CBD: A New Divergence Measure for Complex Mass Function and its Application in Pattern Recognition

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Research Article

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Posted Date: July 19th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-579531/v1>

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CBD: A new divergence measure for complex mass function and its application in pattern recognition

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Abstract

The theory of complex mass function is an effective method to deal with uncertainty information, and it is a generalized of Dempster-Shafer evidence theory. However, divergence measure is still an open issue in the realm of complex mass function theory. The main contribution of our paper is to propose a generalized divergence measure for complex mass function that is called complex belief divergence (CBD), which has the properties of symmetry, nonnegativity, nondegeneracy. When complex mass function degenerates into classical mass function, the CBD will degenerate into classical belief divergence, which has a better ability to measure uncertainty of information. Finally, a pattern recognition algorithm based on CBD is designed and applied to a medical diagnosis problem, which proves its practical prospect.

Keywords: Evidence theory, Complex mass function, Complex belief divergence, Uncertainty information, Pattern recognition

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1. Introduction

Information is full of uncertainty in the real world [1–4], so it is important to develop the technology to deal with the uncertain information. Amounts of models and theories are proposed for uncertainty modeling, such as Dempster-Shafer evidence theory [5, 6], Z numbers [7, 8], fuzzy sets theory [9–11], and other models [12–15]. Those models are widely used in real-world applications, including multi-criteria decision making [16–18], reliability analysis [19–22], classification [23, 24], complex network [25–27], medical diagnosis [28, 29], and risk analysis [30, 31].

Dempster-Shafer evidence theory is one of the most effective and efficient models to deal with uncertain information [32–34], and evidence theory is an extension to traditional probability theory as it assigns belief to multi-element subsets [35, 36]. Hence, evidence theory has better ability to deal with uncertain information than traditional probability theory [37, 38]. However, there are many open issues in Dempster-Shafer evidence theory. For example, it can lead to counterintuitive results when the evidence is highly conflicting [39–41], how to determine the basic assignment [42–44], how to measure the uncertainty of the evidence [45], and so on. Many applications are developed under the framework of Dempster-Shafer evidence theory [46–49]. Dempster-Shafer theory has also promoted the development of other theories, such as evidential reasoning [50–53], D numbers theory [54, 55], Pignistic belief transform [56, 57], and so on.

Therefore, a generalized Dempster-Shafer evidence theory is proposed [58, 59], which extends the classic Dempster-Shafer evidence theory to the complex form as a way of quantum processing. There are also many other models and methods for quantum information processing [60]. In this paper, a

novel divergence measure called complex belief divergence (CBD) for complex mass function is proposed.

The following is the rest of paper organised. In Section 2, preliminaries include Dempster-Shafer evidence theory, complex Dempster-Shafer evidence theory and divergence measure in Dempster-Shafer theory will be briefly introduce. In Section 3, complex divergence measure and its properties will be discussed. Some numerical examples will be given in Section 4. And an application will be given in Section 5. In Section 6, conclusions will be made in the end.

2. Preliminaries

2.1. Dempster-Shafer evidence theory

Definition 2.1. *The frame of discernment(FOD), a set of mutually collective and exclusive non-empty events Θ , is defined below [61, 62]:*

$$\Theta = \{p_1, p_2, \dots, p_n\} \quad (1)$$

The power set of Θ is denoted as 2^Θ , and it is defined as:

$$2^\Theta = \{\emptyset, \{p_1\}, \{p_1, p_2\}, \dots, \{p_1, p_2, \dots, p_n\}\} \quad (2)$$

where $\emptyset$ is an empty set.

Definition 2.2. *The FOD Θ must satisfy the following properties [61, 62]:*

$$(1) m(\emptyset) = 0$$

$$(2) \sum_{A \in 2^\Theta} m(A) = 1,$$

where $m : 2^\Theta \rightarrow [0, 1]$ is called the basic probability assignment(BPA), which is

also known as the mass function. And if $m(A) > 0$, $A \in 2^\Theta$, then A is called the focal element.

The classic Dempster combination rule (DCR) is defined by Dempster to combine different evidences, and it is given as follows [61].

Definition 2.3. Given two independent BPAs, m_1, m_2 , the obtained BPA of DCR can be denoted as m , and it can be obtained as follows:

$$\begin{cases} m(\emptyset) = 0 \\ m(A) = \frac{\sum_{B \cap C = A} m_1(B)m_2(C)}{1-K} \end{cases} \quad (3)$$

where $K = \sum_{B \cap C = \emptyset} m_1(B)m_2(C)$ and $K < 1$.

2.2. Complex Dempster-Shafer evidence theory

Complex Dempster-Shafer evidence theory is the generalization of Dempster-Shafer evidence theory as it is defined on complex numbers [58, 59].

Definition 2.4. A set of mutually exclusive and collective non-empty events Θ is called the frame of discernment (FOD). It is defined as follows:

$$\Theta = \{p_1, p_2, \dots, p_n\} \quad (4)$$

The power set of Θ is denoted as 2^Θ , and it is defined as:

$$2^\Theta = \{\emptyset, \{p_1\}, \{p_1, p_2\}, \dots, \{p_1, p_2, \dots, p_n\}\} \quad (5)$$

where $\emptyset$ is an empty set.

Definition 2.5. A complex mass function $\mathbb{M}$ in FOD Θ is represented as a mapping from $2^\Theta \rightarrow \mathbb{C}$, defined by

$$\mathbb{M} : 2^\Theta \rightarrow \mathbb{C} \quad (6)$$

$\mathbb{M}$ satisfies following properties.

1. $\mathbb{M}(\emptyset)=0$
2. $\mathbb{M}(A)=\mathbf{m}(A)e^{i\theta(A)}, A \in 2^\Theta$.
3. $\sum_{A \in 2^\Theta} \mathbb{M}(A) = 1$

where $i=\sqrt{-1}$. $\mathbf{m}(A)$ represents the magnitude of $\mathbb{M}(A)$ while $\theta(A)$ represents the phase of $\mathbb{M}(A)$. $\mathbf{m}(A) \in [0, 1], \theta(A) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ [63].

The following is another expression of $\mathbb{M}$.

Definition 2.6.

$$\mathbb{M}(A) = x + yi, A \in 2^\Theta \quad (7)$$

which satisfies $\sqrt{x^2 + y^2} \leq 1$. Also, $\mathbf{m}(A) = \sqrt{x^2 + y^2}$ and $\theta(A) = \arctan(\frac{y}{x})$.

A is a focal element satisfying $|\mathbb{M}(A)| > 0$ and $A \in 2^\Theta$. When all $\mathbb{M}(A)$ is real number, complex Dempster-Shafer evidence theory degenerates into classical Dempster-Shafer evidence theory.

Definition 2.7. Suppose there are two independent complex mass functions, $\mathbb{M}_1, \mathbb{M}_2$, the obtained BPA of DCR can be denoted as $\mathbb{M}$, and it can be obtained as follows:

$$\begin{cases} \mathbb{M}_1(\emptyset) = 0 \\ \mathbb{M}_1(A) = \frac{\sum_{B \cap C = A} \mathbb{M}_1(B)\mathbb{M}_2(C)}{1-K} \end{cases} \quad (8)$$

where $K = \sum_{B \cap C = \emptyset} \mathbb{M}_1(B)\mathbb{M}_2(C)$ and $K < 1$.

2.3. Divergence measure in Dempster-Shafer evidence theory

Divergence is useful for difference measure of information [64–66], which has been used in many fields [67–70]. For example, relative belief entropy [71] is proposed by Fei and Deng based on Kullback-Leibler divergence [72].

Definition 2.8. *Given two mass functions m_1 and m_2 , the definition of relative belief entropy is as follows.*

$$KL(m_1||m_2) = \sum_{A \in 2^\Theta} m_1(A) \log \left(\frac{m_1(A)}{m_2(A)} \right) \quad (9)$$

It should be noted that when $m_1(A) = 0$, $m_1(A) \log \left(\frac{m_1(A)}{m_2(A)} \right)$ should be interpreted as 0. Also, it can be proved that $KL(m_1||m_2) > 0$. In this paper, a natural number e is taken for all log functions.

Definition 2.9. *Jensen-Shannon belief divergence is defined by Fei and Deng as a distance measure in belief theory [71], and it is presented as follows.*

$$\begin{aligned} JS(m_1||m_2) &= \frac{1}{2} KL(m_1||\frac{m_1+m_2}{2}) + \frac{1}{2} KL(m_2||\frac{m_1+m_2}{2}) \\ &= \frac{1}{2} \sum_{A \in 2^\Theta} \left(m_1(A) \log \frac{2m_1(A)}{m_1(A) + m_2(A)} \right) \\ &\quad + \frac{1}{2} \sum_{A \in 2^\Theta} \left(m_2(A) \log \frac{2m_2(A)}{m_1(A) + m_2(A)} \right) \end{aligned} \quad (10)$$

Inspired by belief entropy [73, 74], Song and Deng improve the relative divergence by considering the uncertainty caused by subsets [75], and the improved relative divergence is given as follows.

Definition 2.10.

$$KL(m_1||m_2) = \sum_{A \in 2^\Theta} \frac{m_1(A)}{2^{|A|} - 1} \log \left(\frac{m_1(A)}{m_2(A)} \right) \quad (11)$$

When all focal elements A are single elements, Eq.(11) degenerates into Eq.(9).

Based on improved relative divergence, the definition of the improved Jensen-Shannon belief entropy is as follows.

Definition 2.11.

$$\begin{aligned} JS(m_1||m_2) &= \frac{1}{2}KL(m_1||\frac{m_1+m_2}{2}) + \frac{1}{2}KL(m_2||\frac{m_1+m_2}{2}) \\ &= \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(m_1(A) \log \frac{2m_1(A)}{m_1(A) + m_2(A)} \right) \\ &\quad + \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(m_2(A) \log \frac{2m_2(A)}{m_1(A) + m_2(A)} \right) \end{aligned} \quad (12)$$

3. Complex belief divergence

In this section, we propose a new divergence measure called CBD. The new measure extends from a real number system to a complex number system and also takes into account the cardinality of subsets.

3.1. Definition of CBD

Based on the relative belief entropy improved by Song and Deng [75], the proposed relative entropy is extended to the complex number system.

Definition 3.1. *There are two complex mass functions $\mathbb{M}_1$ and $\mathbb{M}_2$, and the definition of complex relative belief entropy is as*

$$KL(\mathbb{M}_1||\mathbb{M}_2) = \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(|\mathbb{M}_1(A)| \log_2 \frac{|\mathbb{M}_1(A)|}{|\mathbb{M}_2(A)|} \right) \quad (13)$$

And improved Jensen-Shannon belief divergence is extended to the complex number system as complex belief divergence (CBD) as follows:

Definition 3.2. *There are two complex mass functions $\mathbb{M}_1$ and $\mathbb{M}_2$. The definition of the complex belief divergence(CBD) $JS(\mathbb{M}_1||\mathbb{M}_2)$ is follows.*

$$JS(\mathbb{M}_1||\mathbb{M}_2) = \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(|\mathbb{M}_1(A)| \log_2 \frac{2 |\mathbb{M}_1(A)|}{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|} \right) + \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(|\mathbb{M}_2(A)| \log_2 \frac{2 |\mathbb{M}_2(A)|}{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|} \right) \quad (14)$$

3.2. Properties of CBD

Theorem 3.1. *$JS(\mathbb{M}_1||\mathbb{M}_2)$ is symmetric. The CBBAs, $\mathbb{M}_1$ and $\mathbb{M}_2$, are two BPAs mapping to complex numbers, and JS is the divergence of the two CBBAs, then we get: $JS(\mathbb{M}_1||\mathbb{M}_2) = JS(\mathbb{M}_2||\mathbb{M}_1)$.*

Proof. According to Eq.(10) we have:

$$JS(\mathbb{M}_1||\mathbb{M}_2) = \frac{1}{2} KL(\mathbb{M}_1||\frac{\mathbb{M}_1 + \mathbb{M}_2}{2}) + \frac{1}{2} KL(\mathbb{M}_2||\frac{\mathbb{M}_1 + \mathbb{M}_2}{2}) \quad (15)$$

$$JS(\mathbb{M}_2||\mathbb{M}_1) = \frac{1}{2} KL(\mathbb{M}_2||\frac{\mathbb{M}_1 + \mathbb{M}_2}{2}) + \frac{1}{2} KL(\mathbb{M}_1||\frac{\mathbb{M}_1 + \mathbb{M}_2}{2}) \quad (16)$$

so, in any case $JS(\mathbb{M}_1||\mathbb{M}_2) = JS(\mathbb{M}_2||\mathbb{M}_1)$ □

Theorem 3.2. *$JS(\mathbb{M}_1||\mathbb{M}_2)$ is nonnegative. The CBBAs, $\mathbb{M}_1$ and $\mathbb{M}_2$, are two BPAs mapping to complex numbers and JS is the divergence of the two CBBAs. It comes to a conclusion that $JS(\mathbb{M}_1||\mathbb{M}_2) \geq 0$.*

Proof. According to the knowledge that $x - 1 \geq \log_2(x)$ when $x \in (0, 1]$,

we have:

$$\begin{aligned}
& -KL(\mathbb{M}_1(A) || \frac{\mathbb{M}_1(A) + \mathbb{M}_2(A)}{2}) \\
&= - \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(|\mathbb{M}_1(A)| \log_2 \frac{2 |\mathbb{M}_1(A)|}{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|} \right) \\
&= \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(|\mathbb{M}_1(A)| \log_2 \frac{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|}{2 |\mathbb{M}_1(A)|} \right) \\
&\leq \sum_{A \in 2^\Theta} \frac{|\mathbb{M}_1(A)|}{2^{|A|} - 1} \left(\frac{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|}{2 |\mathbb{M}_1(A)|} - 1 \right) \\
&= \sum_{A \in 2^\Theta} \frac{|\mathbb{M}_1(A)|}{2^{|A|} - 1} \left(\frac{|\mathbb{M}_1(A) + \mathbb{M}_2(A)| - 2 |\mathbb{M}_1(A)|}{2 |\mathbb{M}_1(A)|} \right) \\
&= \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(\frac{|\mathbb{M}_1(A) + \mathbb{M}_2(A)| - 2 |\mathbb{M}_1(A)|}{2} \right)
\end{aligned} \tag{17}$$

To prove in the same way, we can get:

$$- KL(\mathbb{M}_2(A) || \frac{\mathbb{M}_1(A) + \mathbb{M}_2(A)}{2}) = \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(\frac{|\mathbb{M}_1(A) + \mathbb{M}_2(A)| - 2 |\mathbb{M}_2(A)|}{2} \right) \tag{18}$$

Then, sum the Eq.(17) and Eq.(18).

$$\begin{aligned}
& KL(\mathbb{M}_1(A) || \frac{\mathbb{M}_1(A) + \mathbb{M}_2(A)}{2}) + KL(\mathbb{M}_2(A) || \frac{\mathbb{M}_1(A) + \mathbb{M}_2(A)}{2}) \\
&\geq \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} (|\mathbb{M}_1(A)| + |\mathbb{M}_2(A)| - |\mathbb{M}_1(A) + \mathbb{M}_2(A)|)
\end{aligned} \tag{19}$$

And we assume that there are two complex numbers $\mathbb{M}_1 = a + bi$ and

$\mathbb{M}_2 = c + di$, it is easily to prove that $|\mathbb{M}_1| + |\mathbb{M}_2| \geq |\mathbb{M}_1 + \mathbb{M}_2|$.

$$\begin{aligned}
& |\mathbb{M}_1| + |\mathbb{M}_2| - |\mathbb{M}_1 + \mathbb{M}_2| = |a + bi| + |c + di| - |(a + c) + (b + d)i| \\
&= \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} - \sqrt{(a + c)^2 + (b + d)^2} \\
&= \frac{(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2})^2 - (\sqrt{(a + c)^2 + (b + d)^2})^2}{\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a + c)^2 + (b + d)^2}} \\
&= \frac{a^2 + b^2 + c^2 + d^2 + 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} - ((a + c)^2 + (b + d)^2)}{\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a + c)^2 + (b + d)^2}} \\
&= \frac{2(\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} - ac - bd)}{\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a + c)^2 + (b + d)^2}} \\
&= \frac{2(\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} - ac - bd)(\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} + ac + bd)}{(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a + c)^2 + (b + d)^2})(\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} + ac + bd)} \\
&= \frac{2(a^2d^2 + b^2c^2) - 2abcd}{(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a + c)^2 + (b + d)^2})(\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} + ac + bd)} \\
&= \frac{2(ad - bc)^2}{(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a + c)^2 + (b + d)^2})(\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} + ac + bd)} \\
&\geq 0
\end{aligned} \tag{20}$$

So, we have $|\mathbb{M}_1| + |\mathbb{M}_2| - |\mathbb{M}_1 + \mathbb{M}_2| \geq 0$, and use this in Eq.(17). So, we have $JS(\mathbb{M}_1 || \mathbb{M}_2) \geq 0$. $\square$

Theorem 3.3. *The CBBAs, $\mathbb{M}_1$ and $\mathbb{M}_2$, are two BPAs mapping to complex numbers and JS is the divergence of the two CBBAs. It comes to a conclusion that $JS(\mathbb{M}_1 || \mathbb{M}_2) = 0$ only if $\mathbb{M}_1$ and $\mathbb{M}_2$ have same real part and imaginary part.*

Proof.

$$\begin{aligned}
& -KL(\mathbb{M}_1(A) || \frac{\mathbb{M}_1(A) + \mathbb{M}_2(A)}{2}) \\
&= \sum_{A \in 2^\Theta} \frac{1}{2^{|A|} - 1} \left(\frac{|\mathbb{M}_1(A) + \mathbb{M}_2(A)| - 2|\mathbb{M}_1(A)|}{2} \right)
\end{aligned} \tag{21}$$

Because $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ are equal, so $|\mathbb{M}_1(A) + \mathbb{M}_2(A)| = 2|\mathbb{M}_1(A)|$ and $JS(\mathbb{M}_1||\mathbb{M}_2) = 0$. $\square$

4. Numerical examples

Example 4.1. Suppose that $X = \{A, B, C\}$ is a frame of discernment. There are two mass functions as follows:

1. $\mathbb{M}_1(A) = 0.4 + 0.2i, \mathbb{M}_1(B) = 0.3 + 0.2i, \mathbb{M}_1(C) = 0.3 - 0.4i;$
 $\mathbb{M}_2(A) = 0.4 + 0.2i, \mathbb{M}_2(B) = 0.3 + 0.2i, \mathbb{M}_2(C) = 0.3 - 0.4i;$
2. $\mathbb{M}_1(A, B) = 0.4 + 0.2i, \mathbb{M}_1(A, C) = 0.6 - 0.2i;$
 $\mathbb{M}_2(A, B) = 0.4 + 0.2i, \mathbb{M}_2(A, C) = 0.6 - 0.2i;$

The calculation process of divergence are as follows:

$$\begin{aligned}
1. JS(\mathbb{M}_1||\mathbb{M}_2) &= \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|-1}} \left(|\mathbb{M}_1(A)| \log_2 \frac{2|\mathbb{M}_1(A)|}{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|} \right) \\
&+ \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|-1}} \left(|\mathbb{M}_2(A)| \log_2 \frac{2|\mathbb{M}_2(A)|}{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|} \right) \\
&= \frac{1}{2} \times \frac{1}{2^1-1} \times \sqrt{0.4^2 + 0.2^2} \times \log_2 \frac{2 \times \sqrt{0.4^2 + 0.2^2}}{\sqrt{(0.4+0.4)^2 + (0.2+0.2)^2}} \\
&+ \frac{1}{2} \times \frac{1}{2^1-1} \times \sqrt{0.3^2 + 0.2^2} \times \log_2 \frac{2 \times \sqrt{0.3^2 + 0.2^2}}{\sqrt{(0.3+0.3)^2 + (0.2+0.2)^2}} \\
&+ \frac{1}{2} \times \frac{1}{2^1-1} \times \sqrt{0.3^2 + 0.4^2} \times \log_2 \frac{2 \times \sqrt{0.3^2 + 0.4^2}}{\sqrt{(0.3+0.3)^2 + (0.4+0.4)^2}} \\
&= 0 \\
2. JS(\mathbb{M}_1||\mathbb{M}_2) &= \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|-1}} \left(|\mathbb{M}_1(A)| \log_2 \frac{2|\mathbb{M}_1(A)|}{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|} \right) \\
&+ \frac{1}{2} \sum_{A \in 2^\Theta} \frac{1}{2^{|A|-1}} \left(|\mathbb{M}_2(A)| \log_2 \frac{2|\mathbb{M}_2(A)|}{|\mathbb{M}_1(A) + \mathbb{M}_2(A)|} \right) \\
&= \frac{1}{2} \times \frac{1}{2^2-1} \times \sqrt{0.4^2 + 0.2^2} \times \log_2 \frac{2 \times \sqrt{0.4^2 + 0.2^2}}{\sqrt{(0.4+0.4)^2 + (0.2+0.2)^2}} \\
&+ \frac{1}{2} \times \frac{1}{2^2-1} \times \sqrt{0.6^2 + 0.2^2} \times \log_2 \frac{2 \times \sqrt{0.6^2 + 0.2^2}}{\sqrt{(0.6+0.6)^2 + (0.2+0.2)^2}} \\
&= 0
\end{aligned}$$

From the cases of Example 4.1, it gains a conclusion that the divergence is zero when two CBAs have the same real part and imaginary part. And also it proves

that the relative divergence improved by Song and Deng considering the uncertainty caused by subsets [75] can degenerate into the relative divergence without considering the number of elements in the subsets.

Example 4.2. Set that there are two CBBAAs below:

$$\mathbb{M}_1(A) : \mathbb{M}_1(A) = x + yi, \mathbb{M}_1(X_\Theta) = 1 - x - yi,$$

$$\mathbb{M}_2(A) : \mathbb{M}_2(A) = 1 - x + yi, \mathbb{M}_2(X_\Theta) = x - yi$$

where $\Theta \in \{1, 2\}$. $X_1 = \{B\}$ and $X_2 = \{A, B\}$ respectively when $\Theta = 1$ and $\Theta = 2$. The values of $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ vary from the variables x and y .

Suppose $y = 0$, so the CBBAAs $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ degenerate into classical BBAs. The JS divergence change from the variable x within the range $[0, 1]$ depicted in Fig.1(a). The solid line represents $\{B\}$ and the dashed line represents $\{A, B\}$. This finding verifies that when the complex number degenerate into real number, the divergence measure works.

Suppose x within the range of $[0, 1]$ and $y = 0.1$, the two CBBAAs $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ are complex numbers. The value of JS divergences with the variation of x in the range of $[0, 1]$, in the case of the singleton and multiple subsets, is depicted in Fig.1(b). For another case, the divergence is 0 no matter what kind of sets the CBBAAs are when $x = 0.5$ and $y = 0.1$. And for the case of x in the range of $[0, 0.5) \cup (0.5, 1]$, even if $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ have the same value, the divergence between $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ in the case of $X_1 = \{B\}$ is bigger than the case of $X_2 = \{A, B\}$. The reason for intuitive and reasonable result is that $X_2 = \{A, B\}$ has intersection with $\{A\}$.

In addition, from the two results showed in Fig.1, it can gain a conclusion that the divergence is symmetrical and nonnegative.

Example 4.3. Set that there are two CBBAAs in $\omega = \{A, B\}$ below:

$$\mathbb{M}_1(A) : \mathbb{M}_1(A) = x + yi, \mathbb{M}_1(\{B\}) = 1 - x - yi,$$

Figure 1: JS divergence in Example 4.2

$$\mathbb{M}_2(A) : \mathbb{M}_2(A) = 1 - x + (y + a)i, \mathbb{M}_2(\{B\}) = x + (a - y)i.$$

The values of $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ vary from the parameters x and y .

To study the value of CBD how to change with the CBBAs in the real and imaginary part, there are all four kinds of different situations to discuss. Suppose two variables x y respectively in the range of $[0, 1]$ and $[-1, 1]$, meeting the condition that $x^2 + y^2 \leq 1$, $(1 - x)^2 + y^2 \leq 1$, $(1 - x)^2 + (y - a)^2 \leq 1$ and $(1 - x)^2 + (a - y)^2 \leq 1$.

When we set $a = 0$, two CBBAs $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ have the same imaginary part in the Example 4.2. With x , the real part, in the range of $[0, 1]$ in the Fig.2(a), the CBD has the minimum value of zero when $x = 0.5$, exactly two CBBAs have the same real part, such that $\mathbb{M}_1(A)$ is equal to $\mathbb{M}_2(A)$. In Fig.2(b), the JS divergence is greater than zero in other cases of x and y , because of the different imaginary parts between $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$. Moreover, Fig.2(c) shows that as x change from 0 to 0.5, the JS divergence decreases, because two CBBAs gradually become similar. On the contrast, when x change 0.5 to 1, the JS divergence increase.

When we set $y = 0.1$ and a in the range of $[-1, 1]$, two CBBAs $\mathbb{M}_1(A)$ and $\mathbb{M}_2(A)$ have different imaginary part, we have $\mathbb{M}_1(A) = x + 0.1i$, $\mathbb{M}_2(B) = 1 - x - 0.1i$ and $\mathbb{M}_1(A) = x + (0.1 - a)i$, $\mathbb{M}_2(B) = 1 - x - (0.1 - a)i$. The

Figure 2: JS as CBBAs vary in the real part

variation of JS divergence, as the variable a within the range of $[-1, 1]$, is depicted in Fig.3(a). And from the Fig.3(b), we can see that the JS divergence is on the symmetry of $a = 0.1$ and $x = 0.5$. It once again proves that the JS divergence is symmetrical. In addition, Fig.3(c) show that when $a = 0$, we have the minimum JS divergence and the two imaginary parts are same.

In total, the only condition that JS divergence is equal is that the real parts and the imaginary parts are equal at the same time.

Figure 3: JS as CBBAs vary in the real and imaginary part

5. Application

Recently, the research of pattern recognition has attracted many researchers' attention. And in this part, a decision making algorithm based on CBD measure for pattern recognition is designed. The practicability of the algorithm is verified by its application in medical diagnosis. In addition, the ex-

tension of the algorithm and the related work are analyzed and compared to verify the effectiveness of the algorithm.

5.1. Decision-Making Algorithm

We define that $X = \{x_1, \dots, x_i, \dots, x_n\}$ represents attributes, $P = \{P_1, \dots, P_j, \dots, P_m\}$ represents medical patterns in CBBAs of the form $P_j = \{ \langle x_i, \mathbb{M}_{P_j}^{x_i}(Y), \mathbb{M}_{P_j}^{x_i}(N), \mathbb{M}_{P_j}^{x_i}(Y, N) \rangle \mid x_i \in X \}$, and $S = \{S_1, \dots, S_k, \dots, S_\lambda\}$ represents the goal of pattern recognition in CBBAs of the form $S_k = \{ \langle x_i, \mathbb{M}_{S_k}^{x_i}(Y), \mathbb{M}_{S_k}^{x_i}(N), \mathbb{M}_{S_k}^{x_i}(Y, N) \rangle \mid x_i \in X \}$. The goal is to partition the sample $S = \{S_1, \dots, S_k, \dots, S_\lambda\}$ according to a given $P = \{P_1, \dots, P_j, \dots, P_m\}$.

1. The distance between a given pattern P_j and sample S_k measured using the CBD $JS_{CBB A}$

$$\begin{aligned} JS(\mathbb{M}_{P_j} \parallel \mathbb{M}_{S_k}) &= \frac{1}{n} \times \sum_{i=1}^n JS(\mathbb{M}_{P_j}^{x_i} \parallel \mathbb{M}_{S_k}^{x_i}) \\ &= \frac{1}{2} \sum_{i=1}^n KL(\mathbb{M}_{P_j}^{x_i} \parallel \frac{\mathbb{M}_{P_j}^{x_i} + \mathbb{M}_{S_k}^{x_i}}{2}) + \frac{1}{2} \sum_{i=1}^n KL(\mathbb{M}_{S_k}^{x_i} \parallel \frac{\mathbb{M}_{P_j}^{x_i} + \mathbb{M}_{S_k}^{x_i}}{2}) \end{aligned} \quad (22)$$

2. Choose the minimum distance between the CBBAs $\mathbb{M}_{S_k}$ and $\mathbb{M}_{P_j}$

$$JS(\mathbb{M}_{P_\mu} \parallel \mathbb{M}_{S_k}) = \min_{1 \leq j \leq m} JS(\mathbb{M}_{P_j} \parallel \mathbb{M}_{S_k}) \quad (23)$$

3. Sample S_k is classified as P_μ , where

$$\begin{aligned} \mu &= \arg \min_{1 \leq j \leq m} JS(\mathbb{M}_{P_j} \parallel \mathbb{M}_{S_k}) \\ P_\mu &\rightarrow S_k \end{aligned} \quad (24)$$

Table 1: Three patterns and a sample

	Attributes	$\mathbb{M}_Y$	$\mathbb{M}_N$	$\mathbb{M}_{Y,N}$
Pattern1	X1	$0.9901e^{iarctan(0.0101)}$	0	$0.0141e^{iarctan(-1.0000)}$
	X2	$0.8062e^{iarctan(0.1250)}$	0	$0.2236e^{iarctan(-0.5000)}$
	X3	$0.7071e^{iarctan(0.1429)}$	$0.1118e^{iarctan(-0.5000)}$	$0.2062e^{iarctan(-0.2500)}$
Pattern2	X1	$0.99055e^{iarctan(0.1111)}$	$0.1414e^{iarctan(-1.0000)}$	0
	X2	$0.9901e^{iarctan(0.0101)}$	0	$0.0141e^{iarctan(-1.0000)}$
	X3	$0.9055e^{iarctan(0.1111)}$	0	$0.1414e^{iarctan(-1.0000)}$
Pattern3	X1	$0.6083e^{iarctan(0.1667)}$	$0.2062e^{iarctan(-0.2500)}$	$0.2062e^{iarctan(-0.2500)}$
	X2	$0.8062e^{iarctan(0.1250)}$	0	$0.2236e^{iarctan(-0.5000)}$
	X3	$0.9901e^{iarctan(0.0101)}$	0	$0.0141e^{iarctan(-1.0000)}$
Sample1	X1	$0.5099e^{iarctan(0.2000)}$	$0.3041e^{iarctan(-0.1667)}$	$0.2062e^{iarctan(-0.2500)}$
	X2	$0.6083e^{iarctan(0.1667)}$	$0.2062e^{iarctan(-0.2500)}$	$0.2062e^{iarctan(-0.2500)}$
	X3	$0.8062e^{iarctan(0.1250)}$	$0.1118e^{iarctan(-0.5000)}$	$0.1118e^{iarctan(-0.5000)}$

5.2. Application in a medical diagnosis

Suppose a pattern recognition problem given pattern CBBAs $P = \{P_1, P_2, P_3\}$ and a kind of test sample CBBAs S , which relates to three kinds of attributes $\{x_1, x_2, x_3\}$ shown in Table 1. The goal is to determine which of the given pattern $\{P_1, P_2, P_3\}$ is best suited to test sample S . The calculation process is described below.

1. The distance between a given pattern P_1, P_2 , and P_3 and sample S_k measured using the CBD JS_{CBBAs} , as follows:
 $JS(\mathbb{M}_{P_1} || \mathbb{M}_S) = 0.119;$
 $JS(\mathbb{M}_{P_2} || \mathbb{M}_S) = 0.117;$
 $JS(\mathbb{M}_{P_3} || \mathbb{M}_S) = 0.066;$
2. The minimum value is between the sample $\mathbb{M}_{S_k}$ and $\mathbb{M}_{P_3}$
 $JS(\mathbb{M}_{P_3} || \mathbb{M}_S) = 0.066;$
3. Sample S is classified as P_3

$$\mu = 3;$$

$$P_3 \rightarrow S$$

After above algorithm running, the results are presented in Table.2. As is obvious from step.2, P_3 is the smallest distance from S , so the sample S is assigned to pattern P_3 .

5.3. Extension and Comparison

In Section 6.2, the weight of each feature is evenly distributed, so an extended algorithm is proposed to include a weight ratio of each attributes.

$$\begin{aligned} JS^\omega(\mathbb{M}_{P_j} || \mathbb{M}_{S_k}) &= \sum_{i=1}^n \omega_i JS(\mathbb{M}_{P_j}^{x_i} || \mathbb{M}_{S_k}^{x_i}) \\ &= \sum_{i=1}^n \omega_i KL(\mathbb{M}_{P_j}^{x_i} || \frac{\mathbb{M}_{P_j}^{x_i} + \mathbb{M}_{S_k}^{x_i}}{2}) + \sum_{i=1}^n \omega_i KL(\mathbb{M}_{S_k}^{x_i} || \frac{\mathbb{M}_{P_j}^{x_i} + \mathbb{M}_{S_k}^{x_i}}{2}) \end{aligned} \quad (25)$$

where $\sum_{i=1}^n \omega_i = 1$. ω_i can be considered as a subjective or objective factors.

On this basis, the validity of the two methods of average weight distance measurement is further verified, so we compare them with Garg and Rani method and [76] Xiao method [63].

In the medical diagnosis application, the frame of discernment has two elements, namely, Y and N . $\mathbb{M}(Y), \mathbb{M}(N), \mathbb{M}(Y, N)$, three complex intuitionistic fuzzy sets, respectively relate to the membership, nonmembership, and hesitancy degrees. In [76], the weights ω_i are set to $\omega_1 = 0.3, \omega_2 = 0.35, \omega_3 = 0.35$. After the calculation is completed, the results are presented in the Table., it is obvious that $JS^\omega(\mathbb{M}_{P_3} || \mathbb{M}_S) \leq JS^\omega(\mathbb{M}_{P_2} || \mathbb{M}_S) \leq JS^\omega(\mathbb{M}_{P_1} || \mathbb{M}_S)$. The conclusion is that pattern P_3 belongs to the sample S .

In [76], $< K_1, K_2, K_3, K_4 >$ four processed data correlation coefficients with Garg and Rani method [76] and $d_{CBBA}, d_{CBBA}^\omega$ with Xiao method [63], the

Table 2: Results generated by various method

Methods	$(\mathbb{M}_{P_1} \mathbb{M}_S)$	$(\mathbb{M}_{P_2} \mathbb{M}_S)$	$(\mathbb{M}_{P_3} \mathbb{M}_S)$	Classification results
(d_{CBBA})	0.2256	0.2436	0.1526	P_3
d_{CBBA}^ω	0.2156	0.2400	0.1553	P_3
$K1$	0.0907	0.0539	0.0249	P_3
$K2$	0.2458	0.2941	0.1838	P_3
$K3$	0.0868	0.0531	0.0255	P_3
$K4$	0.2317	0.0531	0.0255	P_3
JS	0.1195	0.1175	0.0667	P_3
JS^ω	0.1136	0.1171	0.0695	P_3

distance between complex intuitionistic fuzzy sets, are presented in Table 2. It is obvious that all of them have a result that P_3 belongs to the sample S and the proposed method for pattern recognition based on JS, JS^ω as effective as them. Compared with Garg and Rani method [76], the proposed method can measure the distance when the frame of discernment has two or more elements. Compared with Xiao method [63], the proposed method can take the number of elements in the subset into account. This application shows that the CBD can be used in the real applications under uncertain environments.

6. Conclusions

In this paper, complex belief divergence (CBD), a novel divergence measure, is proposed for complex mass function. This is the first time that divergence measure is studied in the realm of complex mass function. The properties of the CBD are analysed. The effectiveness of the CBD is illustrated by numerical examples. Finally, an application is proposed based on CBD. In the future, we will investigate on how to use CBD in many

more complex applications. Also, the relationship of CBD and other divergence measures, like Rényi divergence and Tsallis divergence, will be also explored.

7. Declarations

7.1. Conflict of interest

The authors declare that they have no conflict of interest.

7.2. Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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