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Hybrid waves consisting of the solitons, breathers and lumps for a (2+1)-dimensional extended shallow water wave equation

Gao-Fu Deng · Yi-Tian Gao · Xin Yu · Cui-Cui Ding · Ting-Ting Jia · Liu-Qing Li

Received: date / Accepted: date

Abstract Shallow water waves are studied for the applications in hydraulic engineering and environmental engineering. In this paper, a (2+1)-dimensional extended shallow water wave equation is investigated. Hybrid solutions consisting of H-soliton, M-breather and J-lump solutions have been constructed via the modified Pfaffian technique, where H, M and J are the positive integers. One-breather solutions with a real function $\phi(y)$ are derived, where y is the scaled space variable, we notice that $\phi(y)$ influences the shapes of the background planes. Discussions on the hybrid waves consisting of one breather and one soliton indicate that the one breather is not affected by one soliton after interaction. One-lump solutions with $\phi(y)$ are obtained with the condition $k_{1R}^2 < k_{1I}^2$, where k_{1R} and k_{1I} are the real constants, we notice that the one lump consists of two low valleys and one high peak, as well as the amplitude and velocity keep invariant during its propagation. Hybrid waves consisting of the one lump and one soliton imply that the shape of the one soliton becomes periodic when $\phi(y)$ is changed from a linear function to a periodic function.

Keywords (2+1)-dimensional extended shallow water wave equation · Solitons; Breathers · Lumps · Modified Pfaffian technique

Gao-Fu Deng · Yi-Tian Gao (Corresponding author) · Xin Yu (Corresponding author) · Cui-Cui Ding · Ting-Ting Jia · Liu-Qing Li
Ministry-of-Education Key Laboratory of Fluid Mechanics and National Laboratory for Computational Fluid Dynamics, Beijing University of Aeronautics and Astronautics, Beijing 100191, China
E-mail: gaoyt126@126.com E-mail: yuxin@buaa.edu.cn

1 Introduction

Nonlinear evolution equations (NLEEs) have been used to describe the wave propagations in fluid mechanics, plasma physics and nonlinear optics [1–7]. Solitions have been found that their shapes and amplitudes unchanged during the propagations [8]. Breathers have depicted the growths of disturbance on the continuous backgrounds related to the modulation instabilities [9]. Lumps, localized in all the directions of the space and possessing the meromorphic structures, have been found to propagate stably and used to describe the nonlinear patterns in fluid mechanics, plasma physics and nonlinear optics [10]. To construct the nonlinear wave solutions of the NLEEs, methods have been proposed, e.g., the Wronskian technique has been used to construct the soliton solutions [11], Darboux transformation has been used to construct the breather solutions [12] and Kadomtsev-Petviashvili hierarchy reduction has been used to construct the lump solutions [13].

Shallow water waves, known as one type of water waves with small depth relative to the water wavelength, have been studied for the applications in hydraulic engineering and environmental engineering [14–19]. In this paper, we will investigate the following (2+1)-dimensional extended shallow water wave equation [20–26],

$$u_{yt} + u_{xxxy} - 3u_x u_{xy} - 3u_{xx}u_y + \alpha u_{xy} = 0, \tag{1}$$

where u is a real function with respect to the scaled space variables x, y and time variable t, α is a real constant and the subscripts represent the partial derivatives. Multiple soliton solutions for Eq. (1) have been derived via the Hereman's simplified method and Cole-Hopf transformation method [20]. Travelling wave solutions for Eq. (1) have been derived via the (G'/G)-

expansion method [22]. Periodic wave solutions for Eq. (1) have been derived via the Hirota direct method and Riemann theta function [21]. Lax pair, Bäcklund transformation and conservation laws for Eq. (1) have been investigated via the binary Bell polynomials method [23]. Lump solutions and interaction behaviors for Eq. (1) have been investigated via the ansätz technique [24]. Abundant wave solutions for Eq. (1) have been discussed via the (G'/G)-expansion method [25]. Ref. [26] has constructed the N-soliton solutions with the Pfaffian form for Eq. (1) as

$$u_N = -2\left(\ln f_N\right)_x + \phi(y), \tag{2}$$

with $\phi(y)$ being a real function of y, f_N being a real function and defined as

$$\begin{aligned} f_N &= (\bullet) \\ &= (1,2,\cdots,2N) \\ &= \sum_{j=2}^{2N}(-1)^j(1,j)(2,3,\cdots,\hat{j},\cdots,2N), \end{aligned} \tag{3}$$

in which

$$(r,j) = c_{r,j} + \int^x \left(\varphi_{r,x}\varphi_j - \varphi_r\varphi_{j,x}\right)dx, \tag{4}$$

where $\hat{j}$ means that the element j is omitted, r, j and N are the positive integers, $c_{r,j}$ is a real constant satisfying the condition $c_{r,j} = -c_{j,r}$, φ_r's and φ_j's are the scalar functions of x, y, t and satisfying the following linear partial differential conditions,

$$\begin{aligned} \varphi_{r,y} &= \phi(y)_y \int^x \varphi_r dx, \quad \varphi_{r,t} = -\alpha\varphi_{r,x} - \varphi_{r,xxx}, \\ \varphi_{j,y} &= \phi(y)_y \int^x \varphi_j dx, \quad \varphi_{j,t} = -\alpha\varphi_{j,x} - \varphi_{j,xxx}. \end{aligned} \tag{5}$$

However, to our knowledge, hybrid solutions for Eq. (1) consisting of the soliton, breather and lump solutions have not been constructed via the modified Pfaffian technique. In Section 2, hybrid solutions consisting of the soliton and breather solutions for Eq. (1) will be constructed via the modified Pfaffian technique. In Section 3, based on the solutions obtained in Section 2, hybrid solutions consisting of the soliton, breather and lump solutions for Eq. (1) will be obtained. In Section 4, our conclusions will be presented.

2 Hybrid solutions consisting of the soliton and breather solutions for Eq. (1)

To construct the breather solutions for Eq. (1), we redefine (r,j) in Eq. (4) as

$$(r,j) = c_{r,j}\frac{k_r - k_j}{k_r + k_j} + \int^x \left(\varphi_{r,x}\varphi_j - \varphi_r\varphi_{j,x}\right)dx, \tag{6}$$

where $c_{2r-1,2r} = 1$ $(r = 1,2,...,N)$, otherwise $c_{r,j} = 0$, k_r's and k_j's are the complex constants, φ_r's and φ_j's are the complex functions. Introducing the definitions in the form of

$$\begin{aligned} (\widetilde{r},j) &= c_{r,j}\frac{k_r^* - k_j}{k_r^* + k_j} + \int^x \left(\varphi_{r,x}^*\varphi_j - \varphi_r^*\varphi_{j,x}\right)dx, \\ (r,\widetilde{j}) &= c_{r,j}\frac{k_r - k_j^*}{k_r + k_j^*} + \int^x \left(\varphi_{r,x}\varphi_j^* - \varphi_r\varphi_{j,x}^*\right)dx, \\ (\widetilde{r},\widetilde{j}) &= c_{r,j}\frac{k_r^* - k_j^*}{k_r^* + k_j^*} + \int^x \left(\varphi_{r,x}^*\varphi_j^* - \varphi_r^*\varphi_{j,x}^*\right)dx, \end{aligned} \tag{7}$$

where $*$ denotes the complex conjugate, $\widetilde{r}$ and $\widetilde{j}$ denote the complex conjugate of the elements of r and j, respectively, we have $(r,j) = (\widetilde{r},\widetilde{j})^*$ and $(\widetilde{r},j) = (r,\widetilde{j})^*$. Assuming that f_N in Eq. (3) has the following form

$$\begin{aligned} f_{2M+H} = (&1,2,\cdots,4p-2,4p-1,4p,\cdots,4M, \\ &4M+1,\cdots,4M+2H), (p = 1,2,\cdots,M), \end{aligned} \tag{8}$$

and

$$\begin{aligned} & k_{4p-1} = k_{4p-3}^*, k_{4p} = k_{4p-2}^*, k_{4M+\iota} = k_{4M+\iota}^*, \\ & \varphi_{4p-1} = \varphi_{4p-3}^*, \varphi_{4p} = \varphi_{4p-2}^*, \varphi_{4M+\iota} = \varphi_{4M+\iota}^*, \\ & (\iota = 1,2,\cdots,2H), \end{aligned} \tag{9}$$

where M and H are the positive integers, we can obtain

$$\begin{aligned} & f_{2M+H} \\ & = (1,2,3,4\cdots,4p-3,4p-2,4p-1,4p, \\ & \quad \cdots,4M,4M+1,\cdots,4M+2H) \\ & = (1,2,\widetilde{1},\widetilde{2}\cdots,4p-3,4p-2,\widetilde{4p-3}, \\ & \quad \widetilde{4p-2},\cdots,4M,4M+1,\cdots,4M+2H) \\ & = (1,2,3,4\cdots,4p-3,4p-2,4p-1,4p, \\ & \quad \cdots,4M,4M+1,\cdots,4M+2H)^* \\ & = f_{2M+H}^*. \end{aligned} \tag{10}$$

Eq. (10) indicates that f_{2M+H} is a real function. Motivated by Ref. [26], we can set φ_r's and φ_j's in Conditions (5) as

$$\begin{aligned} \varphi_r &= e^{k_r x + \frac{\phi(y)}{k_r} - (\alpha k_r + k_r^3)t + \Omega_r}, \\ \varphi_j &= e^{k_j x + \frac{\phi(y)}{k_j} - (\alpha k_j + k_j^3)t + \Omega_j}, \end{aligned} \tag{11}$$

where Ω_r's and Ω_j's are the complex constants. Then we can derive the hybrid solutions consisting of the M-

breather and H-soliton solutions for Eq. (1) as

$$
\begin{aligned}
&u=-2\left(\ln f_{2M+H}\right)_x+\phi(y),\\
&f_{2M+H}=(1,2,\cdots,4p-2,4p-1,4p,\cdots,4M,\\
&\qquad\qquad 4M+1,\cdots,4M+2H),\\
&(r,j)=c_{r,j}\frac{k_r-k_j}{k_r+k_j}+\frac{k_r-k_j}{k_r+k_j}\\
&\qquad e^{(k_r+k_j)x+\frac{\phi(y)}{k_r}+\frac{\phi(y)}{k_j}-(\alpha k_r+k_r^3+\alpha k_j+k_j^3)t+\Omega_r+\Omega_j},\\
&\Omega_{4p-1}=\Omega_{4p-3}^*,\Omega_{4p}=\Omega_{4p-2}^*,\Omega_{4M+\iota}=\Omega_{4H+\iota}^*.
\end{aligned}
\tag{12}
$$

Hereby, taking $M=1$ and $H=0$ in Solutions (12), and noting k_r's and Ω_r's as the forms,

$$
\begin{aligned}
&k_1=k_{1R}+ik_{1I},k_2=k_{2R}+ik_{2I},\\
&k_3=k_{1R}-ik_{1I},k_4=k_{2R}-ik_{2I},\\
&\Omega_1=\Omega_{1R}+i\Omega_{1I},\Omega_2=\Omega_{2R}+i\Omega_{2I},\\
&\Omega_3=\Omega_{1R}-i\Omega_{1I},\Omega_4=\Omega_{2R}-i\Omega_{2I},
\end{aligned}
\tag{13}
$$

where $k_{1R},k_{1I},k_{2R},k_{2I},\Omega_{1R},\Omega_{1I},\Omega_{2R}$ and Ω_{2I} are the real constants, the subscripts R and I indicate the real and imaginary parts, respectively, we can derive the one-breather solutions for Eq. (1) as

$$
\begin{aligned}
&u=-2\left(\ln f_2\right)_x+\phi(y),\\
&f_2=(1,2,3,4)\\
&\quad=(1,2)(3,4)-(1,3)(2,4)+(1,4)(2,3)\\
&\quad=h_1\left(1+2e^{h_2}\cos h_3+h_4e^{2h_2}\right),
\end{aligned}
\tag{14}
$$

with

$$
\begin{aligned}
&h_1=\frac{(k_{1I}-k_{2I})^2+(k_{1R}-k_{2R})^2}{(k_{1I}+k_{2I})^2+(k_{1R}+k_{2R})^2},\\
&h_2=\left(\frac{k_{1R}}{k_{1I}^2+k_{1R}^2}+\frac{k_{2R}}{k_{2I}^2+k_{2R}^2}\right)\phi(y)\\
&\qquad+(k_{1R}+k_{2R})x+w_1t+\Omega_{1R}+\Omega_{2R},\\
&w_1=3k_{1I}^2k_{1R}+3k_{2I}^2k_{2R}-k_{1R}^3-k_{2R}^3\\
&\qquad-\alpha k_{1R}-\alpha k_{2R},\\
&h_3=-\left(\frac{k_{1I}}{k_{1I}^2+k_{1R}^2}+\frac{k_{2I}}{k_{2I}^2+k_{2R}^2}\right)\phi(y)\\
&\qquad+(k_{1I}+k_{2I})x+w_2t+\Omega_{1I}+\Omega_{2I},\\
&w_2=-3k_{1R}^2k_{1I}-3k_{2R}^2k_{2I}+k_{1I}^3+k_{2I}^3\\
&\qquad+\alpha k_{1I}+\alpha k_{2I},\\
&h_4=\frac{k_{1I}k_{2I}\left[(k_{1I}+k_{2I})^2+(k_{1R}-k_{2R})^2\right]}{k_{1R}k_{2R}\left[(k_{1I}-k_{2I})^2+(k_{1R}+k_{2R})^2\right]}.
\end{aligned}
\tag{15}
$$

To construct the hybrid solutions consisting of one-breather and one-soliton solutions for Eq. (1), we can set $M=H=1$ in Solutions (12) as

$$
\begin{aligned}
&u=-2\left(\ln f_3\right)_x+\phi(y),\\
&f_3=(1,2,3,4,5,6),\\
&(r,j)=c_{r,j}\frac{k_r-k_j}{k_r+k_j}+\frac{k_r-k_j}{k_r+k_j}\\
&e^{(k_r+k_j)x+\frac{\phi(y)}{k_r}+\frac{\phi(y)}{k_j}-(\alpha k_r+k_r^3+\alpha k_j+k_j^3)t+\Omega_r+\Omega_j},\\
&(r,j=1,2,\ldots,6),\\
&k_1=k_{1R}+ik_{1I},k_2=k_{2R}+ik_{2I},k_5=k_5^*,\\
&k_3=k_{1R}-ik_{1I},k_4=k_{2R}-ik_{2I},k_6=k_6^*,\\
&\Omega_1=\Omega_{1R}+i\Omega_{1I},\Omega_2=\Omega_{2R}+i\Omega_{2I},\Omega_5=\Omega_5^*,\\
&\Omega_3=\Omega_{1R}-i\Omega_{1I},\Omega_4=\Omega_{2R}-i\Omega_{2I},\Omega_6=\Omega_6^*.
\end{aligned}
\tag{16}
$$

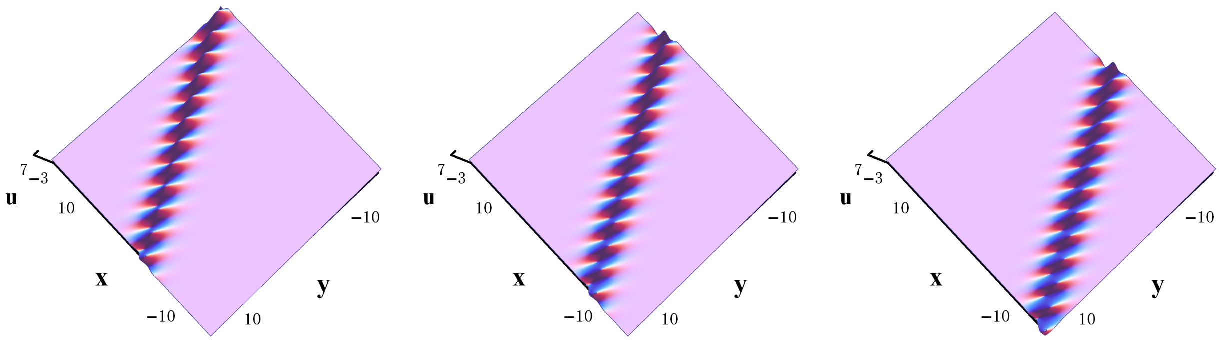
Solutions (14) indicate that the one breather is adjusted by k_1,k_2 and $\phi(y)$, as well as the one breather is localized along the curve $\left(\frac{k_{1R}}{k_{1I}^2+k_{1R}^2}+\frac{k_{2R}}{k_{2I}^2+k_{2R}^2}\right)\phi(y)+(k_{1R}+k_{2R})x+w_1t+\Omega_{1R}+\Omega_{2R}=0$ while periodic along the curve $-\left(\frac{k_{1I}}{k_{1I}^2+k_{1R}^2}+\frac{k_{2I}}{k_{2I}^2+k_{2R}^2}\right)\phi(y)+(k_{1I}+k_{2I})x+w_2t+\Omega_{1I}+\Omega_{2I}=0$. Figs. 1 and 2 present the propagations of the one breather via Solutions (14) with different backgrounds. Comparing Figs. 1 with 2, we notice that the background of the one breather exhibits periodic property along the y direction when $\phi(y)$ is changed from a linear function to a periodic function. Figs. 3 and 4 illustrate the hybrid waves consisting of one breather and one soliton via Solutions (16). Similar to Figs. 1 and 2, the backgrounds of the hybrid waves exhibit periodic property along the y direction when $\phi(y)$ is changed from a linear function to a periodic function. Furthermore, comparing Figs. 1 with 3, we find that the one breather is not affected by one soliton after interaction.

3 Hybrid solutions consisting of the soliton, breather and lump solutions for Eq. (1)

To construct the lump solutions for Eq. (1), we define the set $\mathscr{J}=\{\zeta_1,\zeta_2,\ldots,\zeta_J\}\subseteq\{1,2,\ldots,M,M+1,\ldots,M+J\}$ with $\zeta_1<\zeta_2<\ldots<\zeta_J$, and let

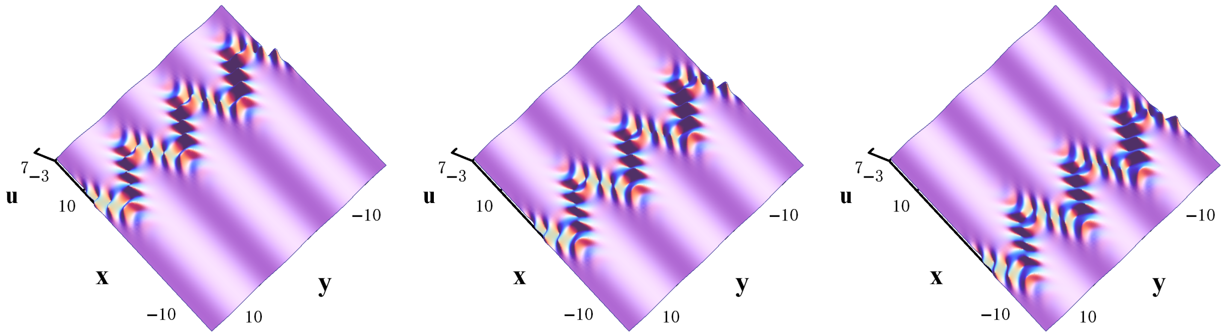
$$
\begin{aligned}
&\Omega_{4\zeta_1-3}=-\Omega_{4\zeta_1-2}+i\pi,\\
&\Omega_{4\zeta_1-1}=-\Omega_{4\zeta_1}-i\pi,
\end{aligned}
\tag{17}
$$

with $i=\sqrt{-1}$, and J is the positive integer. Based on Conditions (9) and taking $k_{4\zeta_1-2}\rightarrow-k_{4\zeta_1-3}$, we can



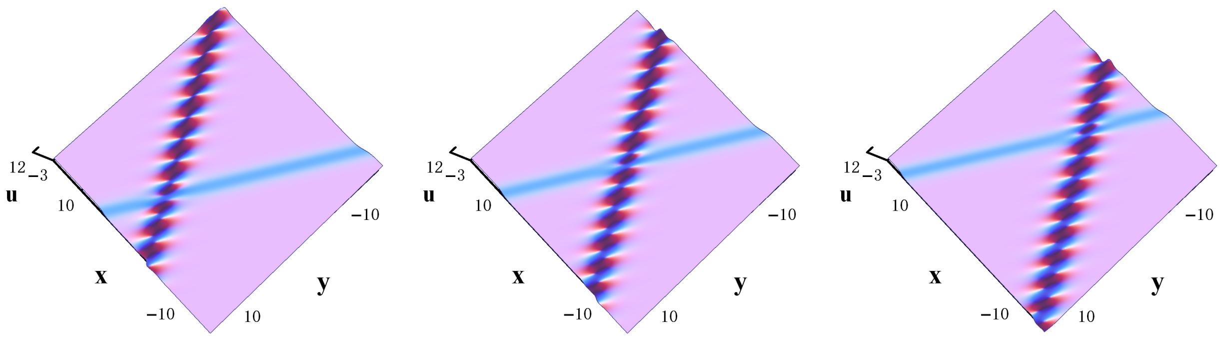
(a) t=-0.6 (b) t=0 (c) t=0.6

Figs. 1 One breather via Solutions (14) with $\alpha = 1, k_{1R} = -0.04, k_{1I} = 0.08, k_{2R} = -1, k_{2I} = 2, \Omega_{1R} = \Omega_{2R} = \Omega_{1I} = \Omega_{2I} = 0$ and $\phi(y) = 0.12y$.



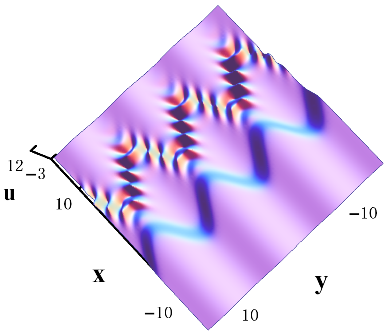
(a) t=-0.6 (b) t=0 (c) t=0.6

Figs. 2 The same as Figs. 1 except that $\phi(y) = 0.5\sin(0.66y)$.

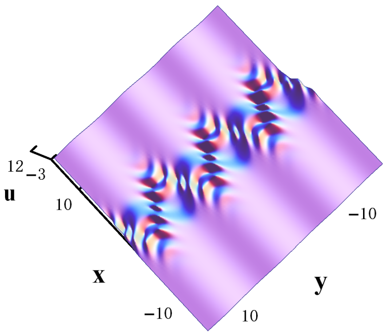


(a) t=-0.6 (b) t=0 (c) t=0.6

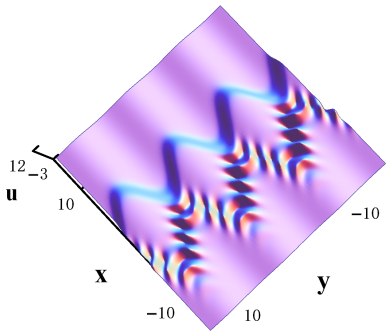
Figs. 3 Hybrid waves consisting of one breather and one soliton via Solutions (16) with $\alpha = 1, k_{1R} = -0.04, k_{1I} = 0.08, k_{2R} = -0.95, k_{2I} = 1.91, k_5 = -2.1, k_6 = 0.1, \Omega_{1R} = \Omega_{2R} = \Omega_{1I} = \Omega_{2I} = 0$ and $\phi(y) = 0.12y$.



(a) t=-0.6



(b) t=0



(c) t=0.6

Figs. 4 The same as Figs. 3 except that $\phi(y) = 0.5\sin(0.66y)$.

derive

$$\begin{aligned}
&\lim_{k_{4\zeta_1-2}\to -k_{4\zeta_1-3}} (4\zeta_1-3, 4\zeta_1-2) \\
&= -2k_{4\zeta_1-3}x + \left(6k^3_{4\zeta_1-3} + 2\alpha k_{4\zeta_1-3}\right)t + \frac{2\phi(y)}{k_{4\zeta_1-3}}, \\
&\lim_{k_{4\zeta_1-2}\to -k_{4\zeta_1-3}} (4\zeta_1-1, 4\zeta_1) \\
&= -2k^*_{4\zeta_1-3}x + \left(6(k^*_{4\zeta_1-3})^3 + 2\alpha k^*_{4\zeta_1-3}\right)t + \frac{2\phi(y)}{k^*_{4\zeta_1-3}}.
\end{aligned} \tag{18}$$

Hence, hybrid solutions consisting of the H-soliton, M-breather and J-lump solutions for Eq. (1) can be derived as

$$u = \lim_{\mathbf{k_1}\to -\mathbf{k_2}} \left[-2\left(\ln f_{2J+2M+H}\right)_x + \phi(y)\right], \tag{19}$$

where the vectors $\mathbf{k_1} = (k_{4\zeta_1-2}, k_{4\zeta_2-2}, \ldots, k_{4\zeta_J-2})$ and $\mathbf{k_2} = (k_{4\zeta_1-3}, k_{4\zeta_2-3}, \ldots, k_{4\zeta_J-3})$.

Let $J = \zeta_1 = 1$ and $M = H = 0$ in Hybrid Solutions (19), we can derive the one-lump solutions for Eq. (1) as

$$\begin{aligned}
u &= \lim_{k_2\to -k_1} \left[-2\left(\ln f_2\right)_x + \phi(y)\right] \\
&= -2\left(\ln \bar{f}_2\right)_x + \phi(y),
\end{aligned} \tag{20}$$

with

$$\begin{aligned}
\bar{f}_2 &= \lim_{k_2\to -k_1} f_2 \\
&= \frac{k^4_{1I} - k^4_{1R}}{k^2_{1I}k^2_{1R}} + 4[(k_{1R} - k_{1I}i)x - 3(k_{1R} - k_{1I}i)^3 t \\
&- \alpha(k_{1R} - k_{1I}i)t - \frac{\phi(y)}{k_{1R} - k_{1I}i}][(k_{1R} + k_{1I}i)x \\
&- 3(k_{1R} + k_{1I}i)^3 t - \alpha(k_{1R} + k_{1I}i)t - \frac{\phi(y)}{k_{1R} + k_{1I}i}].
\end{aligned} \tag{21}$$

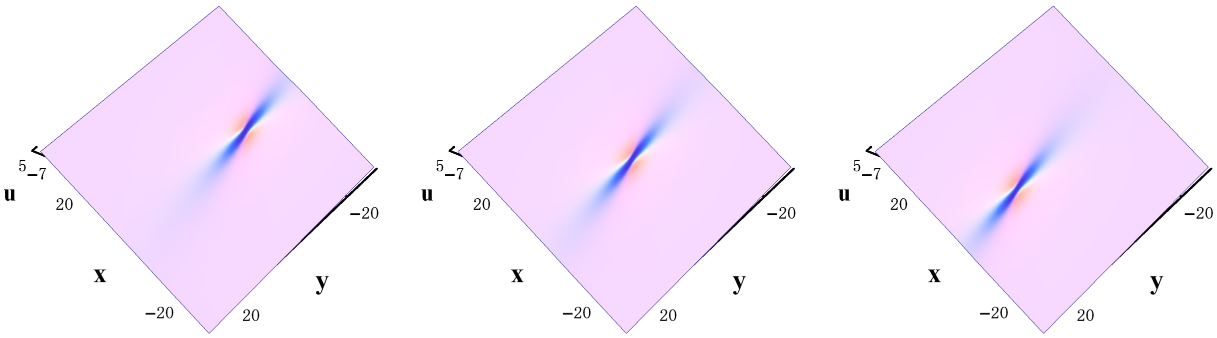
Let $J = H = \zeta_1 = 1$ and $M = 0$ in Hybrid Solutions (19), we can derive the hybrid solutions consisting of the one-lump and one-soliton solutions for Eq. (1) as

$$\begin{aligned}
&u = \lim_{k_2\to -k_1} \left[-2\left(\ln f_3\right)_x + \phi(y)\right], \\
&f_3 = (1,2,3,4,5,6), \\
&(r,j) = c_{r,j}\frac{k_r - k_j}{k_r + k_j} + \frac{k_r - k_j}{k_r + k_j} \\
&e^{(k_r+k_j)x + \frac{\phi(y)}{k_r} + \frac{\phi(y)}{k_j} - (\alpha k_r + k_r^3 + \alpha k_j + k_j^3)t + \Omega_r + \Omega_j}, \\
&(r,j = 1,2,\ldots,6), \\
&k_1 = k_3^* = k_{1R} + ik_{1I}, k_2 = k_4^* = -k_{1R} - ik_{1I}, \\
&\Omega_1 = \Omega_3^* = \Omega_{1R} + i\Omega_{1I}, \Omega_2 = \Omega_4^* = -\Omega_{1R} - i\Omega_{1I}, \\
&k_5 = k_5^*, k_6 = k_6^*, \Omega_5 = \Omega_5^*, \Omega_6 = \Omega_6^*.
\end{aligned} \tag{22}$$

From Solutions (20), we find that the one lump is adjusted by k_1 and $\phi(y)$, and the one lump needs to satisfy the condition $k^2_{1R} < k^2_{1I}$. Figs. 5 present the one lump whose background is a slope plane when $\phi(y) = 0.15y$, we notice that the one lump consists of two low valleys and one high peak, as well as its amplitude and velocity keep invariant during the propagation. Figs. 6 present the one lump whose background is periodic when $\phi(y) = \sin(0.3y)$. Comparing Figs. 5 with 6, we find that the background of the one lump is affected by $\phi(y)$. Figs. 7 and 8 present the hybrid waves consisting of the one lump and one soliton via Solutions (22), we notice that the shape of the one soliton becomes periodic when $\phi(y)$ is changed from a linear function to a periodic function.

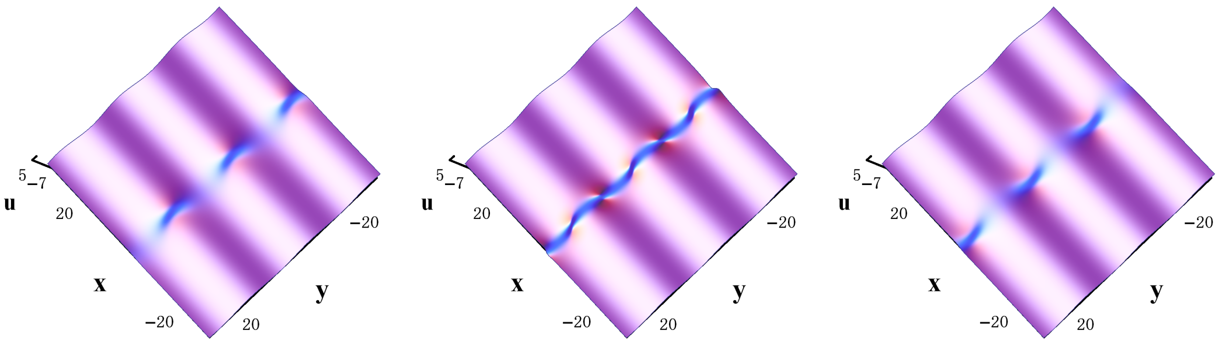
4 Conclusions

Shallow water waves, which refer to the water waves with small depth relative to the water wavelength, have



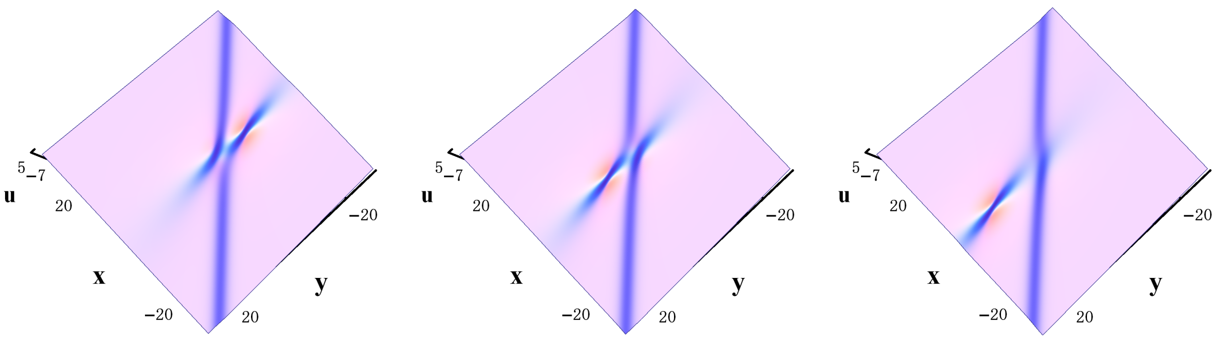
(a) t=-3 (b) t=0 (c) t=3

Figs. 5 One lump via Solutions (20) with $\alpha = 1, k_{1R} = -0.36, k_{1I} = -0.55, \Omega_{1R} = \Omega_{1I} = 0$ and $\phi(y) = 0.15y$.



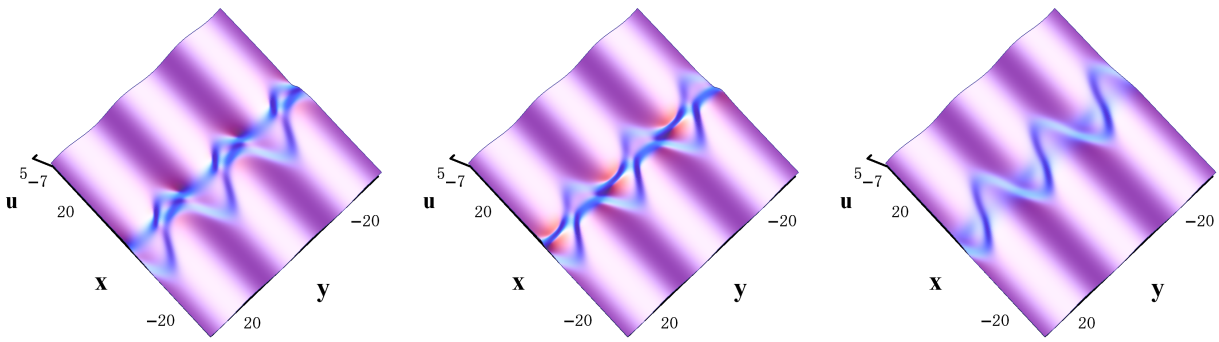
(a) t=-3 (b) t=0 (c) t=3

Figs. 6 The same as Figs. 5 except that $\phi(y) = \sin(0.3y)$.



(a) t=-3 (b) t=0 (c) t=3

Figs. 7 Hybrid waves consisting of the one lump and one soliton via Solutions (22) with $\alpha = 1, k_{1R} = -0.36, k_{1I} = -0.55, k_5 = 0.5, k_6 = 0.3, \Omega_{1R} = \Omega_{1I} = \Omega_5 = \Omega_6 = 0$ and $\phi(y) = 0.15y$.



(a) t=-3 (b) t=0 (c) t=3

Figs. 8 The same as Figs. 6 except that $\phi(y) = \sin(0.3y)$.

been studied for the applications in hydraulic engineering and environmental engineering. In this paper, we have investigated a (2+1)-dimensional extended shallow water wave equation, i.e., Eq. (1). Hybrid Solutions (19) for Eq. (1), consisting of H-soliton, M-breather and J-lump solutions, have been constructed via the modified Pfaffian technique.

From Solutions (14), we have noticed that the one breather is adjusted by k_1, k_2 and $\phi(y)$, as well as the one breather is localized along the curve $(k_{1R}+k_{2R})x+w_1t+\Omega_{1R}+\Omega_{2R}+\left(\frac{k_{1R}}{k_{1I}^2+k_{1R}^2}+\frac{k_{2R}}{k_{2I}^2+k_{2R}^2}\right)\phi(y)=0$ while periodic along the curve $-\left(\frac{k_{1I}}{k_{1I}^2+k_{1R}^2}+\frac{k_{2I}}{k_{2I}^2+k_{2R}^2}\right)\phi(y)+(k_{1I}+k_{2I})x+w_2t+\Omega_{1I}+\Omega_{2I}=0$. Background of the one breather exhibits periodic property along the y direction when $\phi(y)$ is changed from a linear function to a periodic function, as seen in Figs. 1 and 2. Based on Solutions (16), hybrid waves consisting of the one breather and one soliton for Eq. (1) have been presented in Figs. 3 and 4. Comparing Figs. 1 with 3, we have found that the one breather is not affected by one soliton after the interaction.

Solutions (20) have indicated that the one lump is adjusted by k_1 and $\phi(y)$, and the one lump needs to satisfy the condition $k_{1R}^2 < k_{1I}^2$. One lump, whose background is a slope plane when $\phi(y)$ is a linear function, has been presented in Figs. 5. We have noticed that the one lump consists of two low valleys and one high peak, as well as the amplitude and velocity keep invariant during the propagation. One lump, whose background is periodic when $\phi(y)$ is a periodic function, has been presented in Figs. 6. Comparing Figs. 5 with 6, we have found that the background of the one lump is affected by $\phi(y)$. Based on Solutions (22), hybrid waves consisting of the one lump and one soliton have been presented in Figs. 7 and 8, we have noticed that the shape of the one soliton becomes periodic when $\phi(y)$ is changed from a linear function to a periodic function.

Acknowledgements The authors express their sincere thanks to the members of their discussion group for their valuable suggestions. This work has been supported by the National Natural Science Foundation of China under Grant No. 11772017, and by the Fundamental Research Funds for the Central Universities.

5 Conflict of Interest:

The authors declare that they have no conflict of interest.

6 Data Availability Statements:

All data generated or analysed during this study are included in this published article (and its supplementary information files).

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