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Construction of higher-order smooth positons and breather positons via Hirota's bilinear method

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Abstract

Based on the Hirota's bilinear method, a more classic limit technique is perfected to obtain second-order smooth positons. Immediately afterwards, we propose an extremely ingenious limit approach in which higher-order smooth positons and breather positons can be quickly derived from N -soliton solution. Under this ingenious technique, the smooth positons and breather positons of the modified Korteweg-de Vries system are quickly and easily derived. Compared with the generalized Darboux transformation, the approach mentioned in this paper has the following advantages and disadvantages: the advantage is that it is simple and fast; the disadvantage is that this method cannot get a concise general mathematical expression of n th-order smooth positons.

Key words: Hirota's bilinear method; Smooth positons; Breather positons.

1. Introduction

It is well known that there are almost all kinds of nonlinear integrable partial differential equations in many subjects such as optics, plasma physics and fluid mechanics [1–3]. Then, a very important and meaningful link is how to obtain exact solutions of these integrable systems. Initially, many experts are interested in the N -soliton solution in order to explain the solitary wave phenomena in shallow water systems [3, 4]. With the observation of more and more natural phenomena, such as rogue waves and lump waves, scholars gradually pay more and more attention to degenerate solutions of integrable systems [5–8].

Degenerate solutions include rogue wave solutions, lump solutions, higher-order rational solutions, positon solutions, and so on [5–10]. Because this study focuses on a wide variety of positon solutions, so other solutions except positon solutions will not be described in detail. In the beginning, Matveev used the Korteweg-de Vries equation as an example to derive singular positon solutions on the basis of the Wronskian determinant of N -soliton solution [11, 12]. Later, He's team expanded the concept of positon solutions and proposed smooth positon solutions [13, 14] and breather positon solutions [15].

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A n th-order smooth positon solution is actually one of the cases in n -pole solutions [9, 16, 17]. The discovery of breather positons is a very significant thing, which is actually the transition state from higher-order breather waves to rogue waves [15].

However, most of the current studies on smooth positons and breather positons are based on the Wronskian representation of N -soliton solutions [9, 11–15, 18, 19]. How to derive higher-order smooth positons and breather positons from the general N -soliton solution simply and quickly is a problem that is worthy of attention.

Next, this research uses the modified Korteweg-de Vries equation (mKdV equation for short),

$$u_t + 6u^2u_x + u_{xxx} = 0, \quad (1)$$

as an example to illustrate our findings. Since the mKdV equation was proposed, the research on the exact solution of this equation has not been interrupted [20–25]. Hirota obtains the N -soliton solution of mKdV equation by bilinear method [20]. The N pole solution and N th smooth positons of Eq.(1) have also been derived by inverse scattering method, Darboux transformation, and Wronskian representation of the N -soliton solution [9, 14, 16, 23, 24]. Interestingly, Chen et al uses Jacobian elliptic functions as seed solutions to obtain rogue periodic waves by Darboux transformation [25].

By using the bilinear method [20, 21], the N -soliton solution of mKdV equation is represented as

$$u(x, t) = i \left[\ln \frac{g}{f} \right]_x, \quad (2)$$

where

$$g = \sum_{\mu=0,1} \exp \left(\sum_{j<s}^N \mu_j \mu_s A_{js} + \sum_{j=1}^N \mu_j \left(\eta_j - i \frac{\pi}{2} \right) \right), \quad f = \sum_{\mu=0,1} \exp \left(\sum_{j<s}^N \mu_j \mu_s A_{js} + \sum_{j=1}^N \mu_j \left(\eta_j + i \frac{\pi}{2} \right) \right),$$

with

$$\exp(A_{js}) = \left(\frac{k_j - k_s}{k_j + k_s} \right)^2, \quad \eta_j = -k_j^3 t + k_j x + \eta_j^{(0)}.$$

However, the researchers [21, 22] only obtain multiple double pole solutions on the basis of Eq.(2). According to the academic terminology of this study, they actually get the interaction between multiple second-order smooth positons.

Therefore, this paper mainly does two things: one is to improve the method mentioned in Refs. [21, 22, 26] to obtain second-order smooth positons; the other is that we propose an extremely ingenious limit method to derive higher-order smooth positons and breather positons from Eq.(2). In addition, we have carefully compared the results obtained by the bilinear method with those [14, 23] generated by the Darboux transformation to illustrate the advantages and disadvantages of the method proposed in this paper.

2. Higher-order smooth positons

In this section, we first summarize and refine the idea of deriving second-order smooth positons in Refs. [21, 22, 26]. On the basis of obtaining second-order smooth positons, third-order smooth positons are further skillfully constructed. By mathematical induction, we have found a very skillful limit method for deriving higher-order smooth positons from Eq.(2).

Although the ideas in Refs. [21, 22, 26] are clear, these processes are somewhat complex. These ideas can be condensed into the following proposition.

Proposition 2.1: *If some of the parameters in Eq.(2) are assigned as follows:*

$$N = 2, k_2 = k_1 + \epsilon, \eta_1^{(0)} = \xi_1^{(0)} + \ln\left(-\frac{\beta}{\epsilon}\right), \eta_2^{(0)} = \xi_1^{(0)} + \ln\left(\frac{\beta}{\epsilon}\right), \quad (3)$$

then a second-order smooth positon u_{2-sp} will be obtained from the 2-soliton solution when $\epsilon \rightarrow 0$, and its mathematical expression is

$$u_{2-sp} = i \left[\ln \frac{g}{f} \right]_x, \quad (4)$$

where

$$g = 1 + \beta \partial_{k_1} e^{\xi_1 - i\frac{\pi}{2}} - B_{11} \beta^2 e^{2\xi_1 - i\pi}, \quad f = 1 + \beta \partial_{k_1} e^{\xi_1 + i\frac{\pi}{2}} - B_{11} \beta^2 e^{2\xi_1 + i\pi},$$

with

$$\xi_j = -k_j^3 t + k_j x + \xi_j^{(0)}, B_{js} = \begin{cases} \frac{1}{4k_j^2} & j = s \\ \left(\frac{k_j - k_s}{k_j + k_s} \right)^2 & j \neq s \end{cases}.$$

Here $k_j, \xi_j^{(0)}$ and β are all real parameters.

Proof. According to Eq.(3), g in Eq.(1) can be expressed as

$$g = 1 - \frac{\beta}{\epsilon} e^{-k_1^3 t + k_1 x + \xi_1^{(0)} + i\frac{\pi}{2}} + \frac{\beta}{\epsilon} e^{-(k_1 + \epsilon)^3 t + (k_1 + \epsilon)x + \xi_1^{(0)} + i\frac{\pi}{2}} - \frac{\beta^2}{4k_1^2} e^{-k_1^3 t + k_1 x - (k_1 + \epsilon)^3 t + (k_1 + \epsilon)x + 2\xi_1^{(0)} + i\pi},$$

which yields the following semi-rational expression when $\epsilon \rightarrow 0$:

$$g = 1 + \partial_{k_1} e^{-k_1^3 t + k_1 x + \xi_1^{(0)} + i\frac{\pi}{2}} - \frac{\beta^2}{4k_1^2} e^{-2k_1^3 t + 2k_1 x + 2\xi_1^{(0)} + i\pi}.$$

Similarly, f in Eq.(2) can be converted to

$$f = 1 + \partial_{k_1} e^{-k_1^3 t + k_1 x + \xi_1^{(0)} - i\frac{\pi}{2}} - \frac{\beta^2}{4k_1^2} e^{-2k_1^3 t + 2k_1 x + 2\xi_1^{(0)} - i\pi}.$$

Thus, Eq.(4) is easily verified. $\square$

Eq.(4) contains two cases: when $\beta > 0$, a bright second-order smooth positon which has a maximum point will be obtained; when $\beta < 0$, a dark second-order smooth positon with a minimum point will be derived. Using the degenerate Darboux transformation, Refs. [14, 23] only find bright smooth positons as shown in Fig.1 (a), but do not find a dark degenerate solution like Fig.1 (b).

Figure 1: (Color online) Two types of second-order smooth positons: (a) A bright second-order smooth positon described by Eq.(4) with parameters $\{k_1 = 1, \beta = 2, \xi_1^{(0)} = 0\}$; (b) A dark second-order smooth positon described by Eq.(4) with parameters $\{k_1 = 1, \beta = -2, \xi_1^{(0)} = 0\}$.

When $\beta = 2|k_1|$, Eq.(4) will reach the maximum value $2|k_1|$ at the origin; when $\beta = -2|k_1|$, Eq.(4) will achieve the minimum value $-2|k_1|$ at the origin. In addition, Eq.(4) with parameters $\xi_1^{(0)} = 0$ and $\beta = \pm 2k_1$ has the following dynamic properties when $|t| \rightarrow \infty$:

$$|u_{2-sp}| \approx |k_1| \operatorname{sech} \left(k_1 x - k_1^3 t + \frac{\ln(16k_1^6 t^2)}{2} \right) + |k_1| \operatorname{sech} \left(k_1 x - k_1^3 t - \frac{\ln(16k_1^6 t^2)}{2} \right), \quad (5)$$

which means that along the trajectories $x = k_1^2 t \pm \frac{\ln(16k_1^6 t^2)}{2k_1}$, the crest and trough values $\pm |k_1|$ of Eq.(4) can be obtained.

Based on Proposition 2.1, a more ingenious limit technique is proposed to derive higher-order smooth positons quickly and conveniently.

Proposition 2.2: *If some of the parameters in Eq.(2) are assigned as follows:*

$$N = 3, k_2 = k_1 + \epsilon, k_3 = k_1 + 2\epsilon, \eta_1^{(0)} = \xi_1^{(0)} + \ln \left(\frac{\beta}{\epsilon^2} \right), \eta_2^{(0)} = \xi_1^{(0)} + \ln \left(\frac{-2\beta}{\epsilon^2} \right), \eta_3^{(0)} = \xi_1^{(0)} + \ln \left(\frac{\beta}{\epsilon^2} \right), \quad (6)$$

then a third-order smooth positon u_{3-sp} will be obtained from the 3-soliton solution when $\epsilon \rightarrow 0$, and its mathematical expression is

$$u_{3-sp} = i \left[\ln \frac{g}{f} \right]_x, \quad (7)$$

where

$$g = 1 + \partial_{k_1}^2 e^{\xi_1 - i\frac{\pi}{2}} - \frac{\beta^2}{2} \left(4B_{11}(\partial_{k_1} \xi_1)^2 - 4B_{11} \partial_{k_1}^2 \xi_1 + 4(\partial_{k_1} B_{11})(\partial_{k_1} \xi_1) + \partial_{k_1}^2 B_{11} \right) e^{2\xi_1 - i\pi} - 8\beta^3 B_{11}^3 e^{3\xi_1 - i\frac{3\pi}{2}},$$

$$f = 1 + \partial_{k_1}^2 e^{\xi_1 + i\frac{\pi}{2}} - \frac{\beta^2}{2} \left(4B_{11}(\partial_{k_1} \xi_1)^2 - 4B_{11} \partial_{k_1}^2 \xi_1 + 4(\partial_{k_1} B_{11})(\partial_{k_1} \xi_1) + \partial_{k_1}^2 B_{11} \right) e^{2\xi_1 + i\pi} - 8\beta^3 B_{11}^3 e^{3\xi_1 + i\frac{3\pi}{2}}.$$

Here ξ_j and B_{js} have been given in Proposition 2.1. In addition, the proof of Proposition 2.2 is roughly the same as that of Proposition 2.1, except that the calculation of Proposition 2.2 is more complicated.

Therefore, we do not give a specific proof process here.

Figure 2: (Color online) Two types of third-order smooth positons: (a) A bright third-order smooth positon described by Eq.(7) with parameters $\{k_1 = 1, \beta = 2, \xi_1^{(0)} = 0\}$; (b) A dark third-order smooth positon described by Eq.(7) with parameters $\{k_1 = 1, \beta = -2, \xi_1^{(0)} = 0\}$.

When $\beta > 0$, a bright third-order smooth positon composed of two bright parts and one dark part will be derived, as shown in Fig. 2 (a); Otherwise, we can get a dark smooth positon consisting of a bright part and two dark parts like Fig.2 (b).

Eq.(7) with parameters $\xi_1^{(0)} = 0$ and $\beta = \pm 2k_1^2$ has the following dynamic properties when $|t| \rightarrow \infty$:

$$|u_{3-sp}| \approx |k_1| \operatorname{sech} \left(k_1 x - k_1^3 t + \frac{\ln(64k_1^{12}t^4)}{2} \right) + |k_1| \operatorname{sech} \left(k_1 x - k_1^3 t \right) + |k_1| \operatorname{sech} \left(k_1 x - k_1^3 t - \frac{\ln(64k_1^{12}t^4)}{2} \right), \quad (8)$$

which means that along the trajectories $\left\{ x = k_1^2 t \pm \frac{\ln(64k_1^{12}t^4)}{2k_1}, x = k_1^2 t \right\}$, the crest and trough values $\pm |k_1|$ of a third-order smooth positon can be derived.

Summarizing Proposition 2.1 and Proposition 2.2, we have the following conjecture through mathematical induction:

Inference 2.3: If some of the parameters in Eq.(2) are assigned as follows:

$$\begin{aligned} k_2 &= k_1 + \epsilon, k_3 = k_1 + 2\epsilon, k_4 = k_1 + 3\epsilon, \dots, k_N = k_1 + (N-1)\epsilon, \\ \eta_1^{(0)} &= \xi_1^{(0)} + \ln \frac{(-1)^{N+1} C_{N-1}^0}{\epsilon^{N-1}}, \eta_2^{(0)} = \xi_1^{(0)} + \ln \frac{(-1)^{N+2} C_{N-1}^1}{\epsilon^{N-1}}, \\ \eta_3^{(0)} &= \xi_1^{(0)} + \ln \frac{(-1)^{N+3} C_{N-1}^2}{\epsilon^{N-1}}, \dots, \eta_N^{(0)} = \xi_1^{(0)} + \ln \frac{(-1)^{2N} C_{N-1}^{N-1}}{\epsilon^{N-1}}, \end{aligned} \quad (9)$$

then a N th-order smooth positon u_{N-sp} will be obtained from the N -soliton solution when $\epsilon \rightarrow 0$.

If N in Inference 2.3 is equal to 4, we will get a fourth-order smooth positon u_{4-sp} :

$$u_{4-sp} = i \left[\ln \frac{g}{f} \right]_x,$$

where

$$g = 1 + \partial_{k_1}^3 e^{\xi_1 - i\frac{\pi}{2}} + w_1 e^{2\xi_1 - i\pi} + w_2 e^{3\xi_1 - i\frac{3\pi}{2}} + 1296\beta^4 B_{11}^6 e^{4\xi_1 - 2i\pi},$$

$$f = 1 + \partial_{k_1}^3 e^{\xi_1 + i\frac{\pi}{2}} + w_1 e^{2\xi_1 + i\pi} + w_2 e^{3\xi_1 + i\frac{3\pi}{2}} + 1296\beta^4 B_{11}^6 e^{4\xi_1 + 2i\pi},$$

with

$$\begin{aligned} w_1 = & 3\beta^2 \left[-B_{11}(\partial_{k_1}\xi_1)^4 - 3B_{11}(\partial_{k_1}^2\xi_1)^2 + 2B_{11}(\partial_{k_1}\xi_1)(\partial_{k_1}^3\xi_1) - 2(\partial_{k_1}B_{11})(\partial_{k_1}\xi_1)^3 \right. \\ & \left. - \frac{3}{2}(\partial_{k_1}^2B_{11})(\partial_{k_1}\xi_1)^2 - \frac{1}{2}(\partial_{k_1}^3B_{11})(\partial_{k_1}\xi_1) + (\partial_{k_1}^3\xi_1)\partial_{k_1}B_{11} - \frac{1}{16}(\partial_{k_1}^4B_{11}) \right], \\ w_2 = & -18B_{11}\beta^3 \left[2B_{11}^2\partial_{k_1}^3\xi_1 - 6B_{11}^2(\partial_{k_1}\xi_1)(\partial_{k_1}^2\xi_1) + 6B_{11}(\partial_{k_1}B_{11})(\partial_{k_1}\xi_1)^2 + 2B_{11}^2\partial_{k_1}^3\xi_1 \right. \\ & \left. - 6B_{11}(\partial_{k_1}B_{11})(\partial_{k_1}^2\xi_1) + 6(\partial_{k_1}B_{11})^2(\partial_{k_1}\xi_1) - B_{11}\partial_{k_1}^3B_{11} + 3(\partial_{k_1}B_{11})(\partial_{k_1}^2B_{11}) \right]. \end{aligned}$$

It is very concise and clear to obtain the degenerate solution according to Inference 2.3, which is the advantage of this method compared with the Darboux transformation. Because the Darboux transformation requires higher mathematical foundations than the bilinear method. The disadvantage is that we cannot get the general mathematical expression of a n th-order smooth positon, but the Darboux transformation can do this [13–15, 23]. It is possible to obtain general mathematical expressions by doing some normalized treatment of Eq.(9). We have done our best at this point.

3. Higher-order breather positons with a zero background

In fact, there have been some mature schemes for constructing breather positons via degenerate Darboux transformation: Starting from the plane wave solution, breather positons with a nonzero background will be derived when the spectral parameters λ_{2j-1} tend to a specific value λ_1 [15]; Based on degenerate Darboux transformation with a zero seed solution, Ref. [18] describes another mechanism for generating breather positons with a zero background using module resonance conditions.

In this section, a simple and efficient method is given to quickly derive breather positons sitting on a zero background from Eq.(2).

Proposition 3.1: *If some of the parameters in Eq.(2) are assigned as follows:*

$$\begin{aligned} k_2 &= k_1 + \epsilon, \eta_1^{(0)} = \xi_1^{(0)} + \ln\left(-\frac{\beta}{\epsilon}\right), \eta_2^{(0)} = \xi_1^{(0)} + \ln\left(\frac{\beta}{\epsilon}\right), \\ k_4 &= k_3 + \epsilon, \eta_3^{(0)} = \xi_3^{(0)} + \ln\left(-\frac{\beta}{\epsilon}\right), \eta_4^{(0)} = \xi_3^{(0)} + \ln\left(\frac{\beta}{\epsilon}\right), \\ N &= 4, k_1 = k_3^*, \xi_1^{(0)} = \xi_3^{(0)*}, \end{aligned} \quad (10)$$

then a second-order breather positon u_{2-bp} will be obtained from the 4-soliton solution when $\epsilon \rightarrow 0$, and its mathematical expression is

$$u_{2-bp} = i \left[\ln \frac{g}{f} \right]_x, \quad (11)$$

where

$$\begin{aligned} g = & 1 + \beta\partial_{k_1} e^{\xi_1 - i\frac{\pi}{2}} + \beta\partial_{k_3} e^{\xi_3 - i\frac{\pi}{2}} - \beta^2 B_{11} e^{2\xi_1 - i\pi} - \beta^2 B_{33} e^{2\xi_3 - i\pi} + \beta^2 \partial_{k_1} \partial_{k_3} B_{13} e^{\xi_1 + \xi_3 - i\pi} + w_1 e^{2\xi_1 + \xi_3 - i\frac{3\pi}{2}} \\ & + w_2 e^{\xi_1 + 2\xi_3 - i\frac{3\pi}{2}} + \beta^4 B_{11} B_{13}^4 B_{33} e^{2\xi_1 + 2\xi_3 - 2i\pi}, \end{aligned}$$

$$f = 1 + \beta \partial_{k_1} e^{\xi_1 + i\frac{\pi}{2}} + \beta \partial_{k_3} e^{\xi_3 + i\frac{\pi}{2}} - \beta^2 B_{11} e^{2\xi_1 + i\pi} - \beta^2 B_{33} e^{2\xi_3 + i\pi} + \beta^2 \partial_{k_1} \partial_{k_3} B_{13} e^{\xi_1 + \xi_3 + i\pi} + w_1 e^{2\xi_1 + \xi_3 + i\frac{3\pi}{2}} \\ + w_2 e^{\xi_1 + 2\xi_3 + i\frac{3\pi}{2}} + \beta^4 B_{11} B_{13} B_{33} e^{2\xi_1 + 2\xi_3 + 2i\pi},$$

with

$$w_1 = -\beta^3 B_{13} B_{11} (B_{13} \partial_{k_3} \xi_3 + 2 \partial_{k_3} B_{13}), w_2 = -\beta^3 B_{13} B_{33} (B_{13} \partial_{k_1} \xi_1 + 2 \partial_{k_1} B_{13}).$$

Here $k_j, \xi_j^{(0)}$ and $\xi_j^{(0)}$ are all complex parameters. ξ_j and B_{js} have been given in Proposition 2.1. Note that the proof of Proposition 3.1 is similar to that of Proposition 2.1, so it will not be repeated here.

Figure 3: (Color online) (a) A second-order breather positon described by Eq.(11) with parameters $\{k_1 = 1, \beta = 2, \xi_1^{(0)} = 0\}$; (b) A third-order breather positon described by Inference 3.2 with parameters $\{n = 3, k_1 = 1, \beta = 2, \xi_1^{(0)} = 0\}$.

If $\{k_1 = k_2^* = a + ib, \eta_1^{(0)} = 0, \eta_2^{(0)} = 0, N = 2\}$, a breather solution will be easily derived from Eq.(2).

And we can easily know some of its dynamic properties: the position of the wave crest is $\left\{t = \frac{1}{4ab(a^2+b^2)} \left(4\pi an + \pi a + b \ln\left(\frac{a^2}{b^2}\right)\right), x = -\frac{6a^2bt + 2b^3t}{2b}\right\}$, the maximum value is $2|a|$; the minimum value is $-2|a|$; the distance between two adjacent crests is $\frac{\pi \sqrt{a^4 - 6a^2b^2 + 9b^4 + 1}}{|b|(a^2 + b^2)}$.

Combined with the relation between breather solutions and breather positons [15, 18], it is easy to know that the crest and trough values are $2|\Re(k_1)|$ and $-2|\Re(k_1)|$ respectively when $|t| \rightarrow \infty$. Because the absolute values of the maximum and minimum values are equal, breather positons will not be classified as bright or dark. If the period of a breather positon shown in Fig.3 (a) tends to infinity, there will be two situations: A second-order breather positon will slowly convert to a second-order smooth positon when $\Im(k_1) \rightarrow 0$; u_{2-bp} tends to 0 when $\{\Im(k_1) \rightarrow 0, \Re(k_1) \rightarrow 0\}$.

Similar to Proposition 3.1, the following inference gives a simple and effective way to quickly derive breather positons sitting on a zero background like Fig.3 (b) from Eq.(2).

Inference 3.2: *If some of the parameters in Eq.(2) are assigned as follows:*

$$\begin{aligned}
k_2 &= k_1 + \epsilon, k_3 = k_1 + 2\epsilon, \dots, k_n = k_1 + (n-1)\epsilon, \\
k_{n+2} &= k_{n+1} + \epsilon, k_{n+3} = k_{n+1} + 2\epsilon, \dots, k_{2n} = k_{n+1} + (n-1)\epsilon, \\
\eta_1^{(0)} &= \xi_1^{(0)} + \ln \frac{(-1)^{n+1} C_{n-1}^0}{\epsilon^{n-1}}, \eta_2^{(0)} = \xi_1^{(0)} + \ln \frac{(-1)^{n+2} C_{n-1}^1}{\epsilon^{n-1}}, \dots, \eta_n^{(0)} = \xi_1^{(0)} + \ln \frac{(-1)^{2n} C_{n-1}^{n-1}}{\epsilon^{n-1}}, \\
\eta_{n+1}^{(0)} &= \xi_{n+1}^{(0)} + \ln \frac{(-1)^{n+1} C_{n-1}^0}{\epsilon^{n-1}}, \eta_{n+2}^{(0)} = \xi_{n+1}^{(0)} + \ln \frac{(-1)^{n+2} C_{n-1}^1}{\epsilon^{n-1}}, \dots, \eta_{2n}^{(0)} = \xi_{n+1}^{(0)} + \ln \frac{(-1)^{2n} C_{n-1}^{n-1}}{\epsilon^{n-1}}, \\
N &= 2n, k_1 = k_{n+1}^*, \xi_1^{(0)} = \xi_{n+1}^{(0)*},
\end{aligned} \tag{12}$$

then a n th-order breather positon u_{n-bp} will be obtained from the $2n$ -soliton solution when $\epsilon \rightarrow 0$.

4. Conclusion

Based on the Hirota's bilinear method, a unique and novel limit method is proposed to quickly derive smooth positons and breather positons from the N -soliton solution. Inference 2.3 and inference 3.2 are the refinements and summaries of this approach. With this technique, we also find that smooth positons can be divided into two types: dark and bright. As far as we know, studies using the Darboux transformation as a tool have not found this phenomenon shown in Fig.1 and Fig.2. Inferences 2.3 and 3.2 have further significance: N triple-pole solutions can be derived via the limit technique mentioned in this research. Of course, this method can also be extended to other (1+1)-dimensional integrable systems.

Compared with the degenerate Darboux transformation method, simplicity and speed are the advantages of this method. But this method cannot get the general mathematical expression of n th smooth positons like the Darboux transformation method. We have also gone through a lot of attempts: either the result does not have a general law according to Inference 2.3, or it is difficult to find the general formula due to our insufficient professionalism. We hope that relevant experts can solve this problem and make the limit method proposed in this study a complete system.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

Data availability statements All data generated or analyzed during this study are included in this article.

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