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A Non-Field Analytical Method for Solving Some Nonlinear Problems in Heat Transfer

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ABSTRACT

This paper presents an extension of the non-field analytical method – known as the method of Kulish – to some nonlinear problems in heat transfer. In view of the fact that solving nonlinear problems is very complicated in general, the extension of the method is presented in the form of several important illustrative examples. Two classes of problems are considered: first are the problems, in which the heat equation contains nonlinear terms, while the second type of the problems includes some problems with nonlinear boundary conditions. From the practical viewpoint, the case considered in Section 4 is of the greatest interest. In that section, it is shown that, for complex heat transfer problems, where applications of the non-field method are practically impossible due to a large volume of necessary computations, it is still possible to analyse the solution behaviour and automatically determine similarity criteria for the limiting values of the parameters. Wherever possible the obtained solutions are compared with known solutions obtained by other methods. The practical advantages of the non-field method over other analytical methods are emphasised in each case.

1. Introduction

Nonlinear heat transfer problems are of much interest in many applications, among which are heat transfer with nonlinear heating or cooling of the boundary (e.g., by radiation), heat transfer due to chemical reactions, ignition and explosions, combustion, as well as many others.

Nomenclature

a_n	coefficients in fractional series	<i>Greek</i>
C_1, C_2	arbitrary constants	α, β, γ arbitrary functions
F	intensive property	Θ temperature jump
f, g	arbitrary functions of independent variables	κ expansion parameter
Q	volumetric heat source	λ, τ auxiliary variables
q	temperature gradient	ν order of fractional differ-integration
T	temperature	<i>Special symbols</i>
t	time	∂^ν fractional differential operator of order ν
x	spatial variable	ℓ fractional operator
z	arbitrary independent variable	∇ Laplacian

This paper presents an extension of the analytical method for solving energy transport problems, which has been proposed by Kulish and Lage¹. The method renders the so-called non-field solutions of the energy transport equations (e.g., heat and diffusion equations). These solutions are known as non-field, because they relate local values of the unknown variable (e.g., temperature, mass concentration, velocity) and its spatial derivative. Recently, the generalised procedure and theoretical justification of the method have been reported².

It has been shown that the transport equation in the form

$$\left[\frac{\partial}{\partial t} - \alpha(x, t) \frac{\partial^2}{\partial x^2} - \beta(x, t) \frac{\partial}{\partial x} + \gamma(x, t) \right] F = 0 \quad (1)$$

can be reduced to an equivalent equation

$$\left[\frac{\partial}{\partial t} - \sqrt{\alpha} \ell \left(\frac{1}{\sqrt{\alpha}} \ell \right) + \sqrt{\alpha} \left(\frac{\partial \ell}{\partial x} \right) - \frac{1}{2\sqrt{\alpha}} \frac{\partial \alpha}{\partial x} \ell + \frac{\beta}{\sqrt{\alpha}} \ell + \gamma \right] F = 0, \quad (2)$$

where the operator

$$\ell = \sum_{n=0}^{\infty} a_n(x, t) \partial^{1-n/2} \quad (3)$$

is in the form of a series with respect to fractional derivatives $\partial^{1-n/2}$. In the latter equation, fractional derivatives are defined through the Riemann-Liouville integrals as

$$\partial^\nu f(t) = \frac{d^\nu f(t)}{dt^\nu} = \frac{1}{\Gamma(-\nu)} \int_0^t \frac{f(\zeta) d\zeta}{(t-\zeta)^\nu}, \quad \text{Re}(\nu) < 0 \quad (4)$$

It has been shown then that the expanded form of Eq. (2) is

$$\left[\frac{\partial}{\partial t} - \sqrt{\alpha} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{\frac{1-n}{2}}{p} \right) a_n \frac{\partial^p}{\partial t^p} \left(\frac{a_m}{\sqrt{\alpha}} \partial^{1-(m+n)/2-p} \right) + \right. \\ \left. + \sqrt{\alpha} \sum_{n=0}^{\infty} \frac{\partial a_n}{\partial x} \partial^{(1-n)/2} - \frac{1}{2\alpha} \frac{\partial \alpha}{\partial x} \sum_{n=0}^{\infty} a_n \partial^{(1-n)/2} + \frac{\beta}{\sqrt{\alpha}} \sum_{n=0}^{\infty} a_n \partial^{(1-n)/2} + \gamma \right] F = 0 \quad (5)$$

with the recurrent expressions for a_n :

$$\begin{aligned} \partial : 1 - a_0^2 &= 0 \\ \partial^{1/2} : \sqrt{\alpha} \frac{\partial a_0}{\partial x} + \frac{\beta}{\sqrt{\alpha}} a_0 - \frac{1}{2\sqrt{\alpha}} \frac{\partial \alpha}{\partial x} - a_1 a_0 - a_0 a_1 &= 0 \\ \partial^0 : \sqrt{\alpha} \frac{\partial a_1}{\partial x} + \frac{\beta}{\sqrt{\alpha}} a_1 - \frac{1}{2\sqrt{\alpha}} \frac{\partial \alpha}{\partial x} a_1 - \gamma - a_2 a_0 - a_1^2 - a_0 a_2 &= 0 \\ \partial^{(1-k)/2} : \sqrt{\alpha} \frac{\partial a_k}{\partial x} + \frac{\beta}{\sqrt{\alpha}} a_k - \frac{1}{2\sqrt{\alpha}} \frac{\partial \alpha}{\partial x} a_k - \\ - \sqrt{\alpha} \sum_{n=0}^{n+2p} \sum_{p=0}^{\leq k+1} \left(\frac{\frac{1-n}{2}}{p} \right) a_n \frac{\partial^p}{\partial t^p} \left(\frac{a_{k-1-n-2p}}{\sqrt{\alpha}} \right) &= 0, \quad k \geq 2 \end{aligned} \quad (6)$$

In the following sections, it is shown how the above results can be extended to some nonlinear problems in heat transfer. Two classes of problems are considered: first are the problems, in which the original transport equation, Eq. (1), contains nonlinear terms, while the second type of the problems includes some problems with nonlinear boundary conditions.

From the practical viewpoint, the case considered in Section 4 is of the greatest interest. In that section, it is shown that, for complex heat transfer problems, where applications of the non-field method are practically impossible due to a large volume of necessary computations, it is still possible to analyse the solution behaviour and automatically determine similarity criteria for the limiting values of the parameters (both large and small).

In view of the fact that solving nonlinear problems is very complicated in general, the extension of the method to nonlinear problems is presented here in the form of several important illustrative examples.

2. Nonlinear heat equation

Consider the following nonlinear problem

$$\begin{aligned} \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} + \kappa \left(\frac{\partial T}{\partial x} \right)^2 &= 0, & 0 < x < \infty, & \quad 0 < t < \infty \\ T|_{x=0} &= T_s(t); & T|_{x=\infty} &= 0; & T|_{t=0} &= 0 \end{aligned} \quad (7)$$

It is necessary to determine the temperature gradient at the boundary $q_s = (\partial T / \partial x)_{x=0}$.

The latter example is useful, because the exact solution is known³ in the case of $T_s = 1$:

$$\begin{aligned} T &= -\frac{1}{\kappa} \ln \left[1 - (1 - e^{-\kappa}) \operatorname{erfc} \left(\frac{x}{2\sqrt{t}} \right) \right] \\ -q_s &= \frac{e^\kappa - 1}{\kappa} \frac{1}{\sqrt{\pi t}} = \left(1 + \frac{\kappa}{2} + \frac{\kappa^2}{6} + \frac{\kappa^3}{24} + \dots \right) \frac{1}{\sqrt{\pi t}} \\ &\approx (1 + 0.5000\kappa + 0.1667\kappa^2 + 0.0416\kappa^3 + \dots) \frac{1}{\sqrt{\pi t}} \end{aligned} \quad (8)$$

The general method of solving the problem given by Eq. (7) is as follows. Equation (7) becomes identical to Eq. (1) with $\alpha = 1$, $\beta = -\kappa(\partial T / \partial x)$, and $\gamma = 0$. Then the

coefficients a_n can be found from Eq. (6). Obviously, in the given case, these coefficients depend on the unknown function T , namely:

$$\begin{aligned} a_1 &= \frac{\beta}{2} = -\frac{1}{2}\kappa T' \\ a_2 &= \frac{1}{2} \left(\frac{\partial a_1}{\partial x} - a_1^2 \right) = \frac{1}{2} \left(-\frac{1}{2}\kappa T'' + \frac{1}{4}\kappa^2 (T')^2 \right), \end{aligned} \quad (9)$$

where the primes denote derivatives with respect to the spatial variable.

The expression for a_2 includes T'' . The latter can be found from the original equation as

$$T'' = \dot{T} + \kappa (T')^2, \quad (10)$$

where the dot denotes the derivative with respect to time.

Coefficients a_n for all $n > 2$ can be found from Eq. (6) in the same way as a_1 and a_2 , while T'' must be changed as given by Eq. (10). The final expression contains only the first-order derivatives with respect to the spatial variable, but the time derivatives of all orders. That is,

$$\begin{aligned} a_3 &= \frac{1}{8}\kappa^2 T' \dot{T} - \frac{1}{8}\kappa^3 (T')^3 \\ a_4 &= \frac{1}{16}\kappa \ddot{T} - \frac{3}{32}\kappa^2 \dot{T}^2 - \frac{9}{32}\kappa^3 \dot{T} (T')^2 - \frac{25}{128}\kappa^4 (T')^4 \\ a_5 &= -\frac{1}{32}\kappa^2 \dot{T} \dot{T}' + \frac{1}{32}\kappa^2 T' \ddot{T} - \frac{1}{32}\kappa^3 \dot{T}' - \frac{5}{16}\kappa^3 T' \dot{T}^2 - \frac{23}{32}\kappa^4 (T')^3 \dot{T} - \frac{13}{32}\kappa^5 (T')^5 \\ a_6 &= -\frac{1}{32}\kappa \ddot{\ddot{T}} - \frac{1}{32}\kappa^2 (\dot{T}')^2 + \frac{3}{64}\kappa \dot{T} \ddot{T} + O(\kappa^3, \kappa^4, \kappa^5, \kappa^6) \\ &\dots \end{aligned} \quad (11)$$

Notice that, to arrive at the latter expressions, it was assumed that the mixed derivatives $\partial^{m+n}T/(\partial^m x \partial^n t)$ are independent of the order of differentiation with respect to different variables.

Using Eq. (11), it becomes possible to establish an expression for $q_s(T_s)$ in the form

$$\begin{aligned}
-q_s = & \partial^{1/2}T_s - \frac{1}{2}\kappa q_s T_s - \left(\frac{1}{4}\kappa \dot{T}_s + \frac{1}{8}\kappa^2 q_s^2\right) \partial^{-1/2}T_s + \left(\frac{1}{8}\kappa^2 q_s \dot{T}_s - \frac{1}{8}\kappa^3 q_s^3\right) \partial^{-1}T_s + \\
& + \left(\frac{1}{16}\kappa \ddot{T}_s - \frac{3}{32}\kappa^2 \dot{T}_s^2 - \frac{9}{32}\kappa^3 \dot{T}_s q_s^2 - \frac{25}{128}\kappa^4 q_s^4\right) \partial^{-3/2}T_s + \\
& + \left(-\frac{1}{32}\kappa^2 \dot{T}_s \dot{q}_s + \frac{1}{32}\kappa^2 q_s \ddot{T}_s - \frac{1}{32}\kappa^3 \dot{q}_s - \frac{5}{16}\kappa^3 q_s \dot{T}_s^2 - \frac{23}{32}\kappa^4 q_s^3 \dot{T}_s - \frac{13}{32}\kappa^5 q_s^5\right) \partial^{-2}T_s + \\
& + \left[-\frac{1}{32}\kappa \ddot{T}_s - \frac{1}{32}\kappa^2 \dot{q}_s^2 + \frac{3}{64}\kappa \dot{T}_s \ddot{T}_s + O(\kappa^3, \kappa^4, \kappa^5, \kappa^6)\right] \partial^{-5/2}T_s + \dots
\end{aligned} \tag{12}$$

In contrast with the corresponding linear problem, the unknown variable is present in both the sides of Eq. (12), which is an ordinary differential equation for q_s . This differential equation is of an infinite order and, at first sight, looks unsolvable. However, there exists a quite simple way to determine q_s upon assuming that the solution is in the form of a power series with respect to the parameter κ , so that,

$$q_s = \sum_{n=0}^{\infty} \kappa^n q_n \tag{13}$$

Upon substituting Eq. (13) into (12) and equating the expressions at the same powers of κ follow the recurrent expressions for q_n , namely:

$$\begin{aligned}
\kappa^0 : -q_0 &= \partial^{1/2}T_s \\
\kappa^1 : -q_1 &= -\frac{1}{2}q_0 T_s - \frac{1}{4}\dot{T}_s \partial^{-1/2}T_s + \frac{1}{16}\ddot{T}_s \partial^{-3/2}T_s + \dots \\
\kappa^2 : -q_2 &= -\frac{1}{2}q_1 T_s - \frac{1}{8}q_0^2 \partial^{-1/2}T_s + \frac{1}{8}q_0 \dot{T}_s \partial^{-1}T_s - \frac{3}{32}\dot{T}_s^2 \partial^{-3/2}T_s + \\
& + \left(-\frac{1}{32}\dot{T}_s \dot{q}_0 + \frac{1}{32}q_0 \ddot{T}_s\right) \partial^{-2}T_s + \left(-\frac{1}{32}q_0^2 + \frac{3}{64}\dot{T}_s \ddot{T}_s\right) \partial^{-5/2}T_s + \dots \\
\kappa^3 : -q_3 &= -\frac{1}{2}q_2 T_s - \frac{1}{4}q_0 q_1 \partial^{-1/2}T_s + \left(\frac{1}{8}q_1 \dot{T}_s - \frac{1}{8}q_0^3\right) \partial^{-1}T_s - \frac{9}{32}\dot{T}_s q_0^2 \partial^{-3/2}T_s + \\
& + \left(-\frac{1}{32}\dot{T}_s \dot{q}_1 + \frac{1}{32}q_1 \ddot{T}_s - \frac{1}{32}\dot{q}_0 q_0^2 - \frac{5}{16}q_0 \dot{T}_s^2\right) \partial^{-2}T_s + \dots
\end{aligned} \tag{14}$$

Hence, the solution to problem (7) is given by Eqs. (13) and (14).

The solution can be validated by comparing it with the known exact solution for the case when there is a temperature jump at the boundary, that is, $T_s = 1$. Then Eq. (13) becomes

$$-q_s = (1 + 0.50000\kappa + 0.1691\kappa^2 + 0.0421\kappa^3 + \dots) \frac{1}{\sqrt{\pi t}}. \quad (15)$$

Comparison between Eq. (15) and the exact solution given by Eq. (8) shows that the form of the non-field solution is correct. There is some difference only between the values of the coefficients at various powers of κ . Thus, the coefficients at κ^0 and κ^1 have been defined exactly, while the coefficients at κ^2 and κ^3 have been defined with the relative errors of 1.4% and 1.2%, respectively.

In principle, one can look for the solution of the original problem (7) in the form of a power series with respect to κ by defining the entire temperature field for each approximation. However, if doing so in practice, it becomes possible to only the first two terms in Eq. (13) – and even then a complicated integral formula must be used. Calculating higher order terms even for a given $T_s(t)$ requires a large volume of calculations, which, in practice, cannot be carried on. On the contrary, the non-field method, employed here, allows one to easily calculate several terms of the power series, due to the fact that there is no need to define the entire temperature field for each approximation. The latter circumstance is, beyond any doubt, a major advantage of the non-field method.

To conclude this section, it is worth noting that, with some modifications, the same method becomes applicable for equations with a strong nonlinearity. An extension of the method to this type of problems will be reported in one of the following papers.

3. Transient heat transfer with a nonlinear heat sink

The solution obtained in the preceding section has a simple structure and does not provide an idea of some characteristic features of the method. In view of this, it is useful to consider a more complicated example, which possesses all features of an arbitrary nonlinear problem.

Consider a process of heat transfer with a nonlinear heat sink given by $aT + bT^2$. Upon choosing proper scales for the variables, the corresponding problem can be written in the form

$$\begin{aligned} \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} + \kappa \left(T + \frac{T^2}{2} \right) &= 0, & 0 < x < \infty, & \quad 0 < t < \infty \\ T|_{x=0} &= T_s(t); & T|_{x=\infty} &= 0; & T|_{t=0} &= 0 \end{aligned} \quad (16)$$

The heat sink function has been chosen so that it can represent an exponential dependence with small temperature deviations from the initial value, because $T + T^2/2 \approx \exp(T) - 1$. Such a behaviour of the heat sink may be, for example, due to a chemical reaction or radiation with a small temperature difference.

As in the previous problem, it is necessary to determine the temperature gradient at the boundary $q_s = (\partial T / \partial x)_{x=0}$.

Following the same procedure as in Section 2, assume $\alpha = 1$, $\beta = 0$, and $\gamma = \kappa(1 + T/2)$ in Eq. (1). Then the relation between the temperature and its gradient can be written in the form analogous with Eq. (12), that is,

$$\begin{aligned}
-q_s = & \partial^{1/2} T_s + \frac{\kappa}{2} \left(1 + \frac{1}{2} T_s \right) \partial^{-1/2} T_s + \frac{\kappa}{8} q_s \partial^{-1} T_s - \left(\frac{\kappa^2}{8} + \frac{\kappa^2}{16} T_s \right) \partial^{-3/2} T_s - \\
& - \left(\frac{\kappa}{32} \dot{q}_s + \frac{3}{32} \kappa^2 q_s + \frac{\kappa^2}{32} T_s q_s \right) \partial^{-2} T_s - \\
& - \left(\frac{\kappa^2}{64} \ddot{T}_s + \frac{\kappa^2}{64} \dot{T}_s T_s + \frac{3\kappa^2}{128} q_s^2 - \frac{\kappa^3}{16} - \frac{\kappa^3}{64} T_s + \frac{3\kappa^3}{128} T_s^2 + \frac{\kappa^3}{128} T_s^3 \right) \partial^{-5/2} T_s + \\
& + \left(\frac{\kappa}{64} \ddot{q}_s + \frac{3\kappa^2}{64} \dot{q}_s + \frac{\kappa^2}{64} T_s \dot{q}_s - \frac{\kappa^2}{128} \dot{T}_s q_s + \frac{9\kappa^3}{128} q_s - \frac{\kappa^3}{64} T_s^2 q_s \right) \partial^{-3} T_s + \\
& + (\dots) \partial^{-7/2} T_s + \dots
\end{aligned} \tag{17}$$

Upon substituting Eq. (13) into (17) and equating the expressions at the same powers of κ follow the recurrent expressions for q_n , namely:

$$\begin{aligned}
-q_0 = & \partial^{1/2} T_s \\
-q_1 = & \left(\frac{1}{2} + \frac{1}{4} T_s \right) \partial^{-1/2} T_s + \frac{1}{8} q_0 \partial^{-1} T_s - \frac{1}{32} \dot{q}_0 \partial^{-2} T_s + \frac{1}{64} \ddot{q}_0 \partial^{-3} T_s - \dots \\
-q_2 = & \frac{1}{8} q_1 \partial^{-1} T_s - \left(\frac{1}{8} + \frac{1}{16} T_s \right) \partial^{-3/2} T_s - \left(\frac{1}{32} \dot{q}_1 + \frac{3}{32} q_0 + \frac{1}{32} T_s q_0 \right) \partial^{-2} T_s - \\
& - \left(\frac{1}{64} \ddot{T}_s + \frac{1}{64} \dot{T}_s T_s + \frac{3}{128} q_0^2 \right) \partial^{-5/2} T_s + \left(\frac{1}{64} \ddot{q}_1 + \frac{3}{64} \dot{q}_0 - \frac{1}{64} T_s \dot{q}_0 - \frac{1}{128} \dot{T}_s q_0 \right) \partial^{-3} T_s + \\
& + \left(\frac{1}{128} \ddot{T}_s + \frac{1}{128} \dot{T}_s^2 + \frac{1}{128} T_s \ddot{T}_s + \frac{7}{256} q_0 \dot{q}_0 \right) \partial^{-7/2} T_s + \dots \\
-q_3 = & \frac{1}{8} q_2 \partial^{-1} T_s - \left(\frac{1}{32} \dot{q}_2 + \frac{3}{32} q_1 + \frac{1}{32} T_s q_1 \right) \partial^{-2} T_s + \\
& + \left(\frac{1}{16} - \frac{3}{64} q_0 q_1 + \frac{1}{64} T_s - \frac{3}{128} T_s^2 - \frac{1}{128} T_s^3 \right) \partial^{-5/2} T_s + \\
& + \left(\frac{1}{64} \ddot{q}_2 + \frac{3}{64} \dot{q}_1 + \frac{1}{64} T_s \dot{q}_1 - \frac{1}{128} \dot{T}_s q_1 + \frac{9}{128} q_0 - \frac{1}{64} T_s^2 q_0 \right) \partial^{-3} T_s + \\
& + \left(\frac{7}{256} q_0 \dot{q}_1 + \frac{7}{256} q_1 \dot{q}_0 + \frac{5}{128} \dot{T}_s + \frac{9}{256} T_s \dot{T}_s + \frac{1}{128} T_s^2 \dot{T}_s + \frac{3}{128} q_0^2 - \frac{3}{512} T_s q_0^2 \right) \partial^{-7/2} T_s + \dots
\end{aligned} \tag{18}$$

Equations (13) and (18) render the solution of problem given by Eq. (16).

The rest of this section is devoted to finding the temperature gradient at the boundary for the case when the boundary temperature experiences a jump, namely: $T_s = \Theta = \text{const.}$

From Eq. (18), it follows that

$$\begin{aligned}
 -q_0 &= \frac{\Theta}{\sqrt{\pi t}} \\
 -q_1 &= \left[\Theta + \Theta^2 \left(\frac{1}{2} - \frac{1}{8} - \frac{1}{128} - \frac{1}{512} - \dots \right) \right] \sqrt{\frac{t}{\pi}} \approx \\
 &\approx [\Theta + 0.365\Theta^2] \sqrt{\frac{t}{\pi}}
 \end{aligned} \tag{19}$$

As mentioned in the previous section, in principle, the expression for q_1 can be obtained by the power series expansion method with respect to κ . The method yields

$$-q_1 = \left[\Theta + \Theta^2 \left(1 - \frac{2}{\pi} \right) \right] \sqrt{\frac{t}{\pi}} \approx [\Theta + 0.363\Theta^2] \sqrt{\frac{t}{\pi}} \tag{20}$$

Comparison of the latter expression with Eq. (19) shows that the coefficient at Θ has been defined exactly, while the coefficient at Θ^2 differs from the exact value by less than 0.6%.

From Eq. (18) follow expressions for higher order terms in Eq. (13):

$$\begin{aligned}
 -q_2 &= [-0.167\Theta + (-0.125 - 0.083 + 0.055 - 0.004 + \dots)\Theta^2 + \\
 &\quad + (-0.046 + 0.018 - 0.004 + 0.001 - 0.001 + \dots)\Theta^3] \sqrt{\frac{t^3}{\pi}} \\
 -q_3 &= [0.036\Theta + (-0.021 + 0.070 + 0.008 - 0.016 + \dots)\Theta^2 + \\
 &\quad + (-0.019 + 0.036 - 0.015 + 0.001 - \dots)\Theta^3 + \\
 &\quad + (-0.004 + 0.006 - 0.002 + 0.002 - \dots)\Theta^4] \sqrt{\frac{t^5}{\pi}}
 \end{aligned} \tag{21}$$

For example, for $\Theta = 1$:

$$-q_s = \frac{1}{\sqrt{\pi t}} [1 + 1.365\kappa t - 0.348(\kappa t)^2 + 0.079(\kappa t)^3 - \dots] \quad (22)$$

The latter solution can be used for all $\kappa t < 1$.

4. Criteria for modelling

The algorithm, presented in Sections 2 and 3, allows to find asymptotic solutions without the need to conduct laborious computations. For instance, from Eq. (22) follows that, for a given boundary temperature jump Θ ,

$$-q_s \sim \frac{1}{\sqrt{t}} f(\kappa t), \quad (23)$$

where f is an unknown function.

Assume not that is is necessary to find the form of an asymptotic solution for $\Theta \rightarrow \infty$. Because Θ is a dimensionless quantity, similarity theory, with no further assumptions, yields

$$-q_s \sim \frac{1}{\sqrt{t}} f(\kappa t, \Theta). \quad (24)$$

The method presented here, on the other hand, renders – Eqs. (13) and (21) – the form

$$-q_s \sim \frac{\Theta}{\sqrt{t}} f(\Theta \kappa t), \quad (25)$$

which is much more informative than Eq. (24).

It is worth noting here that the asymptotic form for $\Theta \rightarrow 0$ also follows from Eqs. (13) and (21), although, in this case, is readily obtainable from solving the corresponding linear problem.

Finally, consider the general formula for an arbitrary value of Θ :

$$-q_s \sim \frac{\Theta}{\sqrt{t}} [1 + (a_1 + b_1 \Theta)(\kappa t) + (a_2 + b_2 \Theta + c_2 \Theta^2)(\kappa t)^2 + \dots] \quad (26)$$

Information, which can be recovered from Eq. (26), is not trivial: the function $q_s \sqrt{t}/\Theta$ depends on integers powers of κt only and the highest power of Θ in each term does not exceed the corresponding power of κt .

Assume now that, for the original problem given by Eq. (16), with $\kappa < 0$, it is necessary to determine the ignition time, t_{ign} , which implies $q_s(t_{\text{ign}}) = 0$. It follows from Eq. (26) that

$$\begin{aligned} t_{\text{ign}} &\approx \frac{C_1}{\kappa}, & \Theta \rightarrow 0 \\ t_{\text{ign}} &\sim \frac{C_2}{\kappa\Theta}, & \Theta \rightarrow \infty \end{aligned} \quad (27)$$

Hence, for an arbitrary value of Θ , it becomes possible to offer an interpolation formula

$$t_{\text{ign}} = \frac{C_1}{\kappa} \left(1 + \frac{\Theta}{C_2} \right)^{-1}. \quad (28)$$

The constant coefficient C_1 can be found from the corresponding linear problem, for which the term $T^2/2$ is absent. The constant coefficient C_2 is to be found either by the presented method, or numerically, or from experimental data.

As the concluding example in this section consider the case when the thermal conductivity varies with temperature linearly, namely:

$$\begin{aligned} \frac{\partial T}{\partial t} - (a + bT) \frac{\partial^2 T}{\partial x^2} - b \left(\frac{\partial T}{\partial x} \right)^2 &= 0, & 0 < x < \infty, & \quad 0 < t < \infty \\ T|_{x=0} &= T_s(t); & T|_{x=\infty} &= 0; & T|_{t=0} &= 0 \end{aligned} \quad (29)$$

The latter problem is equivalent to Eq. (1), in which $\alpha = a + bT$, $\beta = b(\partial T/\partial x)$, and $\gamma = 0$. The relationship between the boundary values of T_s and q_s is then

$$\begin{aligned} -q_s \sqrt{a + bT_s} &= \left\{ \partial^{1/2} + \frac{1}{4} \frac{b}{\sqrt{a + bT_s}} q_s + \left(\frac{1}{4} \frac{b}{a + bT_s} T_s - \frac{5}{32} \frac{b^2}{a + bT_s} q_s^2 \right) \partial^{-1/2} + \right. \\ &\quad \left. + \left[\frac{1}{16} \frac{b}{\sqrt{a + bT_s}} \dot{q}_s - \frac{7}{32} \frac{b^2}{(a + bT_s)^{3/2}} q_s \dot{T}_s + \frac{15}{64} \frac{b^3}{(a + bT_s)^{3/2}} q_s^3 \right] \partial^{-1} + \dots \right\} T_s \end{aligned} \quad (30)$$

For the boundary condition of a temperature jump, $T_s = \Theta = \text{const}$, the latter equation becomes

$$\begin{aligned}
-q_s \sqrt{a + b\Theta} = & \frac{\Theta}{\sqrt{\pi t}} + \frac{\Theta}{4} \frac{b}{\sqrt{a + b\Theta}} q_s - \frac{5\Theta}{16\sqrt{\pi}} \frac{b^2}{a + b\Theta} q_s^2 \sqrt{t} + \\
& + \left[\frac{\Theta}{16} \frac{b}{\sqrt{a + b\Theta}} \dot{q}_s + \frac{15\Theta}{64} \frac{b^3}{(a + b\Theta)^{3/2}} q_s^3 \right] t + \dots
\end{aligned} \tag{31}$$

If only the first term in the right side of the latter equation is taken into account, an approximate expression for q_s becomes

$$-q_s \approx \frac{\Theta}{\sqrt{\pi t}} \frac{1}{\sqrt{a + b\Theta}}. \tag{32}$$

A second approximation follows from substituting Eq. (32) into the second term in the right side of Eq. (31) and truncating all the subsequent terms

$$-q_s \approx \frac{\Theta}{\sqrt{\pi t}} \frac{1}{\sqrt{a + b\Theta}} \left(1 - \frac{1}{4} \frac{b\Theta}{a + b\Theta} \right). \tag{33}$$

Continuing this iteration process without calculating exact values of the coefficients, one arrives at the expression for q_s in the form of power series with respect to $z = b\Theta/(a + b\Theta)$, that is,

$$-q_s = \frac{\Theta}{\sqrt{\pi t}} \frac{1}{\sqrt{a + b\Theta}} (1 - \kappa_1 z - \kappa_2 z^2 - \dots). \tag{34}$$

Approximate values of the coefficients κ_n can be obtained by substituting the latter expression into Eq. (31). This is beyond the scope of this example, the main aim of which is to show that q_s depends on Θ and t in the form

$$-q_s = \frac{\Theta}{\sqrt{\pi t}} \frac{1}{\sqrt{a + b\Theta}} f\left(\frac{b\Theta}{a + b\Theta}\right), \tag{35}$$

where f is an unknown function of z .

If similarity theory is used to tackle the same problem, the following result can be found

$$-q_s = \frac{\Theta}{\sqrt{\pi a t}} g\left(\frac{b\Theta}{a}\right), \quad (36)$$

because $q_s\sqrt{at}$ and $b\Theta/a$ are the two dimensionless combinations of the parameters involved. In the latter expression, g is an unknown function of $b\Theta/a$.

However, Eq. (35) is more informative than Eq. (36) due to the following reasons. From considering the limiting cases, it follows that $f(0) = 1$, $\lim_{z \rightarrow 1} f(z) = f(1) = \text{const}$ and $0 \leq f(1) < 1$. The last inequality can be established from comparing with the solution to the auxiliary problem, in which the thermal conductivity remains constant and equals $b\Theta$.

In the same way, for the function g , it follows $g(0) = 1$ and $g(\infty) = 0$.

Hence, both the functions change within finite limits: f changes within the finite interval $0 \leq z < 1$, while g changes within the semi-infinite interval $0 \leq b\Theta/a < \infty$. Due to this difference, the procedure of finding f from additional calculations or experimental data is much simpler than the procedure of finding g .

5. Heat transfer with nonlinear effects on the surface

In this section, two examples of heat transfer with nonlinear effects on the surface are considered, namely: surface radiative cooling and heating, respectively.

In the first case, the boundary value problem is in the form

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} \right) T &= 0, \quad 0 < x < \infty, \quad 0 < \tau < \infty \\ q_s = \frac{\partial T}{\partial x} \Big|_{x=0} &= \lambda T_s^4, \quad T \Big|_{x=\infty} = T \Big|_{\tau=0} = T_0 = \text{const} \end{aligned} \quad (37)$$

where $\tau = \kappa t / (\rho c_p)$ and $\lambda = \sigma / \kappa$.

For the given problem, it is necessary to determine the temperature evolution, $T_s(\tau)$, on the surface $x = 0$.

Notice that a similar problem with $q_s = \lambda T_s^2$ was considered earlier as an illustrative example². In the given case, the equation for the surface temperature is in the form (compare this with Eq. (26) in [2])

$$T_s(\tau) = T_0 - \lambda \partial^{-1/2} T_s^4(\tau) \quad (38)$$

As in the previous study², the solution is sought in the form

$$T = \sum_{n=0}^{\infty} a_n \tau^{n/2}, \quad a_n = \text{const.} \quad (39)$$

Then, by complete analogy with the case $q_s = \lambda T_s^2$, the solution becomes

$$\frac{T_s}{T_0} = 1 - \frac{2}{\sqrt{\pi}} (\lambda T_0^3) \tau^{1/2} + 4 (\lambda T_0^3)^2 \tau - \left(\frac{4}{3\sqrt{\pi}} + \frac{2}{\pi^{3/2}} \right) (\lambda T_0^3)^3 \tau^{3/2} + \dots \quad (40)$$

For practical calculations, the latter expression is valid for small values of τ .

The asymptotic solution for $\tau \rightarrow \infty$ can be easily obtained in the same way as for the case $q_s = \lambda T_s^2$, that is,

$$T_s \approx [T_0 / (\lambda \sqrt{\pi})]^{1/4} \tau^{-1/8}. \quad (41)$$

Now consider the case of radiative surface heating by an external volumetric heat source of the constant intensity Q . The heat transfer process at the surface is governed by the Stefan-Boltzmann law. It is necessary to determine the temperature evolution, $T_s(\tau)$, on the surface $x = 0$.

In the given case, the boundary value problem becomes

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} \right) T &= Q, \quad 0 < x < \infty, \quad 0 < \tau < \infty \quad Q = \text{const} \\ \frac{\partial T}{\partial x} \Big|_{x=0} &= \lambda T_s^4; \quad T \Big|_{x=\infty} = Q\tau, \quad T \Big|_{\tau=0} = 0 \end{aligned} \quad (42)$$

Upon eliminating $(\partial T / \partial x)_{x=0}$ from the boundary condition, the equation for T_s follows:

$$\lambda T_s^4 + \partial^{1/2} T_s = 2Q \sqrt{\frac{\tau}{\pi}}. \quad (43)$$

Upon substituting Eq. (43) into Eq. (39) and equating the coefficients at equal powers of τ , the solution becomes

$$T_s = Q\tau - \lambda Q^4 \frac{\Gamma(5)}{\Gamma(11/2)} \tau^{9/2} + \lambda^2 Q^7 \frac{4\Gamma(17/2)\Gamma(5)}{\Gamma(9)\Gamma(11/2)} \tau^8 - \dots \quad (44)$$

For practical calculations, the latter expression is valid for small values of τ .

From the physics of the problem, it is clear that, at $\tau \rightarrow \infty$, the value of T_s increases slower than τ . In view of this, the term $\partial^{1/2}T_s$ can be neglected in comparison with $2Q\sqrt{\tau/\pi}$. Then, the asymptotic solution for large values of τ becomes

$$T_s \approx [2Q/(\lambda\sqrt{\pi})]^{1/4} \tau^{1/8}. \quad (45)$$

6. General discussion

Some attempts to extend the method of Kulish to nonlinear problems have been made earlier⁴. In particular, the Burgers and Frank-Kamenetskii equations were considered. However, the reported solutions are not completely in a non-field form – a narrow interval within the field is needed to express the corresponding gradient.

The approach, presented in this work, on the other hand, allows not only to extend the method of Kulish to a wide class of nonlinear heat transfer problems, but also obtain non-field solutions, which relate local values of the temperature and its gradient. In most of the cases, these solutions cannot be obtained by other analytical methods, or the procedures leading to these solutions are too laborious (e.g., involve a large volume of calculations), which makes them impractical.

Last but not least, it has been shown that, for complex heat transfer problems, where applications of the non-field method are practically impossible due to a large volume of necessary computations, it is still possible to analyse the solution behaviour and automatically determine similarity criteria for the limiting values of the parameters.

In the following study, the method presented here will be extended, with some modifications, to problems with strong nonlinearities.

References

1. Kulish, V. V. & Lage, L. J. Fractional-diffusion solutions for transient local temperature and heat flux. *J. Heat Transfer*. **122**(2), 372–376. <https://doi.org/10.1115/1.521474> (2000).
2. Kulish, V. A non-field analytical method for solving energy transport equations. *J. Heat Transfer*. **142**(4), 042102. <https://doi.org/10.1115/1.4046301> (2020).
3. Hahn, D.W. & Özisik, M.N. *Heat Conduction* (Wiley, New York, 2012).
4. Novozhilov, V. & Kulish, V. Extension of relations between local values of intensive properties and corresponding fluxes to some nonlinear processes. *J. MESA*. **5**(1), 97–109 (2014).

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Author Contributions

Both the authors equally contributed into this study.

Additional Information

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