Stability and Extraction of New Optical Wave Solutions of Cascaded System in $(2 + 1)$ -dimensions

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Stability and extraction of new optical wave solutions of cascaded system in $(2 + 1)$ -dimensions

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Abstract

This paper uses the new modified sub-ODE method, the new extended auxiliary equation method, and the new Jacobi elliptic function expansion method to revisit the $(2+1)$ -dimensional coupled nonlinear Schrödinger equation with cascading effect. As a consequence, dark, bright, kink-antikink, singular solitons, Weierstrass elliptic function, doubly periodic, and complex optical soliton solutions are retrieved. All solutions are described, along with the existence criterion on the parameters. As solitons are used for data transfer, the obtained results may be found usage in optical couplers, birefringent fibres, and optoelectronic devices. A comparison of the obtained results with those found in the literature is given. The dynamical behaviour of some of the obtained solutions has been explored for suitable choices of the parameters. Using the property of Hamiltonian systems, the solitons stability is determined.

Keywords: Nonlinear Schrödinger equation; Cascaded system; New Jacobi elliptic function expansion method; New extended auxiliary equation method; New modified sub-ODE method.

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1. Introduction

During the past decades, a complete mathematical description of solitons has been developed which make the soliton concept one of the most remarkable developments in modern mathematical physics. A solitary wave is well-defined as non-singular spatially localized solution of a non-dispersive nonlinear equation and emerges in many contexts, including the elevation of the surface of water and the intensity of light in optical fibres. A solitary wave exists due to a fine balance between nonlinearity and dispersion present in the medium. Owing to the robustness of solitons in nonlinear systems, solitonic structures are seen in diverse areas of science such as molecular biology, plasma physics, oceanography, geology and nonlinear field theory [1,2]. But solitons

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which arise in optics called optical solitons or the light wave envelopes in fibres have drawn a significant attention of scientists during the past few decades due to its tremendous applications in telecommunication systems. The optical soliton occurs in a fiber due to the exact balance between the group velocity dispersion (GVD) and the self-phase modulation (SPM). So other than communication, they find applications in logic gates, construction of optical switches, pulse compression and amplification techniques [3], etc. The nonlinear Schrödinger equation (NLSE), which is the celebrated model of contemporary nonlinear physics, governs the propagation of optical solitons [4-16]. Because of their extraordinary stability properties and technological characteristics, optical solitons are attracting a lot of attention and the existence of optical solitons gives a great deal of information about the mechanism of the dynamical optical processes that the NLSE modeled. In this paper, we obtain optical solitons for the governing NLSE in (2+1)-dimensions with cascading effect [17-20]. Such a study is being conducted with three prolific integration approaches such as the new extended auxiliary equation method, the new Jacobi elliptic function expansion method, and the new modified sub-ODE method [21-28].

The layout of the paper is as follows: The mathematical analysis of (2+1)-dimensional coupled NLSE with cascading effect is given in Sect. 2. In Sect. 3, we use the new modified sub-ODE method, the new extended auxiliary equation method, and the new Jacobi elliptic function expansion method to solve NLSE with cascading effect. In Sect. 4, we present some graphical representations of our obtained solutions. In Sect. 5 we present the stability analysis of the obtained solutions. Conclusions are drawn in Sect. 6.

2. Governing Equation

The cascaded (2+1)-dimensional system governing by the coupled NLSE with Kerr-law nonlinearity is written as follows:

$$iq_t + A_1 q_{xx} + B_1 q_{yy} + C_1 q_{xt} + D_1 q_{yt} + E_1 |r|^2 q = 0, \quad (1)$$

and

$$ir_t + A_2 r_{xx} + B_2 r_{yy} + C_2 r_{xt} + D_2 r_{yt} + (E_2 |q|^2 + E_3 |r|^2) r = 0, \quad (2)$$

where $i = \sqrt{-1}$ and A_j, B_j, C_j, D_j, E_j and E_3 are non zero constants for $j = 1, 2$, while the spatial variables x and y are independent variables and t stands as a temporal variable. The complex valued wave profiles are expressed by the dependent variables $q(x, y, t)$ and $r(x, y, t)$. The coefficients of group velocity dispersions (GVD) in the direction of x and y coordinates are A_1, A_2, B_1 and B_2 , respectively, while the coefficients of spatio-temporal dispersion (STD) in the direction of x and y coordinates are C_1, C_2, D_1 and D_2 , respectively. The parameters E_1 and E_2 represent cross phase modulation (XPM) while E_3 represents the SPM coefficient. The linear evolution term is represented by the first term in Eqs. (1) and (2).

In this section, we assume the formal solutions to Eqs. (1) and (2) as :

$$q(x, y, t) = \chi_1(\xi) \exp[i\phi_1(x, y, t)], \quad (3)$$

$$r(x, y, t) = \chi_2(\xi) \exp[i\phi_2(x, y, t)], \quad (4)$$

and

$$\xi = K_1x + K_2y - Ct, \quad \phi_j(x, y, t) = -\eta_jx - \lambda_jy + \omega_jt + \theta_j, \quad j = 1, 2, \quad (5)$$

where $K_1, K_2, C, \eta_j, \lambda_j, \omega_j$ and θ_j are all non-zero constants to be evaluated. The parameters K_1 and K_2 are related to the soliton's inverse widths in the x and y directions, respectively, and C is the soliton's velocity. The frequencies of the solitons along the x and y directions are given by the phase components η_j and λ_j , respectively, while ω_j is the wave number and θ_j stands for the phase constant. In this case, $\chi_1(\xi)$ and $\chi_2(\xi)$ are real-valued functions that represent the pulse shapes. Replacing Eqs. (3),(4) and (5) into the system Eqs. (1) and (2) and segregating the real and imaginary parts respectively, we get

$$(A_1K_1^2 + B_1K_2^2 - K_1CC_1 - D_1K_2C)\chi_1'' - [\omega_1(1 - \eta_1C_1 - \lambda_1D_1) + A_1\eta_1^2 + B_1\lambda_1^2]\chi_1 + E_1\chi_2^2\chi_1 = 0, \quad (6)$$

$$(A_2K_1^2 + B_2K_2^2 - K_1CC_2 - D_2K_2C)\chi_2'' - [\omega_2(1 - \eta_2C_2 - \lambda_2D_2) + A_2\eta_2^2 + B_2\lambda_2^2]\chi_2 + (E_2\chi_1^2 + E_3\chi_2^2)\chi_2 = 0, \quad (7)$$

and

$$[C(1 - \lambda_1D_1 - \eta_1C_1) - \omega_1(C_1K_1 + D_1K_2) + 2A_1\eta_1K_1 + 2B_1\lambda_1K_2]\chi_1' = 0, \quad (8)$$

$$[C(1 - \lambda_2D_2 - \eta_2C_2) - \omega_2(C_2K_1 + D_2K_2) + 2A_2\eta_2K_1 + 2B_2\lambda_2K_2]\chi_2' = 0. \quad (9)$$

The velocity of the soliton can be calculated using Eqs. (8) and (9) as

$$C = \frac{\omega_1(C_1K_1 + D_1K_2) - 2A_1\eta_1K_1 - 2B_1\lambda_1K_2}{(1 - \lambda_1D_1 - \eta_1C_1)}, \quad (10)$$

and

$$C = \frac{\omega_2(C_2K_1 + D_2K_2) - 2A_2\eta_2K_1 - 2B_2\lambda_2K_2}{(1 - \lambda_2D_2 - \eta_2C_2)}. \quad (11)$$

The constraint condition is derived from Eqs. (10) and (11) as:

$$(1 - \lambda_1D_1 - \eta_1C_1)[\omega_2(C_2K_1 + D_2K_2) - 2A_2\eta_2K_1 - 2B_2\lambda_2K_2] \\ = (1 - \lambda_2D_2 - \eta_2C_2)[\omega_1(C_1K_1 + D_1K_2) - 2A_1\eta_1K_1 - 2B_1\lambda_1K_2]. \quad (12)$$

Setting

$$\chi_1(\xi) = \delta\chi_2(\xi), \quad (13)$$

where δ is a non zero constant, such that $\delta \neq 1$. As a result, Eqs. (6) and (7) are reduced to

$$(A_1K_1^2 + B_1K_2^2 - K_1CC_1 - D_1K_2C)\chi_1'' - [\omega_1(1 - \eta_1C_1 - \lambda_1D_1) + A_1\eta_1^2 + B_1\lambda_1^2]\chi_1 \\ + E_1\delta^2\chi_1^3 = 0, \quad (14)$$

and

$$(A_2K_1^2 + B_2K_2^2 - K_1CC_2 - D_2K_2C)\chi_1'' - [\omega_2(1 - \eta_2C_2 - \lambda_2D_2) + A_2\eta_2^2 + B_2\lambda_2^2]\chi_1 \\ + (E_2 + E_3\delta^2)\chi_1^3 = 0, \quad (15)$$

Under the constraint conditions the Eqs. (14) and (15) have the same structure:

$$\frac{A_1 K_1^2 + B_1 K_2^2 - K_1 C C_1 - D_1 K_2 C}{A_2 K_1^2 + B_2 K_2^2 - K_1 C C_2 - D_2 K_2 C} = \frac{\omega_1(1 - \eta_1 C_1 - \lambda_1 D_1) + A_1 \eta_1^2 + B_1 \lambda_1^2}{\omega_2(1 - \eta_2 C_2 - \lambda_2 D_2) + A_2 \eta_2^2 + B_2 \lambda_2^2} = \frac{E_1 \delta^2}{E_2 + E_3 \delta^2}, \quad (16)$$

which leads to the following relations

$$\omega_1 = \frac{E_1 \delta^2 [\omega_2(1 - \eta_2 C_2 - \lambda_2 D_2) + A_2 \eta_2^2 + B_2 \lambda_2^2]}{(E_2 + E_3 \delta^2)(1 - \eta_1 C_1 - \lambda_1 D_1)} - \frac{A_1 \eta_1^2 + B_1 \lambda_1^2}{1 - \eta_1 C_1 - \lambda_1 D_1}, \quad (17)$$

and

$$\delta^2 = \frac{(A_1 K_1^2 + B_1 K_2^2 - K_1 C C_1 - D_1 K_2 C) E_2}{(A_2 K_1^2 + B_2 K_2^2 - K_1 C C_2 - D_2 K_2 C) E_1 - (A_1 K_1^2 + B_1 K_2^2 - K_1 C C_1 - D_1 K_2 C) E_3} > 0. \quad (18)$$

As a consequence, Eqs. (14) and (15) can be written as a single equation:

$$\beta_1 \chi_1'' - \beta_2 \chi_1 + \beta_3 \chi_1^3 = 0, \quad (19)$$

where

$$\begin{aligned} \beta_1 &= A_1 K_1^2 + B_1 K_2^2 - K_1 C C_1 - D_1 K_2 C, \\ \beta_2 &= \omega_1(1 - \eta_1 C_1 - \lambda_1 D_1) + A_1 \eta_1^2 + B_1 \lambda_1^2, \\ \beta_3 &= E_1 \delta^2. \end{aligned} \quad (20)$$

In Eq. (19) comparing $\chi_1''(\xi)$ with $\chi_1^3(\xi)$ yields $N = 1$. The next task is to solve and analyze Eq. (19) in next subsections using the new modified sub-ODE method, the new extended auxiliary equation method, and the new Jacobi elliptic function expansion method, subsequently.

3. Application of the methods

4. The new modified Sub-ODE method

We assume that Eq. (19) has the formal solution

$$\chi_1(\xi) = \mu K(\xi)^M, \mu > 0, \quad (21)$$

where M is a parameter and determined by the homogeneous balance principle. The function $K(\xi)$ satisfies a generalized Sub-ODE in the form:

$$K'^2(\xi) = \alpha_1 K^{2-2M}(\xi) + \alpha_2 K^{2-M}(\xi) + \alpha_3 K^2(\xi) + \alpha_4 K^{2+M}(\xi) + \alpha_5 K^{2+2M}(\xi), M > 0, \quad (22)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are constants. With the modified solutions of Eq.(22) given in [27], we will construct the optical soliton solutions of system (1) and (2). Substituting Eq.(21) along with

Eq.(22) into Eq. (19), collecting all the coefficients of $K^{iM}(\xi)$ ($i = 0, 1, 2, 3$), we get the following set of algebraic equations:

$$K^0(\xi) : \frac{1}{2} \beta_1 \alpha_2 \mu M^2 = 0, \quad (23a)$$

$$K^M(\xi) : \beta_1 \alpha_3 \mu M^2 - \beta_2 \mu = 0, \quad (23b)$$

$$K^{2M}(\xi) : \frac{3}{2} \beta_1 \alpha_4 \mu M^2 = 0, \quad (23c)$$

$$K^{3M}(\xi) : \beta_3 \mu^3 + 2\beta_1 \alpha_5 \mu M^2 = 0. \quad (23d)$$

From Eqs. (23a) - (23d), we find that $\alpha_1 = 0$. Solving Eqs. (23a) - (23d), we obtain

$$\alpha_2 = 0, \alpha_4 = 0, \alpha_3 = \frac{\beta_2}{\beta_1 M^2}, \mu = \epsilon M \sqrt{-\frac{2\beta_1 \alpha_5}{\beta_3}}, \quad (24)$$

provided $\beta_1 \beta_3 < 0$, $\alpha_5 > 0$ and $\epsilon = \pm 1$. We now have the following variety of traveling wave solutions of Eqs.(1) and (2) using the solutions of Eq. (22) provided in [27].

Type 1 : If $\alpha_2 = \alpha_4 = 0$, Eq.(22) gives the following type of Weierstrass elliptic function solutions:

Case (1): Eq.(22) has the Weierstrass elliptic function solution

$$K(\xi) = \left[\frac{1}{\alpha_5} \wp(M\xi, g_2, g_3) - \frac{\alpha_3}{3\alpha_5} \right]^{\frac{1}{2M}}, \quad (25)$$

where $g_2 = \frac{4\alpha_3^2 - 12\alpha_1\alpha_5}{3}$ and $g_3 = \frac{4\alpha_3(-2\alpha_3^2 + 9\alpha_1\alpha_5)}{27}$. Here, $\wp(M\xi, g_2, g_3)$ is called a Weierstrass elliptic function which satisfies the ODE $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, where g_2 and g_3 are called invariants of the Weierstrass elliptic function. Since $\alpha_1 = 0$, then we have Weierstrass elliptic function solution of the coupled system (1) and (2) in the form

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 \wp(M(K_1 x + K_2 y - Ct), g_2, g_3)}{3\beta_3}} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 \wp(M(K_1 x + K_2 y - Ct), g_2, g_3)}{3\beta_3}} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (26)$$

where $g_2 = \frac{4\beta_2^2}{3\beta_1^2 M^4}$, $g_3 = -\frac{8\beta_2^3}{27\beta_1^3 M^6}$ and $\epsilon = \pm 1$.

In particular, the Weierstrass elliptic function can be written in terms of Jacobi elliptic functions cn and ns as (29)

$$\wp(M\xi, g_2, g_3) = [e_2 - (e_2 - e_3) cn^2(Q\xi; m)], \quad (27)$$

$$\wp(M\xi, g_2, g_3) = [e_3 + (e_1 - e_3) ns^2(Q\xi; m)], \quad (28)$$

where $Q = \sqrt{e_1 - e_3}$, $m^2 = \frac{e_2 - e_3}{e_1 - e_3}$, is the modulus of the Jacobi elliptic functions cn and $ns = \frac{1}{sn}$ such that $0 < m < 1$, $e_i (i = 1, 2, 3; e_1 \geq e_2 \geq e_3)$ are three roots of the cubic equation

$$4z^3 - g_2 z + g_3 = 0. \quad (29)$$

Therefore, the solution (26) can be written as

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_2 - (e_2 - e_3) cn^2(Q\xi; m)]}{3\beta_3}} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_2 - (e_2 - e_3) cn^2(Q\xi; m)]}{3\beta_3}} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (30)$$

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_3 + (e_1 - e_3) ns^2(Q\xi; m)]}{3\beta_3}} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_3 + (e_1 - e_3) ns^2(Q\xi; m)]}{3\beta_3}} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (31)$$

In particular, when $m \rightarrow 1$, (i.e. $e_2 \rightarrow e_1$), then $cn(Q\xi; m) \rightarrow \text{sech}(Q\xi)$ and $ns(Q\xi; m) \rightarrow \coth(Q\xi)$, therefore we have the bright and singular soliton solutions of the coupled system (1) and (2) in the form

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_2 - (e_2 - e_3) \text{sech}^2(Q\xi)]}{3\beta_3}} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_2 - (e_2 - e_3) \text{sech}^2(Q\xi)]}{3\beta_3}} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (32)$$

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_3 + (e_1 - e_3) \coth^2(Q\xi)]}{3\beta_3}} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2 - 6\beta_1 M^2 [e_3 + (e_1 - e_3) \coth^2(Q\xi)]}{3\beta_3}} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (33)$$

Case (2): Eq.(22) has the solution

$$K(\xi) = \left[\frac{3 \sqrt{\alpha_5^{-1}} \wp'(M\xi, g_2, g_3)}{6 \wp(M\xi, g_2, g_3) + \alpha_3} \right]^{\frac{1}{M}}, \quad g_2 = \frac{\alpha_3^2}{12} + \alpha_1 \alpha_5, \quad g_3 = \frac{\alpha_3 (36 \alpha_1 \alpha_5 - \alpha_3^2)}{216}. \quad (34)$$

Since $\alpha_1 = 0$, then we have Weierstrass elliptic function solution of the coupled system (1) and (2) in the form

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{-\frac{2\beta_1}{\beta_3}} \left(\frac{3 M^3 \beta_1 \wp'(M(K_1 x + K_2 y - Ct), g_2, g_3)}{6 M^2 \beta_1 \wp(M(K_1 x + K_2 y - Ct), g_2, g_3) + \beta_2} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{-\frac{2\beta_1}{\beta_3}} \left(\frac{3 M^3 \beta_1 \wp'(M(K_1 x + K_2 y - Ct), g_2, g_3)}{6 M^2 \beta_1 \wp(M(K_1 x + K_2 y - Ct), g_2, g_3) + \beta_2} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (35)$$

provided $\beta_1 \beta_3 < 0$, $g_2 = \frac{\beta_2^2}{12\beta_1^2 M^4}$, $g_3 = \frac{\beta_2^3}{216\beta_1^3 M^6}$, and $\epsilon = \pm 1$.

Type 2 : If $\alpha_1 = \alpha_2 = \alpha_4 = 0$, then Eq.(22) have following cases:

Case (1): Eq.(22) has the bright soliton solution

$$K(\xi) = \left[\epsilon \sqrt{-\frac{\alpha_3}{\alpha_5}} \operatorname{sech}(M \sqrt{\alpha_3} \xi) \right]^{\frac{1}{M}}, \quad \alpha_3 > 0, \alpha_5 < 0, \epsilon = \pm 1. \quad (36)$$

The bright soliton solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = - \left[\epsilon^2 \sqrt{\frac{2\beta_2}{\beta_3}} \operatorname{sech} \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = -\frac{1}{\delta} \left[\epsilon^2 \sqrt{\frac{2\beta_2}{\beta_3}} \operatorname{sech} \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (37)$$

provided $\beta_2 \beta_3 > 0$, $\beta_1 \beta_2 > 0$ and $\epsilon = \pm 1$.

Case (2): Eq.(22) has the periodic solution

$$K(\xi) = \left[\epsilon \sqrt{-\frac{\alpha_3}{\alpha_5}} \sec(M \sqrt{-\alpha_3} \xi) \right]^{\frac{1}{M}}, \quad \alpha_3 < 0, \alpha_5 > 0, \epsilon = \pm 1. \quad (38)$$

The periodic solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = - \left[\epsilon^2 \sqrt{\frac{2\beta_2}{\beta_3}} \sec \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = -\frac{1}{\delta} \left[\epsilon^2 \sqrt{\frac{2\beta_2}{\beta_3}} \sec \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (39)$$

provided $\beta_2 \beta_3 > 0$, $\beta_1 \beta_2 > 0$, and $\epsilon = \pm 1$.

Case (3): Eq.(22) has the rational solution

$$K(\xi) = \left[\frac{\epsilon}{M \sqrt{\alpha_5} \xi} \right]^{\frac{1}{M}}, \quad \alpha_3 = 0, \alpha_5 > 0, \epsilon = \pm 1. \quad (40)$$

We have the rational solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = \left[\epsilon^2 \sqrt{-\frac{2\beta_1}{\beta_3}} \left(\frac{1}{K_1x + K_2y - Ct} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon^2 \sqrt{-\frac{2\beta_1}{\beta_3}} \left(\frac{1}{K_1x + K_2y - Ct} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (41)$$

provided $\beta_1\beta_3 < 0$, and $\epsilon = \pm 1$.

Type 3 : If $\alpha_1 = \alpha_2 = 0$, $\alpha_3 > 0$.

Case (1): Eq.(22) has the hyperbolic solution

$$K(\xi) = \left[\frac{2\alpha_3 \operatorname{sech}^2(\frac{M}{2} \sqrt{\alpha_3} \xi)}{2\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} - [\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} + \alpha_4] \operatorname{sech}^2(\frac{M}{2} \sqrt{\alpha_3} \xi)} \right]^{\frac{1}{M}}, \alpha_4^2 - 4\alpha_3\alpha_5 > 0, \quad (42)$$

Since $\alpha_4 = 0$, the bright soliton solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\operatorname{sech}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))}{2 - \operatorname{sech}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\operatorname{sech}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))}{2 - \operatorname{sech}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (43)$$

provided $\beta_1\beta_2 > 0$, $\beta_2\beta_3 > 0$, and $\epsilon = \pm 1$.

In Fig. 1, we depicts intensity structures of the bright solitary wave solution (43) for the different choice of parameter values.

Case (2): Eq.(22) has the hyperbolic solution

$$K(\xi) = \left[\frac{2\alpha_3 \operatorname{csch}^2(\frac{M}{2} \sqrt{\alpha_3} \xi)}{2\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} + [\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} - \alpha_4] \operatorname{csch}^2(\frac{M}{2} \sqrt{\alpha_3} \xi)} \right]^{\frac{1}{M}}, \alpha_4^2 - 4\alpha_3\alpha_5 > 0, \quad (44)$$

Since $\alpha_4 = 0$, the singular soliton solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\operatorname{csch}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))}{2 + \operatorname{csch}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\operatorname{csch}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))}{2 + \operatorname{csch}^2(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (45)$$

provided $\beta_1\beta_2 > 0, \beta_2\beta_3 > 0$, and $\epsilon = \pm 1$.

Case (3): Eq.(22) has the hyperbolic solution

$$K(\xi) = \left[\frac{2\alpha_3}{\epsilon \sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} \cosh(M \sqrt{\alpha_3}\xi) - \alpha_4} \right]^{\frac{1}{M}}, \alpha_4^2 - 4\alpha_3\alpha_5 > 0, \quad (46)$$

Since $\alpha_4 = 0$, the bright soliton solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = \left[\sqrt{\frac{2\beta_2}{\beta_3}} \left(\operatorname{sech} \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct) \right) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\sqrt{\frac{2\beta_2}{\beta_3}} \left(\operatorname{sech} \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct) \right) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (47)$$

provided $\beta_1\beta_2 > 0, \beta_2\beta_3 > 0$, and $\epsilon = \pm 1$.

Figure 2, illustrates the intensity profile of the bright solitary wave solution (47) for the arbitrary choice of parameter values.

Case (4): Eq.(22) has the hyperbolic solution

$$K(\xi) = \left[\frac{2\alpha_3}{\epsilon \sqrt{-(\alpha_4^2 - 4\alpha_3\alpha_5)} \sinh(M \sqrt{\alpha_3}\xi) - \alpha_4} \right]^{\frac{1}{M}}, \alpha_4^2 - 4\alpha_3\alpha_5 < 0, \quad (48)$$

Since $\alpha_4 = 0$, the singular soliton solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = \left[\sqrt{-\frac{2\beta_2}{\beta_3}} \left(\operatorname{csch} \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct) \right) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\sqrt{-\frac{2\beta_2}{\beta_3}} \left(\operatorname{csch} \left(\sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct) \right) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (49)$$

provided $\beta_1\beta_2 > 0, \beta_2\beta_3 < 0$, and $\epsilon = \pm 1$.

Case (5): Eq.(22) has the hyperbolic solution

$$K(\xi) = \left[-\frac{\alpha_3 \operatorname{sech}^2(\frac{M}{2} \sqrt{\alpha_3}\xi)}{\alpha_4 + 2\epsilon \sqrt{\alpha_3\alpha_5} \tanh(\frac{M}{2} \sqrt{\alpha_3}\xi)} \right]^{\frac{1}{M}}, \alpha_5 > 0, \epsilon = \pm 1, \quad (50)$$

Since $\alpha_4 = 0$, the combined hyperbolic solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = - \left[\sqrt{\frac{-\beta_2}{2\beta_3}} \coth\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct)\right) \operatorname{sech}^2\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct)\right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = -\frac{1}{\delta} \left[\sqrt{\frac{-\beta_2}{2\beta_3}} \coth\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct)\right) \operatorname{sech}^2\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}} (K_1x + K_2y - Ct)\right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (51)$$

provided $\beta_1\beta_2 > 0, \beta_2\beta_3 < 0$, and $\epsilon = \pm 1$.

The hyperbolic solution (51) is identical to the solution (51) under some trigonometric identities.

Type 4: If $\alpha_1 = \alpha_2 = 0, \alpha_3 < 0$.

Case (1): Eq.(22) has the periodic wave solution

$$K(\xi) = \left[\frac{-2\alpha_3 \sec^2(\frac{M}{2} \sqrt{-\alpha_3}\xi)}{2\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} - [\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} - \alpha_4] \sec^2(\frac{M}{2} \sqrt{-\alpha_3}\xi)} \right]^{\frac{1}{M}}, \alpha_4^2 - 4\alpha_3\alpha_5 > 0, \quad (52)$$

Since $\alpha_4 = 0$, the periodic solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\sec^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))}{-2 + \sec^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\sec^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))}{-2 + \sec^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (53)$$

provided $\beta_1\beta_2 < 0, \beta_2\beta_3 > 0$, and $\epsilon = \pm 1$.

In Fig. 3, we illustrate intensity structures of the singular periodic wave solution (53) for the arbitrary choice of parameter values.

Case (2): Eq.(22) has the periodic wave solution

$$K(\xi) = \left[\frac{2\alpha_3 \csc^2(\frac{M}{2} \sqrt{-\alpha_3}\xi)}{2\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} - [\sqrt{\alpha_4^2 - 4\alpha_3\alpha_5} + \alpha_4] \csc^2(\frac{M}{2} \sqrt{-\alpha_3}\xi)} \right]^{\frac{1}{M}}, \alpha_4^2 - 4\alpha_3\alpha_5 > 0, \quad (54)$$

Since $\alpha_4 = 0$, the singular periodic wave solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = - \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\csc^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))}{-2 + \csc^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = -\frac{1}{\delta} \left[\epsilon \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\csc^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))}{-2 + \csc^2(\frac{1}{2} \sqrt{-\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct))} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (55)$$

provided $\beta_1\beta_2 < 0, \beta_2\beta_3 > 0$, and $\epsilon = \pm 1$.

Case (3): Eq.(22) has the hyperbolic solution

$$K(\xi) = \left[- \frac{\alpha_3 \sec^2(\frac{M}{2} \sqrt{-\alpha_3}\xi)}{\alpha_4 + 2\epsilon \sqrt{\alpha_3\alpha_5} \tan(\frac{M}{2} \sqrt{-\alpha_3}\xi)} \right]^{\frac{1}{M}}, \alpha_5 > 0, \epsilon = \pm 1, \quad (56)$$

Since $\alpha_4 = 0$, the complex periodic solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = \left[\sqrt{\frac{-\beta_2}{2\beta_3}} \sec^2\left(\frac{i}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \coth\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\sqrt{\frac{-\beta_2}{2\beta_3}} \sec^2\left(\frac{i}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \coth\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (57)$$

provided $\beta_1\beta_2 > 0, \beta_2\beta_3 < 0$, and $\epsilon = \pm 1$.

Case (4): Eq.(22) has the hyperbolic solution

$$K(\xi) = \left[-\frac{\alpha_3 \csc^2\left(\frac{M}{2} \sqrt{-\alpha_3}\xi\right)}{\alpha_4 + 2\epsilon \sqrt{\alpha_3\alpha_5} \cot\left(\frac{M}{2} \sqrt{-\alpha_3}\xi\right)} \right]^{\frac{1}{M}}, \quad \alpha_5 > 0, \epsilon = \pm 1, \quad (58)$$

Since $\alpha_4 = 0$, the complex periodic solution of the coupled system (1) and (2) are then given in the form

$$\begin{cases} q(x, y, t) = - \left[\sqrt{\frac{-\beta_2}{2\beta_3}} \csc^2\left(\frac{i}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \tanh\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = -\frac{1}{\delta} \left[\sqrt{\frac{-\beta_2}{2\beta_3}} \csc^2\left(\frac{i}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \tanh\left(\frac{1}{2} \sqrt{\frac{\beta_2}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (59)$$

provided $\beta_1\beta_2 > 0, \beta_2\beta_3 < 0$, and $\epsilon = \pm 1$.

Type 5 : If $\alpha_1 = \alpha_2 = 0$.

Case (1). Eq.(22) has the solution

$$K(\xi) = \left[\frac{4 \alpha_3 M^2 e^{\epsilon M \sqrt{\alpha_3} \xi}}{\left(e^{\epsilon M \sqrt{\alpha_3} \xi} - \alpha_4 M^2\right)^2 - 4 \alpha_3 \alpha_5 M^4} \right]^{\frac{1}{M}}, \quad \alpha_3 > 0, \epsilon = \pm 1. \quad (60)$$

Since $\alpha_4 = 0$, the exponential function solution of the coupled system (1) and (2) in the following form

$$\begin{cases} q(x, y, t) = \left[\sqrt{\frac{-2\alpha_5\beta_1}{\beta_3}} \left(\frac{4\epsilon M\beta_2 e^{\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)}}{\beta_1 e^{2\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)} - 4\alpha_5\beta_2 M^2} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\sqrt{\frac{-2\alpha_5\beta_1}{\beta_3}} \left(\frac{4\epsilon M\beta_2 e^{\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)}}{\beta_1 e^{2\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)} - 4\alpha_5\beta_2 M^2} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (61)$$

provided $\beta_1\beta_2 > 0, \alpha_5\beta_1\beta_3 < 0$, and $\epsilon = \pm 1$.

Case (2). Eq.(22) has the solution

$$K(\xi) = \left[\frac{4 \alpha_3 M^2 e^{\epsilon M \sqrt{\alpha_3} \xi}}{-1 + 4 \alpha_3 \alpha_5 M^4 e^{2\epsilon M \sqrt{\alpha_3} \xi}} \right]^{\frac{1}{M}}, \quad \alpha_3 > 0, \alpha_4 = 0, \epsilon = \pm 1. \quad (62)$$

Therefore, we have the exact solution of the coupled system (1) and (2) in the following form

$$\begin{cases} q(x, y, t) = - \left[\sqrt{\frac{-2\alpha_5\beta_1}{\beta_3}} \left(\frac{4\epsilon M\beta_2 e^{\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)}}{\beta_1 - 4\alpha_5\beta_2 M^2 e^{2\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)}} \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = -\frac{1}{\delta} \left[\sqrt{\frac{-2\alpha_5\beta_1}{\beta_3}} \left(\frac{4\epsilon M\beta_2 e^{\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)}}{\beta_1 - 4\alpha_5\beta_2 M^2 e^{2\epsilon \sqrt{\frac{\beta_2}{\beta_1}}(K_1x+K_2y-Ct)}} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (63)$$

provided $\beta_1\beta_2 > 0$, $\alpha_5\beta_1\beta_3 < 0$, and $\epsilon = \pm 1$.

4.1. The new extended auxiliary equation method

In this approach, we assume that Eq. (19) has the formal solution:

$$\chi(\xi) = A_0 + A_1 F(\xi) + A_2 F(\xi)^2, \quad (64)$$

where A_0 , A_1 and A_2 are constants to be determined, such that A_1 and A_2 cannot all be zero at the same time, and the function $F(\xi)$ satisfies the first-order ODE:

$$F'(\xi)^2 = \gamma_0 + \gamma_2 F(\xi)^2 + \gamma_4 F(\xi)^4 + \gamma_6 F(\xi)^6, \quad (65)$$

where γ_j ($j = 0, 2, 4, 6$) are arbitrary constants to be determined. It is established that Eq. (65) provides the following solutions [24]:

$$F(\xi) = \frac{1}{2} \left[-\frac{\gamma_4}{\gamma_6} \left(1 \pm f_i(\xi) \right) \right]^2, \quad (66)$$

where the function $f_i(\xi)$ ($i = 1, 2, \dots, 12$) could be specified using the Jacobi elliptic function $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$, and so forth, where m ($0 < m < 1$) defines the modulus of the Jacobi elliptic functions. The Jacobi elliptic functions transformed into hyperbolic and trigonometric functions when $m \rightarrow 1$ or $m \rightarrow 0$, respectively.

The functions $f_i(\xi)$ ($i = 1, 2, \dots, 12$) given by Eq. (66) provides 12 forms of Jacobi elliptic functions.

From Eq. (64), we get

$$\chi'(\xi) = A_1 F'[\xi] + 2F[\xi]A_2 F'[\xi], \quad (67)$$

$$\chi''(\xi) = A_1 \left(\gamma_2 F(\xi) + 2\gamma_4 F(\xi)^3 + 3\gamma_6 F(\xi)^5 \right) + 2A_2 \left(\gamma_0 + 2\gamma_2 F(\xi)^2 + 3\gamma_4 F(\xi)^4 + 4\gamma_6 F(\xi)^6 \right). \quad (68)$$

We obtain a system of algebraic equations by plugging Eqs. (64), (67), (68) along with (65) into Eq. (19), accumulating the coefficients of each power of F^i ($i = 0, 1, \dots, 6$) and setting them to

zero, as follows:

$$F(\xi)^0 : -A_0\beta_2 + A_0^3\beta_3 + 2A_2\beta_1\gamma_0 = 0, \quad (69a)$$

$$F(\xi)^1 : -A_1\beta_2 + 3A_0^2A_1\beta_3 + A_1\beta_1\gamma_2 = 0, \quad (69b)$$

$$F(\xi)^2 : -A_2\beta_2 + 3A_0A_1^2\beta_3 + 3A_0^2A_2\beta_3 + 4A_2\beta_1\gamma_2 = 0, \quad (69c)$$

$$F(\xi)^3 : A_1^3\beta_3 + 6A_0A_1A_2\beta_3 + 2A_1\beta_1\gamma_4 = 0, \quad (69d)$$

$$F(\xi)^4 : 3A_1^2A_2\beta_3 + 3A_0A_2^2\beta_3 + 6A_2\beta_1\gamma_4 = 0, \quad (69e)$$

$$F(\xi)^5 : 3A_1A_2^2\beta_3 + 3A_1\beta_1\gamma_6 = 0, \quad (69f)$$

$$F(\xi)^6 : A_2^3\beta_3 + 8A_2\beta_1\gamma_6 = 0. \quad (69g)$$

Using Mathematica to solve the algebraic equations Eqs. (69a)- (69g), the following results were obtained:

$$A_1 = 0, \gamma_0 = -\frac{A_0(-\beta_2 + A_0^2\beta_3)}{2A_2\beta_1}, \gamma_2 = \frac{\beta_2 - 3A_0^2\beta_3}{4\beta_1}, \gamma_4 = -\frac{A_0A_2\beta_3}{2\beta_1}, \gamma_6 = -\frac{A_2^2\beta_3}{8\beta_1}. \quad (70)$$

By using Eq. (70) in Eq. (64), along with Eq. (66) we get

$$\chi(\xi) = A_0 + \frac{A_2}{4} \left[-\frac{4A_0}{A_2} (1 \pm f_i(\xi)) \right]. \quad (71)$$

Now, we have the following Jacobi elliptic function solutions of coupled Eqs.(1) and (2):

Type 1: If $\gamma_0 = \frac{\gamma_4^3(m^2-1)}{32\gamma_6^2m^2}$, $\gamma_2 = \frac{\gamma_4^2(5m^2-1)}{16\gamma_6m^2}$, $\gamma_6 > 0$, then

$$f_1(\xi) = \text{sn}(\sigma\xi), f_2(\xi) = \frac{1}{m \text{sn}(\sigma\xi)}, \sigma = \frac{\gamma_4}{2m} \sqrt{\frac{1}{\gamma_6}}. \quad (72)$$

Therefore, the corresponding solution of coupled Eqs.(1) and (2) is given by

$$\begin{cases} q(x, y, t) = \left[\mp A_0 \text{sn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 \text{sn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (73)$$

and

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0}{m \text{sn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0}{m \text{sn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (74)$$

provided $\beta_3\beta_1 < 0$.

If $m \rightarrow 1$, then $sn(\xi) \rightarrow \tanh(\xi)$, and hence coupled Eqs.(1) and (2) has the hyperbolic function solutions

$$\begin{cases} q(x, y, t) = \left[\mp A_0 \tanh \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 \tanh \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (75)$$

provided $\beta_3\beta_1 < 0$.

Fig. 4 depicts the intensity profile of the dark solitary wave solution (75) for arbitrary values of the parameters.

Also

$$\begin{cases} q(x, y, t) = \left[\mp A_0 \coth \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 \coth \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (76)$$

provided $\beta_3\beta_1 < 0$.

Fig. 5 depicts the intensity profile of the singular solitary wave solution (76) for arbitrary choice of the parameters.

Type 2: If $\gamma_0 = \frac{\gamma_4^2(1-m^2)}{32\gamma_6^2}$, $\gamma_2 = \frac{\gamma_4^2(5-m^2)}{16\gamma_6}$, $\gamma_6 > 0$, then

$$f_3(\xi) = m \operatorname{sn}(\sigma\xi), f_4(\xi) = \frac{1}{\operatorname{sn}(\sigma\xi)}, \sigma = \frac{\gamma_4}{2} \sqrt{\frac{1}{\gamma_6}}. \quad (77)$$

Therefore, the corresponding solution of coupled Eqs.(1) and (2) is given by

$$\begin{cases} q(x, y, t) = \left[\mp A_0 m \operatorname{sn} \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 m \operatorname{sn} \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (78)$$

provided $\beta_3\beta_1 < 0$.

The intensity profile of the doubly periodic wave solution (78) is depicted in Fig.6 for arbitrary choice of the parameters.

Also

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0}{\operatorname{sn} \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right)} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0}{\operatorname{sn} \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1 x + K_2 y - Ct) \right)} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (79)$$

provided $\beta_3\beta_1 < 0$.

If $m \rightarrow 0$, then $sn(\xi) \rightarrow \sin(\xi)$, and solution (78) vanishes. From (79), we have

$$\begin{cases} q(x, y, t) = \left[\mp A_0 \csc \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 \csc \left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (80)$$

provided $\beta_3\beta_1 < 0$.

Fig.7 depicts the intensity profile of the singular periodic wave solution (80) for selected values of the parameters.

When $m \rightarrow 1$, then we recover solutions (75) and (76), respectively.

Type 3: If $\gamma_0 = \frac{\gamma_4^3}{32\gamma_6^2m^2}$, $\gamma_2 = \frac{\gamma_4^2(4m^2+1)}{16\gamma_6m^2}$, $\gamma_6 < 0$, then

$$f_5(\xi) = cn(\sigma\xi), f_6(\xi) = \frac{\sqrt{1-m^2} sn(\sigma\xi)}{dn(\sigma\xi)}, \sigma = \frac{\gamma_4}{2m} \sqrt{-\frac{1}{\gamma_6}}. \quad (81)$$

Therefore, the corresponding solution of coupled Eqs.(1) and (2) is given by

$$\begin{cases} q(x, y, t) = \left[\mp A_0 cn \left(-\frac{A_0}{2m} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 cn \left(-\frac{A_0}{2m} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (82)$$

and

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0 \sqrt{1-m^2} sn \left(-\frac{A_0}{2m} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right)}{dn \left(-\frac{A_0}{2m} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right)} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0 \sqrt{1-m^2} sn \left(-\frac{A_0}{2m} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right)}{dn \left(-\frac{A_0}{2m} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right)} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (83)$$

provided $\beta_3\beta_1 > 0$.

If $m \rightarrow 1$, then $cn(\xi) \rightarrow \text{sech}(\xi)$, and solution (83) vanishes while (82) have the hyperbolic function solutions

$$\begin{cases} q(x, y, t) = \left[\mp A_0 \text{sech} \left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 \text{sech} \left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1}} (K_1x + K_2y - Ct) \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (84)$$

provided $\beta_3\beta_1 > 0$.

Fig.8 depicts the intensity profile of the bright solitary wave solution (84) for selected values of the parameters.

Type 4: If $\gamma_0 = \frac{\gamma_4^3 m^2}{32\gamma_6^2(m^2-1)}$, $\gamma_2 = \frac{\gamma_4^2(5m^2-4)}{16\gamma_6(m^2-1)}$, $\gamma_6 < 0$, then

$$f_7(\xi) = \frac{dn(\sigma\xi)}{\sqrt{1-m^2}}, f_8(\xi) = \frac{1}{dn(\sigma\xi)}, \sigma = \frac{\gamma_4}{2} \sqrt{-\frac{1}{\gamma_6(1-m^2)}}. \quad (85)$$

Therefore, the corresponding solution of coupled Eqs.(1) and (2) is given by

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0 dn\left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1(1-m^2)}}(K_1x + K_2y - Ct)\right)}{\sqrt{1-m^2}} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0 dn\left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1(1-m^2)}}(K_1x + K_2y - Ct)\right)}{\sqrt{1-m^2}} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (86)$$

provided $\beta_3\beta_1(1-m^2) > 0$.

Also

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0}{dn\left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1(1-m^2)}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0}{dn\left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1(1-m^2)}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (87)$$

provided $\beta_3\beta_1(1-m^2) < 0$.

If $m \rightarrow 0$, then $dn(\xi) \rightarrow 1$ and hence solutions (86) and (87) appear as constants.

Type 5: If $\gamma_0 = \frac{\gamma_4^3}{32\gamma_6^2(1-m^2)}$, $\gamma_2 = \frac{\gamma_4^2(4m^2-5)}{16\gamma_6(m^2-1)}$, $\gamma_6 > 0$, then

$$f_9(\xi) = \frac{1}{cn(\sigma\xi)}, f_{10}(\xi) = \frac{dn(\sigma\xi)}{\sqrt{1-m^2}sn(\sigma\xi)}, \sigma = \frac{\gamma_4}{2} \sqrt{\frac{1}{\gamma_6(1-m^2)}}. \quad (88)$$

Therefore, the corresponding solution of coupled Eqs.(1) and (2) is given by

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0}{cn\left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1(1-m^2)}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0}{cn\left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1(1-m^2)}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (89)$$

provided $\beta_3\beta_1(1-m^2) < 0$.

Also

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0 \operatorname{dn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)}{\sqrt{1-m^2} \operatorname{sn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0 \operatorname{dn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)}{\sqrt{1-m^2} \operatorname{sn}\left(-\frac{A_0}{2m} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (90)$$

provided $\beta_3\beta_1 < 0$.

If $m \rightarrow 0$, then $\operatorname{dn}(\xi) \rightarrow 1$, $\operatorname{cn}(\xi) \rightarrow \cos(\xi)$, $\operatorname{sn}(\xi) \rightarrow \sin(\xi)$ and hence from solution (90), we recover singular periodic wave solution (80). From (89), we have the periodic wave solution

$$\begin{cases} q(x, y, t) = \left[\mp A_0 \sec\left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \left[\frac{1}{\delta} \left[\mp A_0 \sec\left(-\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (91)$$

provided $\beta_3\beta_1 < 0$.

The intensity profile of the periodic wave solution (91) is depicted in Fig.9.

Type 6: If $\gamma_0 = \frac{m^2\gamma_4^3}{32\gamma_6^2}$, $\gamma_2 = \frac{\gamma_4^2(m^2+6)}{16\gamma_6}$, $\gamma_6 < 0$, then

$$f_{11}(\xi) = \operatorname{dn}(\sigma\xi), f_{12}(\xi) = \frac{\sqrt{1-m^2}}{\operatorname{dn}(\sigma\xi)}, \sigma = \frac{\gamma_4}{2} \sqrt{-\frac{1}{\gamma_6}}. \quad (92)$$

Therefore, the corresponding solution of coupled Eqs.(1) and (2) is given by

$$\begin{cases} q(x, y, t) = \left[\mp A_0 \operatorname{dn}\left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp A_0 \operatorname{dn}\left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (93)$$

and

$$\begin{cases} q(x, y, t) = \left[\mp \frac{A_0 \sqrt{1-m^2}}{\operatorname{dn}\left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\mp \frac{A_0 \sqrt{1-m^2}}{\operatorname{dn}\left(-\frac{A_0}{2} \sqrt{\frac{2\beta_3}{\beta_1}}(K_1x + K_2y - Ct)\right)} \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (94)$$

provided $\beta_3\beta_1 > 0$.

If $m \rightarrow 1$, then $dn(\xi) \rightarrow \text{sech}(\xi)$ and solution (94) vanishes while from solutions (93) we recover the bright solitary wave solution (84). If $m \rightarrow 0$, then $dn(\xi) \rightarrow 1$ and the solutions of Eqs.(1) and (2) appears as constants.

4.2. The new Jacobi elliptic function expansion method

Using this method, we infer that Eq. (19) has the formal solution:

$$\chi(\xi) = A_0 + A_1 \frac{F(\xi)}{1 + F(\xi)^2} + A_2 \frac{1 - F(\xi)^2}{1 + F(\xi)^2}, \quad (95)$$

where A_0, A_1 and A_2 are constants to be determined, such that A_1 and A_2 both can not be simultaneously zero, while the function $F(\xi)$ satisfies the first-order ODE:

$$F'(\xi)^2 = \gamma_0 + \gamma_2 F(\xi)^2 + \gamma_4 F(\xi)^4, \quad (96)$$

where γ_j ($j = 0, 2, 4$) are arbitrary constants to be determined.

Incorporating (96) along with Eq.(95) into Eq.(19) and accumulating all the coefficients of $F_i(\xi)$, ($i = 0, 1, \dots, 6$) and putting them to zero, we have the following algebraic equations:

$$F(\xi)^0 : -A_0\beta_2 - A_2\beta_2 + A_0^3\beta_3 + 3A_0^2A_2\beta_3 + 3A_0A_2^2\beta_3 + A_2^3\beta_3 - 4A_2\beta_1\gamma_0 = 0, \quad (97a)$$

$$F(\xi)^1 : -A_1\beta_2 + 3A_0^2A_1\beta_3 + 6A_0A_1A_2\beta_3 + 3A_1A_2^2\beta_3 - 6A_1\beta_1\gamma_0 + A_1\beta_1\gamma_2 = 0, \quad (97b)$$

$$F(\xi)^2 : -3A_0\beta_2 - A_2\beta_2 + 3A_0^3\beta_3 + 3A_0A_1^2\beta_3 + 3A_0^2A_2\beta_3 + 3A_1^2A_2\beta_3 - 3A_0A_2^2\beta_3 - 3A_2^3\beta_3 + 12A_2\beta_1\gamma_0 - 8A_2\beta_1\gamma_2 = 0, \quad (97c)$$

$$F(\xi)^3 : -2A_1\beta_2 + 6A_0^2A_1\beta_3 + A_1^3\beta_3 - 6A_1A_2^2\beta_3 + 2A_1\beta_1\gamma_0 - 6A_1\beta_1\gamma_2 + 2A_1\beta_1\gamma_4 = 0, \quad (97d)$$

$$F(\xi)^4 : -3A_0\beta_2 + A_2\beta_2 + 3A_0^3\beta_3 + 3A_0A_1^2\beta_3 - 3A_0^2A_2\beta_3 - 3A_1^2A_2\beta_3 - 3A_0A_2^2\beta_3 + 3A_2^3\beta_3 + 8A_2\beta_1\gamma_2 - 12A_2\beta_1\gamma_4 = 0, \quad (97e)$$

$$F(\xi)^5 : -A_1\beta_2 + 3A_0^2A_1\beta_3 - 6A_0A_1A_2\beta_3 + 3A_1A_2^2\beta_3 + A_1\beta_1\gamma_2 - 6A_1\beta_1\gamma_4 = 0, \quad (97f)$$

$$F(\xi)^6 : -A_0\beta_2 + A_2\beta_2 + A_0^3\beta_3 - 3A_0^2A_2\beta_3 + 3A_0A_2^2\beta_3 - A_2^3\beta_3 + 4A_2\beta_1\gamma_4 = 0. \quad (97g)$$

Type 1: If we incorporate $\gamma_0 = 1$, $\gamma_2 = -(1 + m^2)$, $\gamma_4 = m^2$ into the algebraic equations (97a-97g) and solve them by Mathematica for $m \rightarrow 1$, we get the following two results:

Case 1:

$$A_0 = 0, A_1 = 0, A_2^2 = \frac{8\beta_1}{\beta_3}, \beta_2 = 4\beta_1. \quad (98)$$

In this case, we deduce that $F(\xi) = \text{sn}(\xi, 1) \rightarrow \tanh(\xi)$. Now, Eqs.(1) and (2) have the solitary wave solution as

$$\begin{cases} q(x, y, t) = \pm \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{1 - \tanh^2(\xi)}{1 + \tanh^2(\xi)} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{1 - \tanh^2(\xi)}{1 + \tanh^2(\xi)} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (99)$$

provided $\beta_2\beta_3 > 0$.

Fig.10 depicts the intensity profile of the hyperbolic wave solution (99) for arbitrary choice of parameters.

Case 2:

$$A_0 = 0, A_1 = 2iA_2, A_2^2 = \frac{2\beta_1}{\beta_3}, \beta_2 = -2\beta_1. \quad (100)$$

In this case, we deduce that $F(\xi) = \text{sn}(\xi, 1) \rightarrow \tanh(\xi)$. Now, Eqs.(1) and (2) has the complex hyperbolic solution of the form

$$\begin{cases} q(x, y, t) = \pm 2 \sqrt{-\frac{\beta_2}{\beta_3}} \left(\frac{1 - \tanh^2(\xi) + 2i \tanh(\xi)}{\text{sech}^2(\xi)} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm 2 \sqrt{-\frac{\beta_2}{\beta_3}} \left(\frac{1 - \tanh^2(\xi) + 2i \tanh(\xi)}{\text{sech}^2(\xi)} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (101)$$

provided $\beta_2\beta_3 < 0$.

Type 2: If we incorporate $\gamma_0 = 1 - m^2$, $\gamma_2 = 2m^2 - 1$, $\gamma_4 = -m^2$ into the algebraic equations(97a-97g) and solving them by Mathematica for $m = 1/2$, we get the following results:

$$A_0 = \frac{A_2}{2}, A_1 = 0, A_2^2 = \frac{4\beta_2}{5\beta_3}, \beta_2 = \frac{5}{2}\beta_1. \quad (102)$$

In this case, we deduce that $F(\xi) = \text{sn}(\xi, \frac{1}{2}) \rightarrow \text{cn}(\xi)$. Now, Eqs.(1) and (2) has the doubly periodic wave solution as

$$\begin{cases} q(x, y, t) = \pm \sqrt{\frac{4\beta_2}{5\beta_3}} \left(\frac{1}{2} + \frac{1 - \text{cn}^2(\xi, \frac{1}{2})}{1 + \text{cn}^2(\xi, \frac{1}{2})} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm \sqrt{\frac{4\beta_2}{5\beta_3}} \left(\frac{1}{2} + \frac{1 - \text{cn}^2(\xi, \frac{1}{2})}{1 + \text{cn}^2(\xi, \frac{1}{2})} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (103)$$

provided $\beta_2\beta_3 > 0$.

Fig.11 depicts the intensity profile of the doubly periodic wave solution (103) for the different choices of the parameters values.

Type 3: If we incorporate $\gamma_0 = m^2$, $\gamma_2 = -(1 + m^2)$, $\gamma_4 = 1$ into the algebraic equations (97a-97g) and solve them by Mathematica for $m \rightarrow 1$, we have the following two subsequent results:

Case 1:

$$A_0 = 0, A_1 = 0, A_2^2 = \frac{8\beta_1}{\beta_3}, \beta_2 = 4\beta_1. \quad (104)$$

In this case, we deduce that $F(\xi) = \text{sn}(\xi, 1) \rightarrow \tanh(\xi)$. Now, Eqs.(1) and (2) has the dark soliton solution as

$$\begin{cases} q(x, y, t) = \pm \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\tanh^2(\xi) - 1}{\tanh^2(\xi) + 1} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm \sqrt{\frac{2\beta_2}{\beta_3}} \left(\frac{\tanh^2(\xi) - 1}{\tanh^2(\xi) + 1} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (105)$$

provided $\beta_2\beta_3 > 0$.

Case 2:

$$A_0 = 0, A_1 = 2iA_2, A_2^2 = \frac{2\beta_1}{\beta_3}, \beta_2 = -2\beta_1. \quad (106)$$

In this case, we deduce that $F(\xi) = \text{sn}(\xi, 1) \rightarrow \tanh(\xi)$. Now, Eqs.(1) and (2) has the complex hyperbolic solution as

$$\begin{cases} q(x, y, t) = \pm 2 \sqrt{-\frac{\beta_2}{\beta_3}} \left(\frac{\tanh^2(\xi) + 2i \tanh(\xi) - 1}{\text{sech}^2(\xi)} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm 2 \sqrt{-\frac{\beta_2}{\beta_3}} \left(\frac{\tanh^2(\xi) + 2i \tanh(\xi) - 1}{\text{sech}^2(\xi)} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (107)$$

provided $\beta_2\beta_3 < 0$.

Type 4: If we incorporate $\gamma_0 = -m^2$, $\gamma_2 = 2m^2 - 1$, $\gamma_4 = 1 - m^2$ into the algebraic equations (97a-97g) and solving them by Mathematica for $m = \frac{\sqrt{3}}{2}$, we have the following subsequent case:

$$A_0 = \frac{A_2}{2}, A_1 = 0, A_2^2 = \frac{4\beta_2}{5\beta_3}, \beta_2 = -\frac{5}{2}\beta_1. \quad (108)$$

In this case, we deduce that $F(\xi) = \text{nc}(\xi, \frac{\sqrt{3}}{2}) \rightarrow \text{cn}^{-1}(\xi, \frac{\sqrt{3}}{2})$. Now, Eqs.(1) and (2) has the doubly periodic wave solution as

$$\begin{cases} q(x, y, t) = \pm \sqrt{\frac{4\beta_2}{5\beta_3}} \left(\frac{1}{2} + \frac{\text{cn}^2(\xi, \frac{\sqrt{3}}{2}) - 1}{\text{cn}^2(\xi, \frac{\sqrt{3}}{2}) + 1} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm \sqrt{\frac{4\beta_2}{5\beta_3}} \left(\frac{1}{2} + \frac{\text{cn}^2(\xi, \frac{\sqrt{3}}{2}) - 1}{\text{cn}^2(\xi, \frac{\sqrt{3}}{2}) + 1} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (109)$$

provided $\beta_2\beta_3 > 0$.

Type 5: If we incorporate $\gamma_0 = \frac{1}{4}$, $\gamma_2 = \frac{1-2m^2}{2}$, $\gamma_4 = \frac{1}{4}$ into the algebraic equations (97a-97g) and solving them by Mathematica, we have the following result:

$$A_0 = 0, A_1^2 = \frac{8m^2\beta_2}{(1+m^2)\beta_3}, A_2 = 0, \beta_2 = -(1+m^2)\beta_1. \quad (110)$$

For $m \rightarrow 1$, we have

$$A_0 = 0, A_1^2 = \frac{4\beta_2}{\beta_3}, A_2 = 0, \beta_2 = -2\beta_1. \quad (111)$$

In this case, we deduce that $F(\xi) = ns(\xi, 1) \pm cs(\xi, 1) \rightarrow \coth(\xi) \pm \operatorname{csch}(\xi)$. Now, Eqs.(1) and (2) has the combined hyperbolic function solution as

$$\begin{cases} q(x, y, t) = \pm \sqrt{\frac{4\beta_2}{\beta_3}} \left(\frac{\coth(\xi) \pm \operatorname{csch}(\xi)}{1 + (\coth(\xi) \pm \operatorname{csch}(\xi))^2} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm \sqrt{\frac{4\beta_2}{\beta_3}} \left(\frac{\coth(\xi) \pm \operatorname{csch}(\xi)}{1 + (\coth(\xi) \pm \operatorname{csch}(\xi))^2} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (112)$$

provided $\beta_2\beta_3 > 0$.

Type 6: If we incorporate $\gamma_0 = \frac{1-m^2}{4}$, $\gamma_2 = \frac{1+m^2}{2}$, $\gamma_4 = \frac{1-m^2}{4}$ into the algebraic equations(97a-97g) and solving them by Mathematica, we have the following subsequent cases:

Case 1:

$$A_0 = 0, A_1 = 0, A_2^2 = \frac{2m^2\beta_2}{(1+m^2)\beta_3}, \beta_2 = -(1+m^2)\beta_1. \quad (113)$$

For $m \rightarrow 1$, we have

$$A_0 = 0, A_1 = 0, A_2^2 = \frac{4\beta_2}{\beta_3}, \beta_2 = -2\beta_1. \quad (114)$$

In this case, we deduce that $F(\xi) = nc(\xi, 1) \pm sc(\xi, 1) \rightarrow \cosh(\xi) \pm \sinh(\xi)$. Now, Eqs.(1) and (2) has combined hyperbolic function solution as

$$\begin{cases} q(x, y, t) = \pm \sqrt{\frac{\beta_2}{\beta_3}} \left(\frac{1 - (\cosh(\xi) \pm \sinh(\xi))^2}{1 + (\cosh(\xi) \pm \sinh(\xi))^2} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm \sqrt{\frac{\beta_2}{\beta_3}} \left(\frac{1 - (\cosh(\xi) \pm \sinh(\xi))^2}{1 + (\cosh(\xi) \pm \sinh(\xi))^2} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (115)$$

provided $\beta_2\beta_3 > 0$.

Figs. 12 and 13 depicts the amplitude and intensity profile of the solitary wave solution (115) respectively, for the suitable values of the parameters.

Case 2:

$$A_0 = 0, A_1^2 = \frac{8m^2\beta_1}{\beta_3}, A_2 = 0, \beta_2 = (-1 + 2m^2)\beta_1. \quad (116)$$

For $m \rightarrow 1$, we have

$$A_0 = 0, A_1^2 = \frac{8\beta_2}{\beta_3}, \beta_2 = \beta_1. \quad (117)$$

In this case, we deduce that $F(\xi) = nc(\xi, 1) \pm sc(\xi, 1) \rightarrow \cosh(\xi) \pm \sinh(\xi)$. Now, Eqs.(1) and (2) has combined hyperbolic function solution as

$$\begin{cases} q(x, y, t) = \pm \sqrt{\frac{8\beta_2}{\beta_3}} \left(\frac{\cosh(\xi) \pm \sinh(\xi)}{1 + (\cosh(\xi) \pm \sinh(\xi))^2} \right) \exp[i\phi_1(x, y, t)], \\ r(x, y, t) = \frac{1}{\delta} \left[\pm \sqrt{\frac{8\beta_2}{\beta_3}} \left(\frac{\cosh(\xi) \pm \sinh(\xi)}{1 + (\cosh(\xi) \pm \sinh(\xi))^2} \right) \right] \exp[i\phi_2(x, y, t)], \end{cases} \quad (118)$$

provided $\beta_2\beta_3 > 0$.

Fig.14 depicts the intensity profile of the bright solitary wave solution (118) for different choice of parameters values.

5. Graphical results and discussions

We have shown some graphs of the exact solutions in this section. Hyperbolic, trigonometric, and periodic solutions are examples of these solutions and explain various nonlinear waves. For the known exact solutions, hyperbolic solutions are special forms of solitary waves solutions that have the extraordinary property of maintaining their identity when interacting with other solitons. We plotted some of the acquired solutions' dynamic behaviors in figures 1-15 to demonstrate their transient characteristics.

Figure 1 shows two and three dimensional intensity structures of the bright solitary wave solution (43) for the choice of parameter values $\epsilon = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the bright solitary wave solution (43) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 2 illustrates the intensity profile of the bright solitary wave solution (47) for the choice of parameter values $\epsilon = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 2, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the bright solitary wave solution (47) is illustrated for $y = 1$ (red curve), $y = 2$ (blue curve), $y = 3$ (green curve) at $t = 0$.

Figure 3 shows the two and three dimensional intensity structures of the singular periodic wave solution (53) for the choice of parameter values $\epsilon = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 1.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the singular periodic wave solution (53) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 4 shows two and three-dimensional intensity structures of the dark solitary wave solution (75) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 =$

1, $D_1 = 1$, $\delta = 1.1$, $y = 1$, and $E_1 = -1$. The 2D profile of the dark solitary wave solution (75) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve), respectively.

Figure 5 shows two and three-dimensional intensity structures of the singular solitary wave solution (76) for different parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = -1$. The 2D structure of the dark solitary wave solution (76) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 6 shows two and three-dimensional intensity structures of the doubly periodic wave solution (78) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, E_1 = -1$, and $m = 0.5$. The 2D structure of the doubly periodic wave solution (78) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 7 shows two and three-dimensional intensity structures of the singular periodic wave solution (80) for various parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = -1$. The 2D structure of the singular periodic wave solution (80) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 8 shows two and three-dimensional intensity structures of the bright solitary wave solution (84) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = 1$. The 2D structure of the bright solitary wave solution (84) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 9 shows two and three dimensional intensity structures of the singular periodic wave solution (91) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = -1$. The 2D structure of the singular periodic wave solution (91) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 10 shows two and three dimensional intensity structures of the hyperbolic wave solution (99) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the hyperbolic wave solution (99) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 11 shows two and three dimensional intensity structures of the doubly periodic wave solution (103) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the doubly periodic wave solution (103) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 12 shows two and three dimensional structures of the amplitude of the solitary wave solution (115) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the amplitude of the kink solitary wave solution (115) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 13 shows two and three dimensional intensity structures of the solitary wave solution (115) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the dark solitary wave solution (115) is illustrated at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 14 shows two and three dimensional intensity structures of the bright solitary wave solution (118) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the bright

solitary wave solution (118) is illustrated at $x = 0$ (red curve), $x = 1$ (blue curve) and $x = 2$ (green curve).

We may deduce from these figures that they reflect the kink, bright and dark, periodic, singular periodic, doubly periodic, and trigonometric wave solutions. The bright soliton solution gets a bell-shaped form and propagates over extremely long distances without distortion or alteration of form. In nonlinear optics, optical dark solitons are also recognized as topological solitons. The time jitter in a dark soliton is smaller than in the corresponding bright soliton, as seen numerically and analytically. Dark soliton could be used in fibre optics communications, signal processing, and in switching applications. The dispersion effect usually allows the pulse to disperse, while nonlinearity causes it to converge. Higher-dimensional model studies are important from a realistic and physical context because nonlinear physical phenomena, such as ultrafast nonlinear optics, are modeled by multi-dimensional systems.

6. Solutions stability analysis

The Hamiltonian method, which describes the evolution of a physical system, is introduced in this section. In its most generic definition, the Hamiltonian system is given by

$$M(\nu) = \frac{1}{2} \left(\lim_{a_0 \rightarrow -\infty} \int_{a_0}^0 |\Psi(\theta)|^2 d\theta + \lim_{a_1 \rightarrow \infty} \int_0^{a_1} |\Psi(\theta)|^2 d\theta \right), \quad (119)$$

where $M(\nu)$ represents momentum, and $\Psi(\theta)$ corresponds to a traveling wave solution. The occurrence of a sufficient condition for the stability of solitary wave solutions is:

$$\frac{\partial M(\nu)}{\partial \nu} > 0, \quad (120)$$

where ν denotes the frequency. We discuss the criteria for stability of solutions for the cascaded (2+1)-dimensional coupled NLSE in given intervals by using Eqs.(119) and (120), respectively. The momentum M_{11} for the solution (47) is given by

$$M_{11} = \frac{1}{2} \int_{-\infty}^{\infty} |r_{11}(\xi)|^2 d\xi, \quad (121)$$

which leads to

$$\frac{\partial M_{11}}{\partial C} = \frac{[-2\beta_1 t + \xi(K_1 C_1 + DK_2)]\beta_2}{\beta_1 \beta_3 \delta^2} \operatorname{sech}^2 \left(\sqrt{\frac{\beta_2}{\beta_1}} \xi \right) - \sqrt{\frac{\beta_2}{\beta_1}} \frac{(K_1 C_1 + DK_2)}{\beta_3 \delta^2} \tanh \left(\sqrt{\frac{\beta_2}{\beta_1}} \xi \right) > 0, \quad (122)$$

for $\beta_1 \beta_2 > 0$. This is the necessary and sufficient condition for soliton stability in a specified interval.

The momentum M_{12} for the solution (47) is given by

$$M_{12} = \frac{1}{2} \int_{-\infty}^{\infty} |r_{12}(\xi)|^2 d\xi, \quad (123)$$

which leads to

$$\begin{aligned} \frac{\partial M_{12}}{\partial C} = & -\frac{A_0^2[-2\beta_1 t\beta_3 + \xi E_1 \delta^2 (K_1 C_1 + DK_2)]}{2\beta_1 \beta_3 \delta^2} \operatorname{sech}^2 \left(\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} \xi \right) \\ & - \frac{E_1 A_0}{\beta_1 \beta_3} \sqrt{-\frac{1}{\beta_1 \beta_3}} (K_1 C_1 + DK_2) \tanh \left(\frac{A_0}{2} \sqrt{-\frac{2\beta_3}{\beta_1}} \xi \right) > 0, \end{aligned} \quad (124)$$

for $\beta_1 \beta_3 < 0$. This is the necessary condition for soliton stability in a specified interval.

7. Conclusion

We successfully applied the new techniques to the the (2+1)-dimensional coupled nonlinear Schrödinger equation and generated new optical solitary wave solutions. This model arises in nonlinear optics and the optical solitary wave solutions of this model can be useful in a wide range of optoelectronic systems, such as optical couplers, birefringent fibers, DWDM devices and magneto-optic waveguides as solitons are used for transmission of data. We obtained exact solutions in the form of solitary wave and elliptic function solutions by using the new extended auxiliary equation method, the new Jacobi elliptic function expansion method, and the new modified sub-ODE method. The new optical solitary wave solutions come in a number of forms, including bright and dark solitons, periodic waves, singular solitary wave solutions which are expressed in different ways, including hyperbolic and trigonometric functions, rational, exponential, Weierstrass, and Jacobian elliptic functions. These findings illustrate the effectiveness and precision of the proposed methods, which can be replicated to other nonlinear models in future. The solutions found in this paper vary from those achieved by other researchers using different approaches [17]. We displayed the results graphically in two and three dimensions for various parameter values. Symbolic computation is used to confirm the obtained results. In this field, the results are novel and insightful.

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Figure 1: 3D profile of the intensity of the bright solitary wave solution (43) for the choice of parameter values $\epsilon = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the bright solitary wave solution (43) is depicted at different times $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 2: 3D profile of the intensity of the bright solitary wave solution (47) for the choice of parameter values $\epsilon = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 2, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the bright solitary wave solution (47) is depicted for $y = 1$ (red curve), $y = 2$ (blue curve), $y = 3$ (green curve) at $t = 0$.

Figure 3: 3D profile of the intensity of the singular periodic wave solution (53) for the choice of parameter values $\epsilon = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 1.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D structure of the intensity of the singular periodic wave solution (53) is depicted at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 4: 3D profile of the dark solitary wave solution (75) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = -1$. The 2D profile of the dark solitary wave solution (75) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 5: 3D profile of the singular solitary wave solution (76) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = -1$. The 2D profile of the dark solitary wave solution (76) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 6: 3D profile of the doubly periodic wave solution (78) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, E_1 = -1$, and $m = 0.5$. The 2D profile of the doubly periodic wave solution (78) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 7: 3D profile of the singular periodic wave solution (80) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = -1$. The 2D profile of the singular periodic wave solution (80) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 8: 3D profile of the bright solitary wave solution (84) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = 1$. The 2D profile of the bright solitary wave solution (84) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 9: 3D profile of the singular periodic wave solution (91) for the choice of parameter values $A_0 = 1, A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1$, and $E_1 = -1$. The 2D profile of the singular periodic wave solution (91) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 10: 3D profile of the hyperbolic wave solution (99) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D profile of the hyperbolic wave solution (99) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 11: 3D profile of the intensity of the doubly periodic wave solution (103) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D profile of the doubly periodic wave solution (103) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 12: 3D profile of the amplitude of the solitary wave solution (115) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D profile of the amplitude of the kink solitary wave solution (115) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 13: 3D profile of the intensity of the solitary wave solution (115) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, y = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D profile of the intensity of the dark solitary wave solution (115) is depicted at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 14: 3D profile of the intensity of the bright solitary wave solution (118) for the choice of parameter values $A_1 = 1, K_1 = 1, K_2 = 1, B_1 = 1, C = 0.5, C_1 = 1, D_1 = 1, \delta = 1.1, \gamma = 1, \lambda_1 = 1, \eta_1 = 1, \omega_1 = 1$, and $E_1 = 1$. The 2D profile of the intensity of the bright solitary wave solution (118) is depicted at $x = 0$ (red curve), $x = 1$ (blue curve) and $x = 2$ (green curve).

Figures

Figure 1

3D profile of the intensity of the bright solitary wave solution (43) for the choice of parameter values $e = 1$; $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 0.5$; $C1 = 1$; $D1 = 1$; $\delta = 1$; $y = 1$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$. The

2D structure of the intensity of the bright solitary wave solution (43) is depicted at different times $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 2

3D profile of the intensity of the bright solitary wave solution (47) for the choice of parameter values $e = 1$; $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 0.5$; $C1 = 1$; $D1 = 1$; $\delta = 1:1$; $y = 2$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$. The

2D structure of the intensity of the bright solitary wave solution (47) is depicted for $y = 1$ (red curve), $y = 2$ (blue curve), $y = 3$ (green curve) at $t = 0$.

Figure 3

3D profile of the intensity of the singular periodic wave solution (53) for the choice of parameter values $e = 1$; $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 1:5$; $C1 = 1$; $D1 = 1$; $\delta = 1:1$; $y = 1$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$.

The 2D structure of the intensity of the singular periodic wave solution (53) is depicted at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 4

3D profile of the dark solitary wave solution (75) for the choice of parameter values $A_0 = 1$; $A_1 = 1$; $K_1 = 1$; $K_2 = 1$; $B_1 = 1$; $C = 0.5$; $C_1 = 1$; $D_1 = 1$; $\delta = 1$; $y = 1$, and $E_1 = -1$. The 2D profile of the dark solitary wave solution (75) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 5

3D profile of the singular solitary wave solution (76) for the choice of parameter values $A_0 = 1$; $A_1 = 1$; $K_1 = 1$; $K_2 = 1$; $B_1 = 1$; $C = 0.5$; $C_1 = 1$; $D_1 = 1$; $\delta = 1:1$; $y = 1$, and $E_1 = -1$. The 2D profile of the dark solitary wave solution (76) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 6

3D profile of the doubly periodic wave solution (78) for the choice of parameter values $A_0 = 1$; $A_1 = 1$; $K_1 = 1$; $K_2 = 1$; $B_1 = 1$; $C = 0.5$; $C_1 = 1$; $D_1 = 1$; $\delta = 1:1$; $y = 1$; $E_1 = -1$, and $m = 0:5$. The 2D profile of the doubly periodic wave solution (78) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 7

3D profile of the singular periodic wave solution (80) for the choice of parameter values $A_0 = 1$; $A_1 = 1$; $K_1 = 1$; $K_2 = 1$; $B_1 = 1$; $C = 0.5$; $C_1 = 1$; $D_1 = 1$; $\delta = 1$; $\gamma = 1$, and $E_1 = -1$. The 2D profile of the singular periodic wave solution (80) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 8

3D profile of the bright solitary wave solution (84) for the choice of parameter values $A_0 = 1$; $A_1 = 1$; $K_1 = 1$; $K_2 = 1$; $B_1 = 1$; $C = 0.5$; $C_1 = 1$; $D_1 = 1$; $\delta = 1:1$; $y = 1$, and $E_1 = 1$. The 2D profile of the bright solitary wave solution (84) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 9

3D profile of the singular periodic wave solution (91) for the choice of parameter values $A_0 = 1$; $A_1 = 1$; $K_1 = 1$; $K_2 = 1$; $B_1 = 1$; $C = 0.5$; $C_1 = 1$; $D_1 = 1$; $\delta = 1:1$; $y = 1$, and $E_1 = -1$. The 2D profile of the singular periodic wave solution (91) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 10

3D profile of the hyperbolic wave solution (99) for the choice of parameter values $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 0.5$; $C1 = 1$; $D1 = 1$; $\delta = 1$; $y = 1$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$. The 2D profile of the hyperbolic wave solution (99) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 11

3D profile of the intensity of the doubly periodic wave solution (103) for the choice of parameter values $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 0.5$; $C1 = 1$; $D1 = 1$; $\delta = 1:1$; $y = 1$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$. The 2D profile of the doubly periodic wave solution (103) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 12

3D profile of the amplitude of the solitary wave solution (115) for the choice of parameter values $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 0.5$; $C1 = 1$; $D1 = 1$; $\delta = 1.1$; $y = 1$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$. The 2D profile of the amplitude of the kink solitary wave solution (115) is shown at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 13

3D profile of the intensity of the solitary wave solution (115) for the choice of parameter values $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 0.5$; $C1 = 1$; $D1 = 1$; $\delta = 1:1$; $y = 1$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$. The 2D profile of the intensity of the dark solitary wave solution (115) is depicted at $t = 0$ (red curve), $t = 1$ (blue curve) and $t = 2$ (green curve).

Figure 14

3D profile of the intensity of the bright solitary wave solution (118) for the choice of parameter values $A1 = 1$; $K1 = 1$; $K2 = 1$; $B1 = 1$; $C = 0.5$; $C1 = 1$; $D1 = 1$; $\delta = 1$; $y = 1$; $\lambda1 = 1$; $\eta1 = 1$; $\omega1 = 1$, and $E1 = 1$. The 2D profile of the intensity of the bright solitary wave solution (118) is depicted at $x = 0$ (red curve), $x = 1$ (blue curve) and $x = 2$ (green curve).