

# Landau-Fermi liquids in disguise

Michele Fabrizio (✉ [fabrizio@sissa.it](mailto:fabrizio@sissa.it))

International School for Advanced Studies (SISSA) <https://orcid.org/0000-0002-2943-3278>

---

## Article

**Keywords:** Landau-Fermi liquids, Fermi surfaces, Luttinger surfaces, quasiparticles

**Posted Date:** June 11th, 2021

**DOI:** <https://doi.org/10.21203/rs.3.rs-569113/v1>

**License:**  This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

---

**Version of Record:** A version of this preprint was published at Nature Communications on March 23rd, 2022. See the published version at <https://doi.org/10.1038/s41467-022-29190-y>.

# Landau-Fermi liquids in disguise

Michele Fabrizio<sup>1</sup>

<sup>1</sup>*International School for Advanced Studies (SISSA),*

*Via Bonomea 265, I-34136 Trieste, Italy*

In periodic systems of interacting electrons, Fermi and Luttinger surfaces refer to the locations within the Brillouin zone of poles and zeros, respectively, of the single-particle Green's function at zero energy and temperature. Such difference in analytic properties underlies the emergence of well-defined quasiparticles close to a Fermi surface, in contrast to their supposed non-existence close to a Luttinger surface, where the single-particle density-of-states vanishes at zero energy.

We here show that, contrary to such common belief, coherent 'quasiparticles' do exist also approaching a Luttinger surface in compressible interacting electron systems. Thermodynamic and dynamic properties of such 'quasiparticles' are just those of conventional ones. For instance, they yield well defined quantum oscillations in Luttinger's surface and linear in temperature specific heat, which is striking given the vanishing density of states of physical electrons, but actually not uncommon in strongly correlated materials.

Singularities in the single particle Green's function  $G(\zeta, \mathbf{k})$  of the complex frequency  $\zeta$  and momentum  $\mathbf{k}$  are known, since the seminal work by Volovik [1], to be associated at  $\zeta = 0$  and temperature  $T = 0$  with topological invariants shared [2, 3] by both Fermi and Luttinger [4] surfaces, i.e., manifolds where  $G(\zeta \rightarrow 0, \mathbf{k})$  has a pole or a zero, respectively. Such invariants are in turn related to Luttinger's theorem [5], which also acquires a topological content at  $T = 0$  [2, 6]. A further link between Fermi and Luttinger surfaces in compressible systems of interacting electrons has been recently uncovered in Ref. [7], according to which both give rise to Landau-Fermi liquid like linear response functions to long-wavelength low-frequency external fields, despite the absence of a quasiparticle spectral peak in Luttinger's case.

Here, we show that the relationship is actually much deeper, and entails the existence in both cases of 'quasiparticle' excitations whose lifetime grows to infinity approaching a Luttinger surface as well as a Fermi surface. Such ubiquitous 'quasiparticles' close to both Fermi and Luttinger

surfaces, which rationalise in simpler terms the results of Ref. [7], may actually provide a new, broader paradigm for strongly correlated electron systems.

Landau-Fermi liquids [8, 9] are commonly associated with the existence of simple poles in the single-particle Green's function close to the Fermi surface, the Landau quasiparticles, with finite residue  $0 < Z < 1$ . Specifically, let us consider a generic non-insulating interacting electron system in three dimensions, whose single-particle Green's function [10] can be written, through Dyson's equation, as

$$G(\zeta, \mathbf{k}) = \frac{1}{\zeta - \epsilon(\mathbf{k}) - \Sigma(\zeta, \mathbf{k})} = \int d\omega \frac{A(\omega, \mathbf{k})}{\zeta - \omega}, \quad (1)$$

where  $\epsilon(\mathbf{k})$  is the non-interacting dispersion measured with respect to the chemical potential,  $\Sigma(\zeta, \mathbf{k})$  the self-energy, and  $A(\omega, \mathbf{k}) > 0$  the single-particle density-of-states (DOS) satisfying  $\int d\omega A(\omega, \mathbf{k}) = 1$ . Taking the analytic continuation on the real axis from above, i.e.,  $G(\zeta = \epsilon + i\eta, \mathbf{k}) \equiv G_+(\epsilon, \mathbf{k})$  the retarded Green's function, and, similarly,  $\Sigma(\zeta = \epsilon + i\eta, \mathbf{k}) \equiv \Sigma_+(\epsilon, \mathbf{k})$ , with infinitesimal  $\eta > 0$ , the following result holds order by order in perturbation theory at zero temperature  $T$ :

$$\text{Im } \Sigma_+(\epsilon \rightarrow 0, \mathbf{k}) = -\gamma(\mathbf{k}) \epsilon^2, \quad (2)$$

with  $\gamma(\mathbf{k}) \geq 0$ . The quasiparticle energy,  $\epsilon_{\text{qp}}(\mathbf{k})$ , and the Fermi surface,  $\mathbf{k} = \mathbf{k}_F$ , are defined, respectively, by

$$\epsilon_{\text{qp}}(\mathbf{k}) - \epsilon(\mathbf{k}) - \Sigma_+(\epsilon_{\text{qp}}(\mathbf{k}), \mathbf{k}) = 0, \quad \epsilon_{\text{qp}}(\mathbf{k}_F) = 0. \quad (3)$$

Expanding  $A(\epsilon, \mathbf{k}) = -\text{Im } G_+(\epsilon, \mathbf{k})/\pi$  for  $\mathbf{k}$  close to the Fermi surface one readily finds, through Eq. (2), that

$$A(\epsilon, \mathbf{k}) \simeq Z(\mathbf{k}) \delta(\epsilon - \epsilon_{\text{qp}}(\mathbf{k})) + A_{\text{inc}}(\epsilon, \mathbf{k}), \quad (4)$$

where the first term is the low energy (coherent) part,  $\epsilon \simeq \epsilon_{\text{qp}}(\mathbf{k}) \simeq 0$ , while the high-energy incoherent component  $A_{\text{inc}}(\epsilon, \mathbf{k})$  carries the rest  $1 - Z(\mathbf{k})$  of the spectral weight. The quasiparticle residue in (4) is formally defined as  $Z(\mathbf{k}) = Z(\epsilon_{\text{qp}}(\mathbf{k}), \mathbf{k})$  for  $\mathbf{k} \simeq \mathbf{k}_F$ , where

$$Z(\epsilon, \mathbf{k})^{-1} \equiv 1 - \frac{\partial \text{Re} \Sigma_+(\epsilon, \mathbf{k})}{\partial \epsilon}. \quad (5)$$

Eq. (4) is actually the starting point of the microscopic derivation of Landau-Fermi liquid theory [11, 12].

However, the validity of Eq. (2) order by order in perturbation theory does not guarantee that the actual interaction strength, especially in strongly correlated electron systems, is within the convergence radius of the perturbation series, nor that the latter converges at all. Nonetheless, it has been recently shown [7] that Eq. (2) is not necessary but only sufficient to microscopically derive Landau-Fermi liquid theory.

Indeed, the condition (2) refers just to the decay rate of the physical electrons. On the contrary, for Landau's Fermi liquid theory to apply, we do have to impose a similar condition of vanishing decay rate, i.e., infinite lifetime, but for the actual quasiparticles, which reads, still at  $T = 0$ ,

$$\lim_{\epsilon \rightarrow 0} \Gamma(\epsilon, \mathbf{k}) \equiv - \lim_{\epsilon \rightarrow 0} Z(\epsilon, \mathbf{k}) \text{Im} \Sigma_+(\epsilon, \mathbf{k}) = \gamma_*(\mathbf{k}) \epsilon^2, \quad (6)$$

where  $Z(\epsilon, \mathbf{k})$  is defined through Eq. (5), and  $\gamma_*(\mathbf{k}) \geq 0$  provided  $Z(\epsilon \rightarrow 0, \mathbf{k}) \geq 0$ , which is generally true even though  $Z(\epsilon, \mathbf{k})$  may well be negative at  $\epsilon \neq 0$ . Such physically sound replacement is actually the *key* of this work, and has huge implications, as we are going to show, despite it may look at first sight innocuous. For instance, Eq. (6) does include Eq. (2) when  $Z$  is finite as special case, but, e.g., remains valid even when  $Z \sim \epsilon^2$  and  $\text{Im} \Sigma_+$  is constant, the furthest possible case from a conventional Fermi liquid. We remark that, while Eq. (2) is a perturbative result, Eq. (6) is a non-perturbative assumption. Moreover, we mention that in two dimensions the analytic  $\epsilon^2$  dependence in Eq. (6) must be substituted by a non analytic  $-\epsilon^2 \ln \epsilon$  one [13], as non analytic are the subleading corrections to Eq. (6) also in three dimensions [14, 15]. However, all the results we are going to derive are valid in three as well as in two dimensions.

According to Luttinger's theorem [4, 5], which we believe is valid beyond perturbation theory [16] provided Eq. (6) holds, see Supplementary Notes, the number of electrons  $N$  at fixed chemical potential can be obtained at low temperature through

$$N = \sum_{\mathbf{k}\alpha} T \sum_n e^{i\epsilon_n 0^+} \frac{\partial \ln G_\alpha^{-1}(i\epsilon_n, \mathbf{k})}{\partial i\epsilon_n} = \sum_{\mathbf{k}\alpha} \int \frac{d\epsilon}{\pi} f(\epsilon) \frac{\partial \delta_\alpha(\epsilon, \mathbf{k})}{\partial \epsilon} \equiv \sum_{\mathbf{k}\alpha} N_\alpha(\mathbf{k}), \quad (7)$$

where  $i\epsilon_n$  are fermionic Matsubara frequencies,  $f(\epsilon)$  is the Fermi distribution function,  $\alpha$  labels the basis in which the Green's function is diagonal, and  $N_\alpha(\mathbf{k})$  must not be confused with the occupation number  $\langle c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha} \rangle$  of the physical electrons. Omitting the label  $\alpha$ , thus  $N_\alpha(\mathbf{k}) \rightarrow N(\mathbf{k})$  and  $\delta_\alpha(\epsilon, \mathbf{k}) \rightarrow \delta(\epsilon, \mathbf{k})$ , the many-body phase-shift is defined through

$$\delta(\epsilon, \mathbf{k}) \equiv \pi + \text{Im} \ln G_+(\epsilon, \mathbf{k}) = \tan^{-1} \frac{-\text{Im} G_+(\epsilon, \mathbf{k})}{-\text{Re} G_+(\epsilon, \mathbf{k})} = \tan^{-1} \frac{\Gamma(\epsilon, \mathbf{k})}{\Xi(\epsilon, \mathbf{k})} \in [0, \pi], \quad (8)$$

satisfying  $\delta(\epsilon \rightarrow -\infty, \mathbf{k}) = 0$  and  $\delta(\epsilon \rightarrow +\infty, \mathbf{k}) = \pi$ , with  $\Gamma(\epsilon, \mathbf{k})$  of Eq. (6), and

$$\Xi(\epsilon, \mathbf{k}) \equiv Z(\epsilon, \mathbf{k}) \left( \epsilon(\mathbf{k}) + \text{Re} \Sigma_+(\epsilon, \mathbf{k}) - \epsilon \right) = - \left( \frac{\partial \ln \text{Re} G_+^{-1}(\epsilon, \mathbf{k})}{\partial \epsilon} \right)^{-1}. \quad (9)$$

Since  $\delta(\epsilon, \mathbf{k})/\pi \in [0, 1]$ , it readily follows from Eq. (7) that also  $N(\mathbf{k}) \in [0, 1]$ .

We mentioned that, as  $T \rightarrow 0$ , Luttinger's theorem acquires a topological content [1–3, 6, 17], which simply reflects the fact that the number of particles must be an integer at fixed chemical potential. In that limit, and assuming Eq. (6) valid,

$$\delta(\epsilon \rightarrow 0, \mathbf{k}) = \tan^{-1} \frac{\gamma_*(\mathbf{k}) \epsilon^2}{\Xi(\epsilon \rightarrow 0, \mathbf{k})}. \quad (10)$$

Let us first consider the manifold of  $\mathbf{k}$ -points such that

$$E(\mathbf{k}) \equiv \lim_{\epsilon \rightarrow 0} \Xi(\epsilon, \mathbf{k}), \quad (11)$$

is finite, in which case  $\delta(\epsilon, \mathbf{k})$  is smooth around  $\epsilon = 0$ , so that, through Eqs. (7)-(8),

$$N(\mathbf{k}) \xrightarrow{T \rightarrow 0} \frac{\delta(0, \mathbf{k})}{\pi} = \frac{1}{\pi} \tan^{-1} \frac{0^+}{E(\mathbf{k})} = \begin{cases} 1 & E(\mathbf{k}) < 0, \\ 0 & E(\mathbf{k}) > 0. \end{cases} \quad (12)$$

Equation (12) defines a surface in the Brillouin zone where  $E(\mathbf{k})$  changes sign. That, because of the definition (9) of  $\Xi(\epsilon, \mathbf{k})$ , occurs only if  $E(\mathbf{k})$  crosses zero. Therefore, in the present formalism there exists a unique surface in  $\mathbf{k}$ -space, which we dub as Fermi-Luttinger surface, where  $\mathbf{k} = \mathbf{k}_{FL}$  such that  $E(\mathbf{k}_{FL}) = 0$ .

If  $\Xi(\epsilon, \mathbf{k}_{FL})$  vanishes linearly at  $\epsilon = 0$ , a reasonable assumption given Eq. (9), then the phase shift  $\delta(\epsilon, \mathbf{k}_{FL})$  has a jump at  $\epsilon = 0$ . More specifically, we can envisage two different scenarios, see Fig. 1:

(F)  $\epsilon(\mathbf{k}_{FL}) + \text{Re} \Sigma_+(0, \mathbf{k}_{FL}) = 0$ , so that

$$\text{Re} G_+^{-1}(\epsilon \rightarrow 0, \mathbf{k}_{FL}) = Z(0, \mathbf{k}_{FL})^{-1} \epsilon, \quad \Xi(\epsilon \rightarrow 0, \mathbf{k}_{FL}) = -\epsilon, \quad (13)$$

and the phase shift  $\delta(\epsilon, \mathbf{k}_{FL})$  jumps from 0 to  $\pi$  at  $\epsilon = 0$ , see left panel in Fig. 1, thus  $N(\mathbf{k}_{FL}) = 1/2$ . This is actually the case of conventional Fermi liquids, where  $\mathbf{k}_{FL}$  belongs to the Fermi surface, the Green's function has a simple pole at  $\epsilon = 0$ , and  $N(\mathbf{k}_{FL}) = 1/2$  simply means that the chemical potential hits an energy level that is therefore half-filled.

(L) the self-energy has a pole at  $\epsilon = 0$ , hence

$$\text{Re} G_+^{-1}(\epsilon \rightarrow 0, \mathbf{k}_{FL}) \simeq - \frac{\Delta(\mathbf{k}_{FL})^2}{\epsilon}, \quad \Xi(\epsilon \rightarrow 0, \mathbf{k}_{FL}) = \epsilon, \quad (14)$$

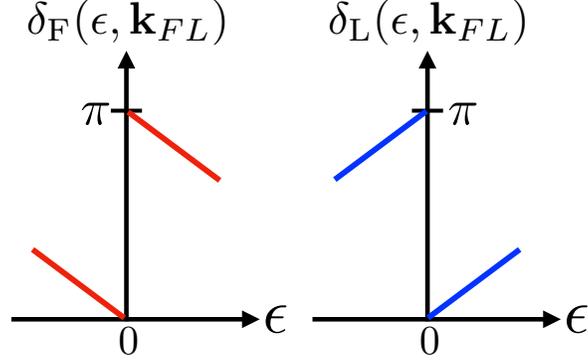


FIG. 1. Sketch of the phase shift versus  $\epsilon$  at  $\mathbf{k}_{FL}$  on the Fermi, left panel, and Luttinger, right panel, surfaces.

and the phase shift  $\delta(\epsilon, \mathbf{k}_{FL})$  jumps from  $\pi$  to 0 at  $\epsilon = 0$ , see right panel in Fig. 1. If we write  $\delta(\epsilon, \mathbf{k}_{FL}) = \delta_c(\epsilon, \mathbf{k}_{FL}) + \delta_d(\epsilon, \mathbf{k}_{FL})$ , where  $\delta_c(\epsilon, \mathbf{k}_{FL}) = \pi + \gamma_*(\mathbf{k}_{FL}) \epsilon$  is continuous, while  $\delta_d(\epsilon, \mathbf{k}_{FL}) = -\pi \theta(\epsilon)$  is discontinuous, then  $N(\mathbf{k}_{FL})$  in Eq. (7) can be written as

$$N(\mathbf{k}_{FL}) = - \int \frac{d\epsilon}{\pi} \frac{\partial f(\epsilon)}{\partial \epsilon} \delta_c(\epsilon, \mathbf{k}_{FL}) - f(0) \xrightarrow{T \rightarrow 0} \frac{\delta_c(0, \mathbf{k}_{FL})}{\pi} - \frac{1}{2} = \frac{1}{2}, \quad (15)$$

the same value as in (F). In this case  $\mathbf{k}_{FL}$  lies on the Luttinger surface, where the Green's function crosses zero at  $\epsilon = 0$ . We note that, if the self-energy has a pole at  $\epsilon = 0$  everywhere in the Brillouin zone, which corresponds to an ideal Mott insulator not breaking spin and translational symmetries, the above result correctly predicts a half-filled band.

We may make a further step forward, and assume that  $\Xi(\epsilon, \mathbf{k})$  can be expanded in Taylor series for small  $\epsilon$  and  $\mathbf{k} \simeq \mathbf{k}_{FL}$ , thus  $E(\mathbf{k}) \simeq 0$ . If so,

$$\Xi(\epsilon \rightarrow 0, \mathbf{k} \simeq \mathbf{k}_{FL}) \simeq E(\mathbf{k}) \mp \epsilon, \quad (16)$$

where the minus and plus signs refer, respectively, to the cases (F) and (L) above. Under such an assumption, we can define, through the single-particle DOS, which can be written as

$$A(\epsilon, \mathbf{k}) = \frac{1}{\pi} Z(\epsilon, \mathbf{k}) \frac{\Gamma(\epsilon, \mathbf{k})}{\Xi(\epsilon, \mathbf{k})^2 + \Gamma(\epsilon, \mathbf{k})^2}, \quad (17)$$

and for  $\epsilon \simeq 0$ , a quasiparticle DOS

$$A_{\text{qp}}(\epsilon, \mathbf{k}) = Z(\epsilon, \mathbf{k})^{-1} A(\epsilon, \mathbf{k}), \quad (18)$$

which, because of Eqs. (6) and (16), becomes at  $\mathbf{k} \simeq \mathbf{k}_{FL}$

$$A_{\text{qp}}(\epsilon, \mathbf{k}) \simeq \delta(\epsilon \mp E(\mathbf{k})). \quad (19)$$

That is exactly what we expect when approaching the Fermi surface in a conventional Fermi liquid, here valid also upon approaching the Luttinger surface. In that case the existence of a ‘quasiparticle’, with  $\delta$ -like DOS is in striking contrast with the behaviour of the experimentally accessible DOS of the physical electron,  $A(\epsilon, \mathbf{k})$  in Eq. (17), which has instead a pseudo-gap since  $Z(\epsilon \rightarrow 0, \mathbf{k}_{FL}) \sim \epsilon^2$ . We emphasise that the definition (18) of the ‘quasiparticle’ DOS is meaningful only at small enough  $\epsilon$  such that  $Z(\epsilon, \mathbf{k})$  is certainly positive.

A notable difference between cases (F) and (L), minus and plus sign in Eq. (19), respectively, is that in the former the quasiparticle peak moves in  $\epsilon$  with  $E(\mathbf{k})$ , just opposite to the (L) case. The consequence is that, if we define the quasiparticle occupation at equilibrium as

$$n_{\text{qp}}(\mathbf{k}) \equiv \int d\epsilon f(\epsilon) A_{\text{qp}}(\epsilon, \mathbf{k}) \simeq f(\pm E(\mathbf{k})), \quad (20)$$

at  $T = 0$  the quasiparticle ‘Fermi’ surface encloses a volume fraction of the whole Brillouin zone equal to the electron filling fraction  $\nu$  in case (F), and the complement hole fraction  $1 - \nu$  in case (L).

Since in both cases Eq. (6) holds, one can derive microscopically a Landau-Fermi liquid theory [7], and, correspondingly, a kinetic equation for the ‘quasiparticle’ distribution function, which looks exactly like the conventional one [8, 9, 12], apart from the fact that the ‘quasiparticle’ energies are  $\epsilon_{\text{qp}}(\mathbf{k}) = E(\mathbf{k})$  and  $\epsilon_{\text{qp}}(\mathbf{k}) = -E(\mathbf{k})$  for case (F) and (L), respectively. For instance, the specific heat can be calculated through the heat density-heat density response function, using the Ward-Takahashi identity [18]. We find, at leading order in  $T$  [19], that

$$c_V = -\frac{1}{V} \sum_{\mathbf{k}} \int \frac{d\epsilon}{T} \frac{\partial f(\epsilon)}{\partial \epsilon} (\Xi(\epsilon, \mathbf{k}) + \epsilon)^2 A_{\text{qp}}(\epsilon, \mathbf{k}) \simeq \frac{\pi^2}{3} \frac{T}{V} \sum_{\mathbf{k}} \delta(\epsilon_{\text{qp}}(\mathbf{k})) \equiv \frac{\pi^2}{3} T A_{\text{qp}}(0), \quad (21)$$

hence the specific heat is still linear in  $T$  even in Luttinger’s case (L), despite the single particle DOS pseudo-gap, a result which is also striking. Similarly, we expect that the coherent ‘quasiparticles’ will give rise to well defined quantum oscillations in both Fermi and Luttinger surfaces, although the amplitudes might be different in the two cases.

To make our point clearer, we present an explicit example based on a toy self-energy vaguely inspired by model calculations for the pseudo-gap phase of underdoped cuprates [20–24].

Assume a 2D square lattice, a less than half-filled band with non-interacting dispersion  $\epsilon(\mathbf{k})$  that gives rise to a closed Fermi surface, see Fig. 2, and a model self-energy [20] at very small  $\epsilon$

$$\Sigma_+(\epsilon, \mathbf{k}) = \frac{\Delta(\mathbf{k})^2}{\epsilon + \epsilon_*(\mathbf{k}) + i\gamma(\mathbf{k})\epsilon^2}, \quad (22)$$

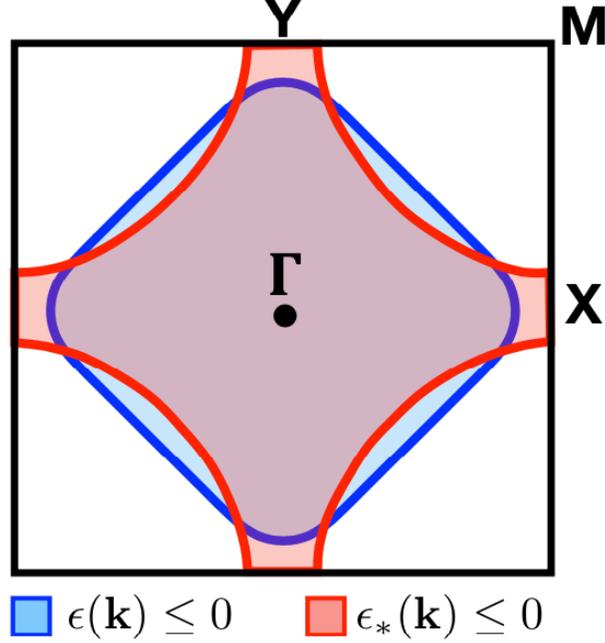


FIG. 2. In black the Brillouin zone with the high symmetry points. The non-interacting Fermi area,  $\epsilon(\mathbf{k}) \leq 0$ , is drawn in blue, while the interacting Luttinger one,  $\epsilon_*(\mathbf{k}) \leq 0$ , in red. They are obtained, respectively, assuming  $\epsilon(\mathbf{k}) = -2 \cos k_x - 2 \cos k_y - \mu$ , with  $\mu = -0.2$ , and  $\epsilon_*(\mathbf{k}) = -2 \cos k_x - 2 \cos k_y + 4t' \cos k_x \cos k_y - \mu_*$ , with  $t' = 0.3$  and  $\mu_*$  such that both areas are equal, thus forcing by hand the same number of electrons in the non-interacting and interacting cases.

which, because of the imaginary term in the denominator, does satisfy Eq. (6) for any  $\mathbf{k}$  [25], even at  $\epsilon_*(\mathbf{k}) = 0$ , where

$$\Sigma_+(\epsilon, \mathbf{k}) = \frac{\Delta(\mathbf{k})^2}{\epsilon + i\gamma(\mathbf{k})\epsilon^2} \underset{\epsilon \rightarrow 0}{\simeq} \frac{\Delta(\mathbf{k})^2}{\epsilon} - i\Delta(\mathbf{k})^2\gamma(\mathbf{k}),$$

is highly singular. We also assume, again unlike real systems,  $\Delta(\mathbf{k}) = \Delta$  and  $\gamma(\mathbf{k}) = \gamma$  to be independent of  $\mathbf{k}$ . We thus readily find that

$$E(\mathbf{k}) = \epsilon_*(\mathbf{k}) \frac{\epsilon_*(\mathbf{k})\epsilon(\mathbf{k}) + \Delta^2}{\Delta^2 + \epsilon_*(\mathbf{k})^2}, \quad (23)$$

vanishes at  $\epsilon_*(\mathbf{k}) = 0$ , which defines the Luttinger surface, case (L) above, and at  $\epsilon_*(\mathbf{k})\epsilon(\mathbf{k}) = -\Delta^2$ , provided the latter equation admits real roots, which would then belong to case (F).

Let us first assume  $\epsilon_*(\mathbf{k})\epsilon(\mathbf{k}) + \Delta^2 > 0$  throughout the Brillouin zone, so that  $E(\mathbf{k})$  just vanishes on the Luttinger surface  $\mathbf{k} = \mathbf{k}_L$  with  $\epsilon_*(\mathbf{k}_L) = 0$ , close to which  $E(\mathbf{k}) \simeq \epsilon_*(\mathbf{k})$ . The Luttinger volume thus comprises all  $\mathbf{k}$  such that  $\epsilon_*(\mathbf{k}) \leq 0$ , which we suppose give rise to an open Luttinger surface, contrary to the closed non-interacting Fermi surface, both shown in Fig. 2.

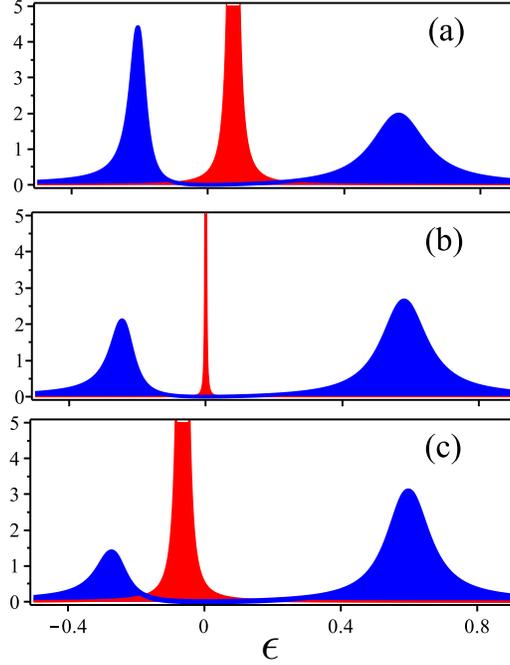


FIG. 3. Physical particle,  $A(\epsilon, \mathbf{k})$  in blue, and ‘quasiparticle’,  $A_{\text{qp}}(\epsilon, \mathbf{k})$  in red, densities of states for  $\mathbf{k}$  along the path  $\mathbf{Y} \rightarrow \mathbf{M}$ . Panels (a), (b) and (c) refer, respectively, to  $\mathbf{k} = (0.8k_L, \pi)$ ,  $\mathbf{k} = (k_L, \pi)$  on the Luttinger surface, and  $\mathbf{k} = (1.1k_L, \pi)$ . We use  $\epsilon(\mathbf{k})$  and  $\epsilon_*(\mathbf{k})$  as in Fig. 2, and take  $\Delta = 0.4$ , so that  $\epsilon(\mathbf{k})\epsilon_*(\mathbf{k}) + \Delta^2 > 0$  throughout the Brillouin zone, and  $\gamma = 1$ . We also added a finite broadening to make the quasiparticle visible on the Luttinger surface.

The equation  $\epsilon = \epsilon(\mathbf{k}) + \text{Re}\Sigma_+(\epsilon, \mathbf{k})$  has two roots,  $\epsilon = \epsilon_-(\mathbf{k}) < 0$  and  $\epsilon = \epsilon_+(\mathbf{k}) > 0$ . Since

$$-\text{Im}\Sigma_+(\epsilon, \mathbf{k}) = \frac{\Gamma \Delta^2 \epsilon^2}{(\epsilon + \epsilon_*(\mathbf{k}))^2 + \gamma^2 \epsilon^4}, \quad (24)$$

is peaked at  $\epsilon = -\epsilon_*(\mathbf{k})$ , the physical particle DOS,  $A(\epsilon, \mathbf{k})$ , displays two asymmetric peaks at  $\epsilon = \epsilon_{\pm}(\mathbf{k})$ , blue in Fig. 3. The negative energy peak is higher than the positive energy one when  $\mathbf{k}$  is inside the Luttinger surface, i.e.,  $\epsilon_*(\mathbf{k}) < 0$ , and the opposite when  $\mathbf{k}$  is outside, much the same as for a conventional Fermi surface, despite here  $A(\epsilon, \mathbf{k}) \sim \epsilon^2$  vanishes quadratically at  $\epsilon = 0$ . The low-energy quasiparticle DOS,  $A_{\text{qp}}(\epsilon, \mathbf{k})$ , red in in Fig. 3, shows a peak that sharpens approaching the Luttinger surface, and moves oppositely from a conventional Fermi liquid: inside the Luttinger surface the peak is at positive energy, while at negative energy outside.

Assume now that the equation  $\epsilon(\mathbf{k})\epsilon_*(\mathbf{k}) = -\Delta^2$  admits two real roots, which create small Fermi pockets in the regions where the Luttinger and the non-interacting Fermi surfaces do not overlap, shown in green in Fig. 4. Since these roots belong to case (F), the physical particle DOS

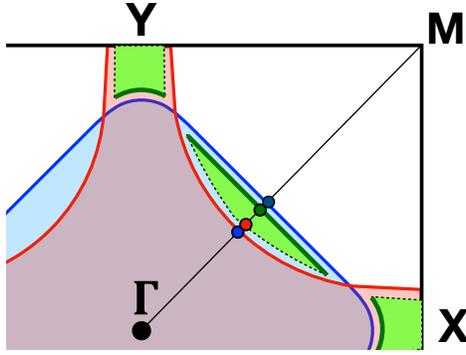


FIG. 4. Fermi pockets, in green, which arise when the equation  $\epsilon(\mathbf{k}) \epsilon_*(\mathbf{k}) = -\Delta^2$  admits real roots. The dots along the  $\Gamma \rightarrow M$  direction are the points at which we calculate the single-particle DOS in Fig. 5. The bold green arcs along the Fermi pockets boundaries are the positions in  $\mathbf{k}$ -space of the highest zero-energy peaks in the physical electron DOS.

should develop peaks at  $\epsilon = 0$  along the borders of such pockets. However, the pseudo-gap on the Luttinger surface implies that the peaks along the arcs closer to the non-interacting Fermi surface, bold green lines in Fig. 4, are much more pronounced. This is explicitly shown in Fig. 5, where we draw the physical particle DOS on the points along  $\Gamma \rightarrow M$  of Fig. 4.

In other words, moving from the case in which  $\epsilon(\mathbf{k}) \epsilon_*(\mathbf{k}) = -\Delta^2$  has no solution to that in

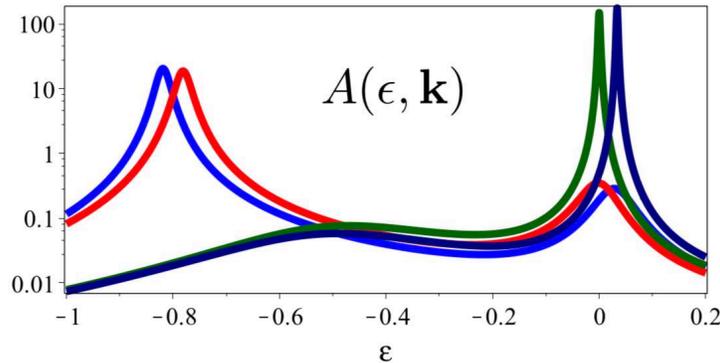


FIG. 5. Physical electron DOS with the same parameters as in Fig. 3 apart from  $\Delta = 0.1$ , a value at which  $\epsilon(\mathbf{k}) \epsilon_*(\mathbf{k}) = -\Delta^2$  has real solutions. The different curves refer to the  $\mathbf{k}$ -points along  $\Gamma \rightarrow M$  shown in Fig. 4. Specifically, the blue curve is on the Luttinger surface, the red at the inner border of the Fermi pocket, the dark green at the outer border, and, finally, the dark blue on the non-interacting Fermi surface. Note the much greater height of the zero-energy peak for  $\mathbf{k}$  close to the non-interacting Fermi surface than to the Luttinger one.

which the solution exists, our toy self-energy describes a kind of Lifshitz's transition resembling that observed by dynamical cluster approximation in the 2D Hubbard model upon increasing hole

doping away from half-filling [22, 23], which in turn has been associated with the Fermi surface evolution across the critical hole-doping level at which the pseudogap vanishes.

We mention that  $E(\mathbf{k})$  of Eq. (11) along, e.g., the path  $\Gamma \rightarrow \mathbf{M}$  in Fig. 4 crosses three zeros, which may evoke the so-called ‘fermion condensation’ [26, 27]. However, in the present case two of the three zeros refer to divergences of  $G_+(\epsilon, \mathbf{k})$ , the boundaries of the Fermi pocket in Fig. 4, while the third to a zero of  $G_+(\epsilon, \mathbf{k})$ , the Luttinger surface, which avoids the band flattening mechanism at the fermion condensation [28].

In conclusion, we have shown that coherent ‘quasiparticles’ emerge both approaching Fermi *and* Luttinger surfaces. This result expands substantially the class of interacting electron systems which are predicted to display conventional Landau-Fermi liquid behaviour, despite very different and anomalous single-particle properties. Moreover, it implies that also close to a Luttinger surface the low-energy physics has the huge emergent symmetry [29] recently discussed in great detail by Ref. [30], which is remarkable since single-particle excitations are absent at the Luttinger surface.

#### ACKNOWLEDGMENTS

We are grateful to Erio Tosatti, Antoine Georges, Marco Schirò, and Grigory Volovik for helpful discussions and comments. This work has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme, Grant agreement No. 692670 “FIRSTORM”.

- 
- [1] Volovik, G. A new class of normal Fermi liquids. *Pis'ma Zh. Eksp. Teor. Fiz.* **53**, 208 (1991). [JETP Lett. **53**, 222 (1991)].
  - [2] Seki, K. & Yunoki, S. Topological interpretation of the luttinger theorem. *Phys. Rev. B* **96**, 085124 (2017). URL <https://link.aps.org/doi/10.1103/PhysRevB.96.085124>.
  - [3] Heath, J. T. & Bedell, K. S. Necessary and sufficient conditions for the validity of luttinger’s theorem. *New Journal of Physics* **22**, 063011 (2020). URL <https://doi.org/10.1088%2F1367-2630%2Fab890e>.
  - [4] Dzyaloshinskii, I. Some consequences of the luttinger theorem: The luttinger surfaces in non-fermi liquids and mott insulators. *Phys. Rev. B* **68**, 085113 (2003). URL <https://link.aps.org/doi/10.1103/PhysRevB.68.085113>.
  - [5] Luttinger, J. M. Fermi surface and some simple equilibrium properties of a system of interacting fermions. *Phys. Rev.* **119**, 1153–1163 (1960). URL <https://link.aps.org/doi/10.1103/PhysRev.119.1153>.

- [6] Oshikawa, M. Topological approach to Luttinger's theorem and the Fermi surface of a Kondo lattice. *Phys. Rev. Lett.* **84**, 3370–3373 (2000). URL <https://link.aps.org/doi/10.1103/PhysRevLett.84.3370>.
- [7] Fabrizio, M. Landau-Fermi liquids without quasiparticles. *Phys. Rev. B* **102**, 155122 (2020). URL <https://link.aps.org/doi/10.1103/PhysRevB.102.155122>.
- [8] Landau, L. The theory of a Fermi liquid. *Zh. Eksp. Teor. Fiz.* **30**, 1058 (1956). [Sov. Phys. JETP **3**, 920 (1957)].
- [9] Landau, L. Oscillations in a Fermi liquid. *Zh. Eksp. Teor. Fiz.* **32**, 59 (1957). [Sov. Phys. JETP **5**, 101 (1957)].
- [10] The Green's function is generically a matrix in the eigenbasis of the non-interacting Hamiltonian. Here we assume, for simplicity, that the Green's function is diagonal in such basis, although our final results do not depend on that choice. Moreover, we do not indicate explicitly the labelling of such eigenbasis, unless necessary.
- [11] Nozières, P. & Luttinger, J. M. Derivation of the Landau theory of Fermi liquids. i. formal preliminaries. *Phys. Rev.* **127**, 1423–1431 (1962). URL <https://link.aps.org/doi/10.1103/PhysRev.127.1423>.
- [12] Luttinger, J. M. & Nozières, P. Derivation of the Landau theory of Fermi liquids. ii. equilibrium properties and transport equation. *Phys. Rev.* **127**, 1431–1440 (1962). URL <https://link.aps.org/doi/10.1103/PhysRev.127.1431>.
- [13] Gangadharaiah, S., Maslov, D. L., Chubukov, A. V. & Glazman, L. I. Interacting fermions in two dimensions: Beyond the perturbation theory. *Phys. Rev. Lett.* **94**, 156407 (2005). URL <https://link.aps.org/doi/10.1103/PhysRevLett.94.156407>.
- [14] Chubukov, A. V. & Maslov, D. L. Nonanalytic corrections to the Fermi-liquid behavior. *Phys. Rev. B* **68**, 155113 (2003). URL <https://link.aps.org/doi/10.1103/PhysRevB.68.155113>.
- [15] Chubukov, A. V. & Maslov, D. L. First-Matsubara-frequency rule in a Fermi liquid. i. fermionic self-energy. *Phys. Rev. B* **86**, 155136 (2012). URL <https://link.aps.org/doi/10.1103/PhysRevB.86.155136>.
- [16] There are circumstances where Luttinger's theorem seem to fail [31–33], which do not occur under our assumption Eq. (6), see Supplementary Notes.
- [17] Volovik, G. E. *Quantum Phase Transitions from Topology in Momentum Space*, 31–73 (Springer Berlin Heidelberg, Berlin, Heidelberg, 2007). URL [https://doi.org/10.1007/3-540-70859-6\\_3](https://doi.org/10.1007/3-540-70859-6_3).
- [18] Narikiyo, O. Ward identities for charge and heat currents of particle-particle and particle-hole pairs. *EPL (Europhysics Letters)* **107**, 17003 (2014). URL <https://doi.org/10.1209/0295-5075/107/17003>.
- [19] Chubukov, A. V., Maslov, D. L., Gangadharaiah, S. & Glazman, L. I. Thermodynamics of a Fermi liquid beyond the low-energy limit. *Phys. Rev. Lett.* **95**, 026402 (2005). URL <https://link.aps.org/doi/10.1103/PhysRevLett.95.026402>.
- [20] Yang, K.-Y., Rice, T. M. & Zhang, F.-C. Phenomenological theory of the pseudogap state. *Phys. Rev. B* **73**, 174501 (2006). URL <https://link.aps.org/doi/10.1103/PhysRevB.73.174501>.

- [21] Rice, T. M., Yang, K.-Y. & Zhang, F. C. A phenomenological theory of the anomalous pseudogap phase in underdoped cuprates. *Reports on Progress in Physics* **75**, 016502 (2011). URL <https://doi.org/10.1088/0034-4885/75/1/016502>.
- [22] Wu, W. *et al.* Pseudogap and fermi-surface topology in the two-dimensional hubbard model. *Phys. Rev. X* **8**, 021048 (2018). URL <https://link.aps.org/doi/10.1103/PhysRevX.8.021048>.
- [23] Scheurer, M. S. *et al.* Topological order in the pseudogap metal. *Proceedings of the National Academy of Sciences* **115**, E3665–E3672 (2018). URL <https://www.pnas.org/content/115/16/E3665>. <https://www.pnas.org/content/115/16/E3665.full.pdf>.
- [24] Robinson, N. J., Johnson, P. D., Rice, T. M. & Tsvelik, A. M. Anomalies in the pseudogap phase of the cuprates: competing ground states and the role of umklapp scattering. *Reports on Progress in Physics* **82**, 126501 (2019). 1906.09005.
- [25] Strictly speaking, in two dimensions the  $\epsilon^2$  dependence in Eqs. (2) and (6) should be replaced by  $-\epsilon^2 \ln \epsilon$ . Here, we still assume  $\epsilon^2$ , as in a quasi two-dimensional model while neglecting a weak dispersion along the third dimension.
- [26] Khodel, V. A. & Shaginyan, V. R. Superfluidity in system with fermion condensate. *Pis'ma Zh. Eksp. Teor. Fiz.* **51**, 488 (1990). [JETP Lett. **51**, 553 (1990)].
- [27] Nozières, P. Properties of fermi liquids with a finite range interaction. *J. Phys. I France* **2**, 443 (1992).
- [28] Khodel, V. A., Clark, J. W. & Zverev, M. V. Topology of the fermi surface beyond the quantum critical point. *Phys. Rev. B* **78**, 075120 (2008).
- [29] Haldane, F. D. M. Luttinger's Theorem and Bosonization of the Fermi Surface (2005). [cond-mat/0505529](https://arxiv.org/abs/cond-mat/0505529).
- [30] Else, D. V., Thorngren, R. & Senthil, T. Non-fermi liquids as ersatz fermi liquids: General constraints on compressible metals. *Phys. Rev. X* **11**, 021005 (2021). URL <https://link.aps.org/doi/10.1103/PhysRevX.11.021005>.
- [31] Georges, A., Parcollet, O. & Sachdev, S. Quantum fluctuations of a nearly critical heisenberg spin glass. *Phys. Rev. B* **63**, 134406 (2001). URL <https://link.aps.org/doi/10.1103/PhysRevB.63.134406>.
- [32] Dave, K. B., Phillips, P. W. & Kane, C. L. Absence of luttinger's theorem due to zeros in the single-particle green function. *Phys. Rev. Lett.* **110**, 090403 (2013). URL <https://link.aps.org/doi/10.1103/PhysRevLett.110.090403>.
- [33] Limtragoon, K., Leong, Z. & Phillips, P. W. Absence of Luttinger's theorem for fermions with power-law Green functions. *SciPost Phys.* **5**, 49 (2018). URL <https://scipost.org/10.21468/SciPostPhys.5.5.049>.

# Supplementary Notes

In these Supplementary Notes we present a proof of Luttinger's theorem [1] somehow complementary to existing ones [1–5], with the purpose of clarifying under which conditions the theorem is valid.

## S1. LUTTINGER'S THEOREM

The first step of Luttinger's theorem is the trivial equivalence

$$\begin{aligned}
 N &= T \sum_{i\epsilon_n} e^{i\epsilon_n 0^+} \text{Tr} \left( G(i\epsilon_n) \right) = -T \sum_{i\epsilon_n} e^{i\epsilon_n 0^+} \frac{\partial}{\partial i\epsilon_n} \text{Tr} \left( \ln G(i\epsilon_n) \right) \\
 &+ T \sum_{i\epsilon_n} e^{i\epsilon_n 0^+} \text{Tr} \left( G(i\epsilon_n) \frac{\partial \Sigma(i\epsilon_n)}{\partial i\epsilon_n} \right),
 \end{aligned} \tag{S1.1}$$

where  $N$  is the number of electrons,  $G(i\epsilon_n)$  and  $\Sigma(i\epsilon_n)$  the Green's function and self-energy matrices, and the trace acts on all internal indices, including momentum, but the Matsubara frequency  $i\epsilon_n$ .

Now, consider the Luttinger Ward functional  $\Phi[G]$  [6], and assume to shift the frequencies of all internal Green's functions by the same amount  $i\omega = i 2m \pi T$ , with  $m$  a non-negative integer. Evidently,  $\Phi[G]$  does not change under such shift, thus

$$\begin{aligned}
 \frac{\delta \Phi[G]}{i\omega} = 0 &= T \sum_{i\epsilon_n} e^{i\epsilon_n 0^+} \text{Tr} \left( \Sigma(i\epsilon_n) \frac{G(i\epsilon_n + i\omega) - G(i\epsilon_n - i\omega)}{2i\omega} \right) \\
 &= -T \sum_{i\epsilon_n} e^{i\epsilon_n 0^+} \text{Tr} \left( G(i\epsilon_n) \frac{\Sigma(i\epsilon_n + i\omega) - \Sigma(i\epsilon_n - i\omega)}{2i\omega} \right).
 \end{aligned} \tag{S1.2}$$

We emphasise that the equivalence between the two sums in Eq. (S1.2) is a trivial consequence of the summation over all Matsubara frequencies  $\epsilon_n = (2n + 1) \pi T$ ,  $n = [-\infty, \infty]$ , and thus no boundary terms are involved.

Luttinger's theorem is valid if, upon defining

$$\begin{aligned}
 I_1 &\equiv T \sum_{i\epsilon_n} e^{i\epsilon_n 0^+} \text{Tr} \left( G(i\epsilon_n) \frac{\Sigma(i\epsilon_n + i\omega) - \Sigma(i\epsilon_n - i\omega)}{2i\omega} \right) = 0, \\
 I_2 &\equiv T \sum_{i\epsilon_n} e^{i\epsilon_n 0^+} \text{Tr} \left( G(i\epsilon_n) \frac{\partial \Sigma(i\epsilon_n)}{\partial i\epsilon_n} \right),
 \end{aligned} \tag{S1.3}$$

the following equivalence holds for sufficiently small  $\omega = 2m\pi T$

$$I_1 \stackrel{?}{=} I_2, \quad (\text{S1.4})$$

in which case we are allowed to drop the last term on the right hand side of Eq. (S1.1), and thus recover the standard Luttinger expression of the number of particles. We note that  $\omega \rightarrow 0$  means  $mT \rightarrow 0$ . Since  $m$  is an integer, rigorously speaking  $\omega \rightarrow 0$  implies  $T \rightarrow 0$ , which we assume hereafter. Nonetheless, we cannot exclude that what follows might remain valid also in the unjustified hypothesis that  $m$  were promoted to a continuous variable and then sent to zero at fixed  $T$ .

The equivalence (S1.4) might seem obvious, but in reality is not so. Indeed, in order to calculate  $I_1$  in Eq. (S1.3) one has first to perform the sum over  $\epsilon_n$  at fixed  $\omega$ , and only after take the limit  $\omega \rightarrow 0$ . Since the the Green's function and the self-energy in the complex frequency  $\zeta$  plane have generally branch cuts on the real axis,  $I_1$  includes three of them, at  $\zeta = \epsilon + i\omega$ ,  $\zeta = \epsilon$  and  $\zeta = \epsilon - i\omega$ , with  $\epsilon \in \mathbb{R}$ , while  $I_2$  just at  $\zeta = \epsilon$ . This implies that the equivalence between  $I_1$  and  $I_2$  depends crucially on the strip in the complex plane between the axes  $\zeta = \epsilon + i\omega$  and  $\zeta = \epsilon - i\omega$ , which, despite a vanishing width for  $\omega \rightarrow 0$ , may have a singular behaviour that makes its contribution finite even in that limit.

We have therefore to directly evaluate  $I_1$  and  $I_2$ . Using standard tricks to compute sums over Matsubara frequencies, we find that

$$\begin{aligned} I_1 &= - \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \text{Tr} \left\{ - 2i \text{Im} \Sigma_+(\epsilon) \frac{G(\epsilon + i\omega) - G(\epsilon - i\omega)}{2i\omega} \right. \\ &\quad \left. + 2i \text{Im} G_+(\epsilon) \frac{\Sigma(\epsilon + i\omega) - \Sigma(\epsilon - i\omega)}{2i\omega} \right\} \\ &\xrightarrow{\omega \rightarrow 0} - \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \text{Tr} \left\{ - 2i \text{Im} \Sigma_+(\epsilon) \frac{\partial \text{Re} G_+(\epsilon)}{\partial \epsilon} + 2i \text{Im} G_+(\epsilon) \frac{\partial \text{Re} \Sigma_+(\epsilon)}{\partial \epsilon} \right\}, \end{aligned} \quad (\text{S1.5})$$

while

$$\begin{aligned} I_2 &= - \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \text{Tr} \left\{ G_+(\epsilon) \frac{\partial \Sigma_+(\epsilon)}{\partial \epsilon} - G_-(\epsilon) \frac{\partial \Sigma_-(\epsilon)}{\partial \epsilon} \right\} \\ &= - \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \text{Tr} \left\{ 2i \text{Re} G_+(\epsilon) \frac{\partial \text{Im} \Sigma_+(\epsilon)}{\partial \epsilon} + 2i \text{Im} G_+(\epsilon) \frac{\partial \text{Re} \Sigma_+(\epsilon)}{\partial \epsilon} \right\}. \end{aligned} \quad (\text{S1.6})$$

It follows that, if

$$\begin{aligned} I_1 - I_2 &= \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} f(\epsilon) \operatorname{Tr} \left\{ \operatorname{Im} \Sigma_+(\epsilon) \frac{\partial \operatorname{Re} G_+(\epsilon)}{\partial \epsilon} + \operatorname{Re} G_+(\epsilon) \frac{\partial \operatorname{Im} \Sigma_+(\epsilon)}{\partial \epsilon} \right\} \\ &= \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} f(\epsilon) \frac{\partial}{\partial \epsilon} \operatorname{Tr} \left( \operatorname{Re} G_+(\epsilon) \operatorname{Im} \Sigma_+(\epsilon) \right) \xrightarrow{T \rightarrow 0} 0, \end{aligned} \quad (\text{S1.7})$$

then Luttinger's theorem holds. Eq. (S1.7) is actually satisfied if  $\operatorname{Tr}(\operatorname{Re} G_+(\epsilon) \operatorname{Im} \Sigma_+(\epsilon))$  is smooth around  $\epsilon = 0$  and

$$\lim_{\epsilon \rightarrow 0} \operatorname{Tr} \left( \operatorname{Re} G_+(\epsilon) \operatorname{Im} \Sigma_+(\epsilon) \right) = 0. \quad (\text{S1.8})$$

We observe that Eq. (S1.8) holds both at a Fermi surface, where the singularity of  $\operatorname{Re} G_+(\epsilon \rightarrow 0)$  is overcompensated by the vanishing  $\operatorname{Im} \Sigma_+(\epsilon \rightarrow 0)$ , and at a Luttinger one, where  $\operatorname{Im} \Sigma_+(\epsilon \rightarrow 0)$  is finite but  $\operatorname{Re} G_+(\epsilon \rightarrow 0)$  vanishes.

We note that Eq. (S1.8) can be rewritten as

$$\lim_{\epsilon \rightarrow 0} \operatorname{Tr} \left( \operatorname{Re} G_+(\epsilon)^{-1} A(\epsilon) \right) = 0, \quad (\text{S1.9})$$

where  $A(\epsilon) = -\operatorname{Im} G_+(\epsilon)$  is the single-particle DOS. Therefore, using the same notations of the article, thus defining

$$\Xi(\epsilon) = -Z(\epsilon) \operatorname{Re} G_+(\epsilon)^{-1}, \quad A_{\text{qp}}(\epsilon) = A(\epsilon) Z(\epsilon), \quad (\text{S1.10})$$

we can also write Eq. (S1.8) as

$$\lim_{\epsilon \rightarrow 0} \operatorname{Tr} \left( \Xi(\epsilon) A_{\text{qp}}(\epsilon) \right) = 0, \quad (\text{S1.11})$$

which is evidently satisfied by

$$A_{\text{qp}}(\epsilon \rightarrow 0) = \delta \left( \Xi(\epsilon \rightarrow 0) \right), \quad (\text{S1.12})$$

i.e., the generalised 'quasiparticle' definition of our work. On the other hand, since we reached Eq. (S1.12) assuming that the 'quasiparticle' decay rate vanishes at  $\epsilon = 0$ , Eq. (6) of the main text, and that Luttinger's theorem holds, we can draw the following conclusion:

- If Eq. (6) of the main text holds, namely

$$-\lim_{\epsilon \rightarrow 0} Z(\epsilon) \operatorname{Im} \Sigma_+(\epsilon) \sim \epsilon^2 \rightarrow 0, \quad (\text{S1.13})$$

then Luttinger's theorem is valid.

This is the main result of this Supplementary Notes.

---

- [1] J. M. Luttinger, Phys. Rev. **119**, 1153 (1960), URL <https://link.aps.org/doi/10.1103/PhysRev.119.1153>.
- [2] I. Dzyaloshinskii, Phys. Rev. B **68**, 085113 (2003), URL <https://link.aps.org/doi/10.1103/PhysRevB.68.085113>.
- [3] K. Seki and S. Yunoki, Phys. Rev. B **96**, 085124 (2017), URL <https://link.aps.org/doi/10.1103/PhysRevB.96.085124>.
- [4] M. Oshikawa, Phys. Rev. Lett. **84**, 3370 (2000), URL <https://link.aps.org/doi/10.1103/PhysRevLett.84.3370>.
- [5] J. T. Heath and K. S. Bedell, New Journal of Physics **22**, 063011 (2020), URL <https://doi.org/10.1088%2F1367-2630%2Fab890e>.
- [6] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960), URL <https://link.aps.org/doi/10.1103/PhysRev.118.1417>.