = (s_i)_{i \in R}$ with $R = [0, N]$.

Assumption 2: *N is not a function of $\boldsymbol{s}$. In particular, if the agent i does not make economic strategy (i.e., inactive), then we order $s_i = 0$, where, for a discrete number of agents, we have $i = 1, \dots, N$, and, for a continuum of agents, we have $i \in R$.*

Assumption 2 eliminates the possibility that N is a function of $\boldsymbol{s}$. To understand this assumption, we simply consider the discrete case $\boldsymbol{s} = (s_1, s_2, \dots, s_N)$. For example, if we order[5] $s_3 = s_4 = \cdots = s_N = 0$, then we have $\boldsymbol{s} = (s_1, s_2, 0 \dots, 0) = (s_1, s_2)$, which has nothing to do with N. Furthermore, by the Assumption 2, we also have the following result: For $\boldsymbol{s} = (s_1, s_2, \dots, s_N)$ and $\boldsymbol{s}' = (s_1, s_2, \dots, s_{N_0})$, if $N_0 < N$, then we may have $T(\boldsymbol{s}) = T(\boldsymbol{s}')$. To see this, let us order $\boldsymbol{s}'' = (s_{N_0+1}, \dots, s_N)$. Then $\boldsymbol{s} = (\boldsymbol{s}', \boldsymbol{s}'')$. In this case, if we consider $\boldsymbol{s}'' = (0, \dots, 0)$, then we have $\boldsymbol{s} = (\boldsymbol{s}', \boldsymbol{s}'') =$

[3] In this paper, we do not distinguish between agent and household.

[4] Here, we have regarded an N-agent market economy as an N-agent non-cooperative game.

[5] $s_3 = s_4 = \cdots = s_N = 0$ means that social members $i = 1, \dots, N$ are inactive.

$(\boldsymbol{s}', \boldsymbol{0}) = \boldsymbol{s}'$; that is, $T(\boldsymbol{s}) = T(\boldsymbol{s}')$. Obviously, these results can be applied to the case of a continuum of agents.

By Proposition 1, we have known that Y is a function of N and T; therefore, by Assumptions 1 and 2 we conclude that T is a function of $\boldsymbol{s}$; that is,

$$T = T(\boldsymbol{s}). \tag{20}$$

For the function $T = T(\boldsymbol{s})$, we further make the continuity assumption as below:

Assumption 3 (continuity): *If there are two collective strategies* $\boldsymbol{s}$ *and* $\boldsymbol{s}'$ *such that* $T(\boldsymbol{s}) < T(\boldsymbol{s}')$, *then we do find a collective strategy* $\boldsymbol{s}''$ *to satisfy* $T(\boldsymbol{s}) < T(\boldsymbol{s}'') < T(\boldsymbol{s}')$, *where* $\boldsymbol{s} \neq \boldsymbol{s}' \neq \boldsymbol{s}''$.

Using equations (20) and Assumption 2, equation (14) can be rewritten as

$$dY\big(N, T(\boldsymbol{s})\big) = \mu\, dN + \theta\, dT(\boldsymbol{s}). \tag{21}$$

Based on Assumptions 1-3, we can prove three lemmas as below:

Lemma 1: *Assume that* N *agents obey the exponential income distribution (3) and that equation (12) holds. For any given number* T_0, *if the probability of each agent acquiring income is independent, the probability of* N *agents taking the collective strategies* $\boldsymbol{s}$ *is denoted by*

$$P(\boldsymbol{s}) = \frac{1}{Z} e^{-\frac{Y(s) - N \cdot \mu}{\theta}} \tag{22}$$

with

$$Z = \sum_{\boldsymbol{s}' \in \{\boldsymbol{r}' | T(\boldsymbol{r}') = T_0\}} e^{-\frac{Y(s') - N \cdot \mu}{\theta}}, \tag{23}$$

where $\boldsymbol{s} \in \{\boldsymbol{r} | T(\boldsymbol{r}) = T_0\}$, $Y(\boldsymbol{s}) = Y\big(N, T(\boldsymbol{s})\big)$, *and the summation* $\sum_{\boldsymbol{s}' \in \{\boldsymbol{r}' | T(\boldsymbol{r}') = T_0\}} \cdot$ *in equation (23) runs over all strategies* $\boldsymbol{s}'$ *in the set* $\{\boldsymbol{r}' | T(\boldsymbol{r}') = T_0\}$.

Proof. The proof can be found in Appendix C. □

Using equations (22) and (23), it is easy to verify

$$\sum_{\boldsymbol{s} \in \{\boldsymbol{r} | T(\boldsymbol{r}) = T_0\}} P(\boldsymbol{s}) = 1. \tag{24}$$

Lemma 2: *Assume that* $Y\big(N, T(\boldsymbol{s})\big)$ *is a differentiable function with respect to* N *and*

$T(\boldsymbol{s})$. *For a continuum of agents, if equation (21) holds, then we have*

$$d\overline{Y} = \mu \, dN + \theta \, d\overline{T}, \tag{25}$$

where

$$\bar{Y} = \sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} Y\big(N, T(\boldsymbol{s})\big) \cdot P(\boldsymbol{s}), \tag{26}$$

$$\bar{T} = \sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} T(\boldsymbol{s}) \cdot P(\boldsymbol{s}), \tag{27}$$

with T_0 *being any given number.*

Proof. The proof can be found in Appendix D. □

Substituting equations (6) and (7) into equations (22) and (23) yields

$$P(\boldsymbol{s}) = \frac{1}{Z} e^{-\beta Y(\boldsymbol{s}) - \alpha N}, \tag{28}$$

$$Z = \sum_{\boldsymbol{s}'\in\{\boldsymbol{r}'|T(\boldsymbol{r}')=T_0\}} e^{-\beta Y(\boldsymbol{s}') - \alpha N}. \tag{29}$$

Lemma 3: *For a given constant* T_0, *if equation (11) holds, by using equations (28) and (29) one has*

$$N = -\frac{\partial}{\partial \alpha} \ln Z, \tag{30}$$

$$\bar{Y} = -\frac{\partial}{\partial \beta} \ln Z, \tag{31}$$

where $\bar{Y} = \sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} Y\big(N, T(\boldsymbol{s})\big) \cdot P(\boldsymbol{s})$ and $Z = \sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} e^{-\beta Y(\boldsymbol{s}) - \alpha N}$.

Proof. The proof can be found in Appendix E. □

Using Lemmas 1-3, we start to prove the central proposition of this paper.

Proposition 2: *For any given number* T_0, *if* N *agents obey the exponential income distribution (3) and if equation (12) holds, then one has*

$$T(\boldsymbol{s}) = -\ln P(\boldsymbol{s}), \tag{32}$$

where $\boldsymbol{s} \in \{\boldsymbol{r}|T(\boldsymbol{r}) = T_0\}$**.**

Proof. It is easy to verify the following two equations:

$$d\left(\alpha \frac{\partial}{\partial \alpha} lnZ\right) = \frac{\partial}{\partial \alpha} lnZ \, d\alpha + \alpha d\left(\frac{\partial}{\partial \alpha} lnZ\right), \tag{33}$$

$$d\left(\beta \frac{\partial}{\partial \beta} lnZ\right) = \frac{\partial}{\partial \beta} lnZ \, d\beta + \beta d\left(\frac{\partial}{\partial \beta} lnZ\right) \tag{34}$$

Substituting equations (30) and (31) into equations (33) and (34) we can obtain

$$d\left(\alpha\frac{\partial}{\partial\alpha}lnZ+\beta\frac{\partial}{\partial\beta}lnZ\right)=dlnZ+\beta d\bar{Y}+\alpha dN, \tag{35}$$

where we have used $\mathrm{dln}Z=\frac{\partial}{\partial\alpha}\mathrm{ln}Z\cdot d\alpha+\frac{\partial}{\partial\beta}\mathrm{ln}Z\cdot d\beta$.

Equation (35) can be written as

$$d\bar{Y}=-\frac{\alpha}{\beta}dN+\frac{1}{\beta}d\left(\mathrm{ln}Z-\alpha\frac{\partial}{\partial\alpha}\mathrm{ln}Z-\beta\frac{\partial}{\partial\beta}\mathrm{ln}Z\right), \tag{36}$$

Because equation (12) holds, by Proposition 1, N is independent of T. Therefore, equation (21) holds. By using Lemma 2 and equation (21), equation (25) holds.

Comparing equations (25) and (36), by using equations (6) and (7) we have

$$\bar{T}=\mathrm{ln}Z-\alpha\frac{\partial}{\partial\alpha}\mathrm{ln}Z-\beta\frac{\partial}{\partial\beta}\mathrm{ln}Z. \tag{37}$$

Substituting equation (29) into equation (37) one has

$$\bar{T}=-\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}}P(\boldsymbol{s})\cdot\mathrm{ln}P(\boldsymbol{s}), \tag{38}$$

where we have used equation (28).

Furthermore, substituting equation (12) into equation (22) yields

$$P(\boldsymbol{s})=\frac{e^{-T(s)}}{\sum_{s'\in\{r'|T(r')=T_0\}}e^{-T(s')}}, \tag{39}$$

where we have used equations (15) and (16).

Combining equations (27) and (38) we have

$$\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}}T(\boldsymbol{s})\,P(\boldsymbol{s})=-\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}}P(\boldsymbol{s})\,\mathrm{ln}P(\boldsymbol{s}). \tag{40}$$

Substituting equation (39) into equation (40) we have

$$\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}}e^{-T(\boldsymbol{s})}=1, \tag{41}$$

which implies that equation (39) can be simplified to

$$P(\boldsymbol{s})=e^{-T(\boldsymbol{s})}; \tag{42}$$

that is,

$$T(\boldsymbol{s})=-\mathrm{ln}P(\boldsymbol{s}).\ \square$$

Equation (32) is the main result of this paper. By equations (24) and (32), we can provide an economic interpretation for the technology factor $T(\boldsymbol{s})$. According to Shannon's information theory (Shannon, 1948), if the probability of an event A

occurring is denoted by $P(A)$, then the information content contained in the event A is equal to $-\ln P(A)$. This means that the lower the probability of an event A occurring, the larger the information content contained in the event A. By Lemma 1, we have acknowledged that $P(\boldsymbol{s})$ denotes the probability of N agents taking the collective decisions $\boldsymbol{s}$. Therefore, by equation (32), the technology factor $T(\boldsymbol{s})$ in the aggregate production function (19) denotes the information content contained in the event of N agents taking the collective decisions $\boldsymbol{s}$. Because each agent i makes decision s_i based on personal information or dispersed knowledge (Hayek, 1945) and because collective decisions $\boldsymbol{s} = (s_1, s_2, \dots, s_N)$ cover all agents in the society, taking collective decisions $\boldsymbol{s}$ can be regarded as a process of integrating all of personal information. Therefore, we denote $T(\boldsymbol{s}) = -\ln P(\boldsymbol{s})$ by this society's information stock. From this sense, the technological progress is a process of integrating all of personal information or dispersed knowledge. Furthermore, by equation (32), the phenomenon of a society evolving to a high technological level should be a low probability event of collective decisions. This result agrees with our observation. To our knowledge, among all the species on the earth, only humans evolved to a high technological civilization.

5. Conclusion

In this paper, we argue that exponential income distribution is a signature of the well-functioning market economy. Empirically, we use the household income data from two representative market-economy countries to support the validity of the exponential income distribution in describing the income structure of the well-functioning market economy. The two countries include one typical developed economy (the United Kingdom) and one typical developing economy (China). In particular, because China is a special sample that has undergone the transition from a planned economy to a market economy, we carefully examine whether there is a transition towards an exponential income distribution after the market-oriented economic reformation. The empirical investigation has verified that the transition of China's income structure evolving to an exponential income distribution occurred after the market-oriented economic reformation (i.e., 1978). The transition of China's income structure is compelling evidence for identifying the exponential income distribution as a signature of the well-functioning market economy. Based on the empirical investigation, we assume that the income structure of the total population in a market economy is approximately described by an exponential distribution. Finally, we strictly prove that, if the income structure of an economy obeys an exponential distribution, the income summation over all households leads to a neoclassical aggregate production function, in which the technology factor is exactly equal to society's information stock. This theoretical finding formalizes the validity of interpreting technology factor in the theory of economic growth as "knowledge" or "information stock".

Appendix A

By equations (8) and (9), one has

$$\frac{\partial N}{\partial \alpha} = -N, \tag{A.1}$$

$$\frac{\partial N}{\partial \beta} = -Y. \tag{A.2}$$

The differential of equation (A.2) yields

$$dY = -d\left(\frac{\partial N}{\partial \beta}\right) = -\frac{1}{\beta} d\left(\beta \frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta}\frac{\partial N}{\partial \beta} d\beta. \tag{A.3}$$

By equation (8), N is a function of α and β; therefore, the complete differential of N is

$$dN = \frac{\partial N}{\partial \alpha} d\alpha + \frac{\partial N}{\partial \beta} d\beta, \tag{A.4}$$

which leads to

$$\frac{\partial N}{\partial \beta} d\beta = dN - \frac{\partial N}{\partial \alpha} d\alpha. \tag{A.5}$$

Substituting equation (A.5) into equation (A.3) yields

$$dY = -\frac{1}{\beta} d\left(\beta \frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta} dN - \frac{1}{\beta}\frac{\partial N}{\partial \alpha} d\alpha. \tag{A.6}$$

On the other hand, we have

$$d\left(\alpha \frac{\partial N}{\partial \alpha}\right) = \alpha d\left(\frac{\partial N}{\partial \alpha}\right) + \frac{\partial N}{\partial \alpha} d\alpha, \tag{A.7}$$

which leads to

$$\frac{\partial N}{\partial \alpha} d\alpha = d\left(\alpha \frac{\partial N}{\partial \alpha}\right) - \alpha d\left(\frac{\partial N}{\partial \alpha}\right). \tag{A.8}$$

Substituting equation (A.8) into equation (A.6) yields

$$\begin{aligned} dY &= -\frac{1}{\beta} d\left(\beta \frac{\partial N}{\partial \beta}\right) + \frac{1}{\beta} dN - \frac{1}{\beta} d\left(\alpha \frac{\partial N}{\partial \alpha}\right) + \frac{\alpha}{\beta} d\left(\frac{\partial N}{\partial \alpha}\right) \\ &= \frac{\alpha}{\beta} d\left(\frac{\partial N}{\partial \alpha}\right) + \frac{1}{\beta} d\left(N - \alpha \frac{\partial N}{\partial \alpha} - \beta \frac{\partial N}{\partial \beta}\right). \end{aligned} \tag{A.9}$$

By equation (A.1), one can rewrite equation (A.9) in the form:

$$dY = -\frac{\alpha}{\beta} dN + \frac{1}{\beta} d\left(N - \alpha \frac{\partial N}{\partial \alpha} - \beta \frac{\partial N}{\partial \beta}\right). \tag{A.10}$$

□

Appendix B

Proof of Proposition 1

Proof. We first verify the sufficiency.

Substituting equations (A.1) and (A.2) into equation (11) we have

$$T = N + \alpha N + \beta Y. \tag{B.1}$$

Because T is independent of N, by equation (10), we get

$$\frac{\partial Y}{\partial N} = -\frac{\alpha}{\beta}, \tag{B.2}$$

$$\frac{\partial Y}{\partial T} = \frac{1}{\beta}. \tag{B.3}$$

Substituting equations (B.2) and (B.3) into equation (B.1) yields equation (12).

Next, we verify the necessity. If equation (12) holds and is solvable, then N is obviously independent of T. Furthermore, Tao (2019) has proved that equation (12) is solvable. □

Appendix C

Proof of Lemma 1

Proof. If the income structure of an N-agent society obeys the exponential income distribution (3), the joint probability distribution among N agents, $P^*(1, \dots, N)$, can be written as

$$P^*(1, \dots, N) = \prod_{j=1}^{n} \left(a_j/N\right)^{a_j}. \tag{C.1}$$

To obtain equation (C.1), we first assume that the probability of each agent acquiring income is independent. By equation (4), the probability of obtaining ε_j units of income is a_j/N. Therefore, by the assumption of independence, the joint probability among a_j agents, each of which obtains ε_j units of income, is $\left(a_j/N\right)^{a_j} = \overbrace{\left(a_j/N\right) \cdot \left(a_j/N\right) \cdots \left(a_j/N\right)}^{a_j}$. Equation (C.1) is the result of considering all income levels $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$.

Substituting equation (3) into equation (C.1) yields

$$P^*(1,\dots,N) = \frac{\prod_{j=1}^{n} N^{-a_j}}{\prod_{j=1}^{n} e^{\frac{a_j\varepsilon_j - a_j\mu}{\theta}}} = \frac{N^{-\sum_{j=1}^{n} a_j}}{e^{\frac{\sum_{j=1}^{n} a_j\varepsilon_j - \left(\sum_{j=1}^{n} a_j\right)\mu}{\theta}}}. \tag{C.2}$$

Finally, we plug equations (4) and (5) into equation (C.2) to obtain

$$P^*(1,\dots,N) = \frac{N^{-N}}{e^{\frac{Y-N\cdot\mu}{\theta}}}. \tag{C.3}$$

By Assumption 1, Y is a function of $\boldsymbol{s}$; that is,

$$Y = Y(\boldsymbol{s}). \tag{C.4}$$

Using equation (C.4), $P^*(1,\dots,N)$ can be written in the form of normalized probability:

$$P(\boldsymbol{s}) = \frac{P^*(1,\dots,N)}{\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} P^*(1,\dots,N)}, \tag{C.5}$$

which by using equation (C.3) can be written as

$$P(\boldsymbol{s}) = \frac{1}{Z} e^{-\frac{Y(\boldsymbol{s})-N\cdot\mu}{\theta}}, \tag{C.6}$$

where

$$Z = \sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} e^{-\frac{Y(\boldsymbol{s})-N\cdot\mu}{\theta}}. \tag{C.7}$$

and the summation $\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}}\cdot$ runs over all possible collective strategies $\boldsymbol{s}$ *that satisfy* $T(\boldsymbol{s}) = T_0$. □

Appendix D

Proof of Lemma 2

Proof. We consider any two collections of collective strategies $\boldsymbol{s}$ and $\boldsymbol{s}'$ and two numbers N and N', respectively; that is, N agents take collective strategy $\boldsymbol{s}$ and N' agents take collective strategy $\boldsymbol{s}'$. Then we order

$$\Delta N = N' - N, \tag{D.1}$$

$$\Delta T = T(\boldsymbol{s}') - T(\boldsymbol{s}), \tag{D.2}$$

where $\Delta N \neq 0$ and $\Delta T \neq 0$. Therefore, we have $\boldsymbol{s}' \neq \boldsymbol{s}$.

Because $Y\big(N, T(\boldsymbol{s})\big)$ is a differentiable function with respect to N and $T(\boldsymbol{s})$, by using the mean value theorem of differential, $Y\big(N', T(\boldsymbol{s}')\big)$ can be written as

$$Y\big(N', T(\boldsymbol{s}')\big)$$

$$= Y\big(N, T(\boldsymbol{s})\big) + \frac{\partial}{\partial N} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta N + \frac{\partial}{\partial T} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta T, \qquad \text{(D.3)}$$

where $0 < \eta < 1$ and, by Assumption 3, the collective strategy $\boldsymbol{s}''$ satisfies $T(\boldsymbol{s}) < T(\boldsymbol{s}'') < T(\boldsymbol{s}')$.

Equation (D.3) can be rewritten as

$$Y\big(N', T(\boldsymbol{s}')\big) - Y\big(N, T(\boldsymbol{s})\big)$$

$$= \frac{\partial}{\partial N} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta N + \frac{\partial}{\partial T} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta T. \qquad \text{(D.4)}$$

Let us order $T(\boldsymbol{s}') = T_0'$ and $T(\boldsymbol{s}) = T_0$ with T_0' and T_0 being two constants. Because $T(\boldsymbol{s}) < T(\boldsymbol{s}'') < T(\boldsymbol{s}')$, we conclude $\boldsymbol{s} \neq \boldsymbol{s}'' \neq \boldsymbol{s}'$. Therefore, by acting the summation operator $\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} \sum_{\boldsymbol{s}'\in\{\boldsymbol{r}'|T(\boldsymbol{r}')=T_0'\}} P(\boldsymbol{s})\, P(\boldsymbol{s}')$ on both sides of equation (D.4) we have

$$\overline{Y}' - \overline{Y} = \frac{\partial}{\partial N} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta N + \frac{\partial}{\partial T} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta\overline{T}, \qquad \text{(D.5)}$$

where

$$\overline{Y}' = \sum_{\boldsymbol{s}'\in\{\boldsymbol{r}'|T(\boldsymbol{r}')=T_0'\}} Y\big(N', T(\boldsymbol{s}')\big)\, P(\boldsymbol{s}'), \qquad \text{(D.6)}$$

$$\overline{Y} = \sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} Y\big(N, T(\boldsymbol{s})\big)\, P(\boldsymbol{s}), \qquad \text{(D.7)}$$

$$\Delta\overline{T} = \sum_{\boldsymbol{s}'\in\{\boldsymbol{r}'|T(\boldsymbol{r}')=T_0'\}} T(\boldsymbol{s}')\, P(\boldsymbol{s}') - \sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} T(\boldsymbol{s})\, P(\boldsymbol{s}). \qquad \text{(D.8)}$$

To derive equation (D.5), we have used $\sum_{\boldsymbol{s}\in\{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} P(\boldsymbol{s}) = 1$ and $\sum_{\boldsymbol{s}'\in\{\boldsymbol{r}'|T(\boldsymbol{r}')=T_0'\}} P(\boldsymbol{s}') = 1$.

Performing the limit operator on both sides of equation (D.5) as below:

$$\underset{\Delta N\to 0}{Lim}\ \lim_{\Delta T\to 0}\left[\overline{Y}' - \overline{Y}\right] = \lim_{\Delta N\to 0}\ \lim_{\Delta T\to 0}\left[\frac{\partial}{\partial N} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta N\right] +$$

$$\lim_{\Delta N\to 0}\ \lim_{\Delta T\to 0}\left[\frac{\partial}{\partial T} Y\big(N + \eta\Delta N, T(\boldsymbol{s}'')\big)\Delta\overline{T}\right]. \qquad \text{(D.9)}$$

Up to the first-order infinitesimal terms, equation (D.9) can be written as

$$d\overline{Y} = \frac{\partial}{\partial N} Y(N, T)\, dN + \frac{\partial}{\partial T} Y(N, T)\, d\overline{T}, \qquad \text{(D.10)}$$

which by equations (15) and (16) yields

$$d\overline{Y} = \mu\, dN + \theta\, d\overline{T}. \qquad \text{(D.11)}$$

□

Appendix E

Proof of Lemma 3

Proof. Substituting equations (A.1) and (A.2) into equation (11) yields

$$T = N + \alpha N + \beta Y. \tag{E.1}$$

Let us order $T = T_0$. By equation (E.1) we have

$$\frac{\partial T_0}{\partial \alpha} = \frac{\partial N}{\partial \alpha} + N + \alpha \frac{\partial N}{\partial \alpha} + \beta \frac{\partial Y}{\partial \alpha} = 0, \tag{E.2}$$

$$\frac{\partial T_0}{\partial \beta} = \frac{\partial N}{\partial \beta} + \alpha \frac{\partial N}{\partial \beta} + Y + \beta \frac{\partial Y}{\partial \beta} = 0. \tag{E.3}$$

By using equations (A.1) and (A.2), equations (E.2) and (E.3) can be rewritten as

$$\alpha \frac{\partial N}{\partial \alpha} + \beta \frac{\partial Y}{\partial \alpha} = 0, \tag{E.4}$$

$$\alpha \frac{\partial N}{\partial \beta} + \beta \frac{\partial Y}{\partial \beta} = 0. \tag{E.5}$$

By using equation (29) we have

$$\frac{\partial}{\partial \alpha} \ln Z = \frac{\sum_{s \in \{r|T(r)=T_0\}} e^{-\beta Y(s) - \alpha N} \left[-\beta \frac{\partial Y(s)}{\partial \alpha} - N - \alpha \frac{\partial N}{\partial \alpha}\right]}{\sum_{s' \in \{r'|T(r')=T_0\}} e^{-\beta Y(s') - \alpha N}}, \tag{E.6}$$

which, by using equation (E.4), yields

$$\frac{\partial}{\partial \alpha} lnZ = \frac{-\sum_{s \in \{r|T(r)=T_0\}} N\, e^{-\beta Y(s) - \alpha N}}{\sum_{s' \in \{r'|T(r')=T_0\}} e^{-\beta Y(s') - \alpha N}} = -\sum_{\boldsymbol{s} \in \{\boldsymbol{r}|T(\boldsymbol{r})=T_0\}} N\, P(\boldsymbol{s}) = -N. \tag{E.7}$$

Similarly, by using equation (29) we have

$$\frac{\partial}{\partial \beta} \ln Z = \frac{\sum_{s \in \{r|T(r)=T_0\}} e^{-\beta Y(s) - \alpha N} \left[-Y(\boldsymbol{s}) - \beta \frac{\partial Y(s)}{\partial \beta} - \alpha \frac{\partial N}{\partial \beta}\right]}{\sum_{s' \in \{r'|T(r')=T_0\}} e^{-\beta Y(s') - \alpha N}}, \tag{E.8}$$

which, by using equation (E.5), yields

$$\frac{\partial}{\partial \beta} \ln Z = \frac{-\sum_{s \in \{r|T(r)=T_0\}} Y(\boldsymbol{s})\, e^{-\beta Y(s) - \alpha N}}{\sum_{s' \in \{r'|T(r')=T_0\}} e^{-\beta Y(s') - \alpha N}} = -\sum_{\boldsymbol{s} \in \{\boldsymbol{s}|T(\boldsymbol{s})=T_0\}} Y(\boldsymbol{s})\, P(\boldsymbol{s}) = -\bar{Y}. \tag{E.9}$$

To derive equations (E.7) and (E.9), we have used equation (28). □

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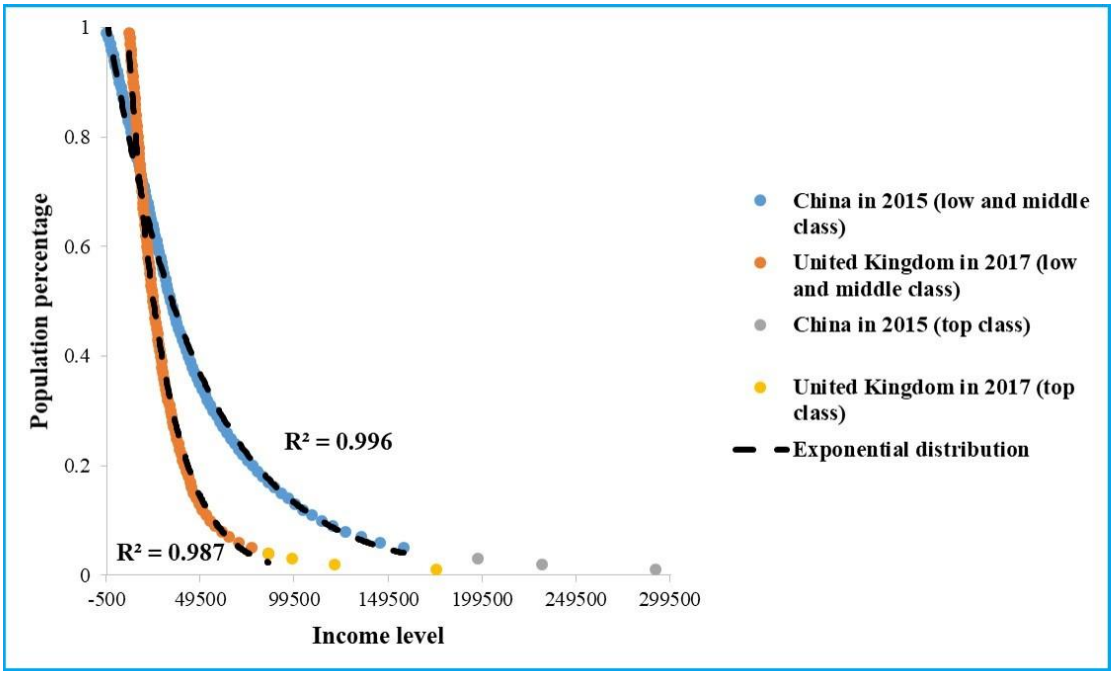

Figure 1: The low- and middle-income classes (about 97% of populations) in China[6] and the United Kingdom[7] uniformly obey the exponential income distribution.

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[6] Data resource for China: http://wid.world/data/

[7] Data resource for the United Kingdom: https://www.gov.uk/government/statistics/percentile-points-from-1-to-99-for-total-income-before-and-after-tax

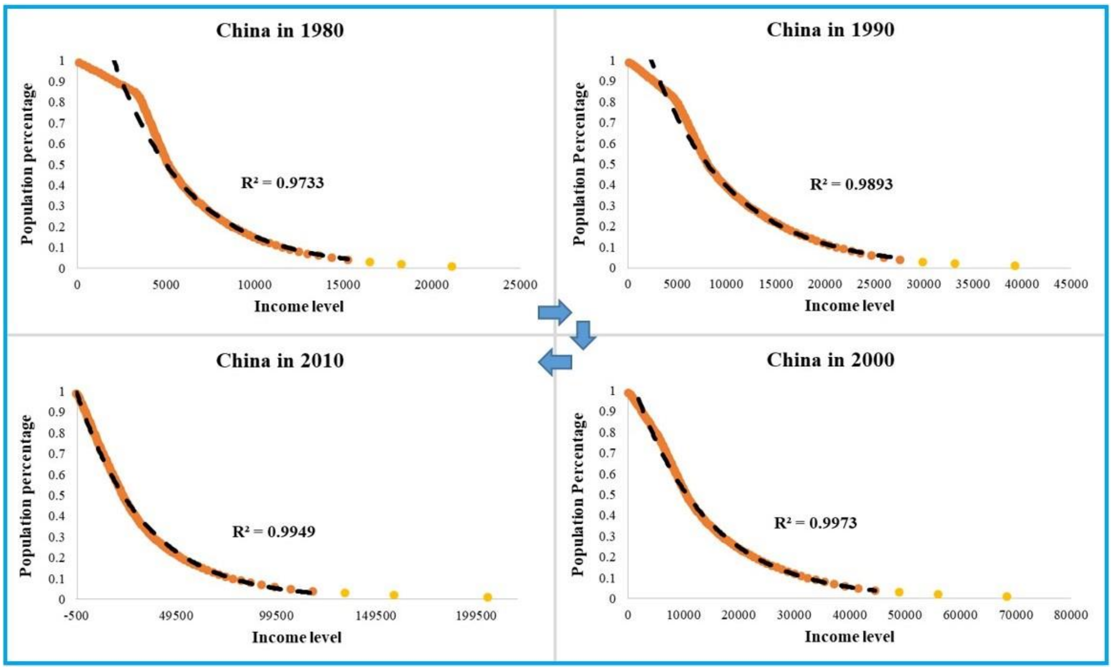

Figure 2: The income structure of China gradually evolved to an exponential distribution after the market-oriented economic reformation[8].

[8] Data resource for China: http://wid.world/data/