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An investigation on Γ -hypermodules and soft Γ -hypermodules

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Abstract

The concept of soft Γ -hypermodules are generalization of soft hypermodules. In this paper, we prove that when the fuzzy Γ -subhypermodule is normal, then equivalence relation μ^* defined on Γ -hypermodules are strongly regular and the additive hyperoperation in quotient Γ -hypermodules (fuzzy subhypermodules) are just operations. Also, we prove that all hyperadditions in the quotient Γ -hypermodule induced by normal Γ -subhypermodule are additions.

1 Introduction

The hypergroup introduced by a French mathematician F. Marty [19], at the 8th Congress of Scandinavian Mathematicians. Hypergraph theory is a useful toll for discrete optimization problems. A comprehensive review of

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the theory of hypergraph appears in [3].

The suitable generalization of classical algebraic structures are algebraic hyperstructures. Several books have been written on hyperstructure theory, see [4, 5, 7, 26].

The fuzzy mathematics started in 1965 after the publication of Zadehs seminal work [23], entitled none other than: Fuzzy Sets. One of the the interesting research topic of fuzzy sets is study of fuzzy hyperstructures.

The notion of Γ -semigroups was introduced by Sen in [21, 22]. In 1964, Nobusawa introduced Γ -rings as a generalization of rings. Barnes [2] weakened slightly the conditions in the definition of Γ -ring in the sense of Nobusawa. Barnes [2], Luh [17] and Kyuno [16] studied the structure of Γ -rings. Recently, Davvaz et. al. [15] discussed the basic properties of in Γ -semihypergroups. After that, Dehkordi et. al. [25, 18] investigated the ideals, rough ideals, homomorphisms and regular relations of Γ -*semihyperrings* and introduced quotient Γ -semihyperrings. Moreover Dehkordi et. al. introduced the notion of *general Γ -hypermodules*[20].

In this paper, we consider the relation μ^* defined by X. Ma et. al. [18] and prove when μ is normal the relation μ^* is strongly regular. On the other hand, the hyperoperation defined on quotient Γ -hypermodules are operations. Also, we consider the hyperoperation μ^* defined by [25] is strongly regular. Also, we consider the equivalence relation defined by Zhan et.al. [24], is a strongly regular relation. Hence, all quotient Γ -subhypermodules are only Γ -modules. Hence, all quotient soft Γ -hypermodules defined by Ma et.al. [18], are soft Γ -modules.

2 Γ -hypermodule

We shall recall some notions and basic results about hyperstructures that we shall use in the following sections.

Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ a hyperoperation. Then, the couple $(H, \circ)$ is called *hypergroupoid*. Let A and B are nonempty subset in H . Then, we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\}, \quad x \circ B = \{x\} \circ B.$$

Definition 2.1. [19] A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all a, b, c of H , we have $(a \circ b) \circ c = a \circ (b \circ c)$. Also, a the couple $(H, \circ)$ is called a *quasihypergroup* if for all a of H , we have $a \circ H = H \circ a$. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a hypergroup.

EXAMPLE 1. Let G be a group and $\{A_g : g \in G\}$ be a collection of disjoint nonempty subsets. Then, $H = \bigcup_{g \in G} A_g$ is a hypergroup by following hyperoperation:

$$x \oplus y = A_t,$$

where $x, y \in H$, $x \in A_{g_1}, y \in A_{g_2}$ and $t = g_1 g_2$.

Definition 2.2. [7] Let $(H, \circ)$ be a hypergroup. Then, we say that $(H, \circ)$ is a *canonical hypergroup* if

- (i) the hyperoperation $\circ$ is commutative,
- (ii) it has a scalar identity (also called scalar unit), which means that there exists $e \in H$ such that

$$\forall x \in H, x = x \circ e = e \circ x,$$

- (iii) every element has a unique inverse, which means that for all $x \in H$, there exists a unique x^{-1} , such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$,
- (iv) it is reversible, which means that if $x \in y \circ z$, then there exist y^{-1} and z^{-1} , such that $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

Definition 2.3. [7] A *polygroup* is an algebraic hyperstructure $(H, +)$ satisfying the following conditions:

- (i) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$,
- (ii) there exists a $0 \in H$ such that $0 + x = x + 0 = x$, for all $x \in H$,
- (iii) for every $x \in H$, there exist a unique element x' such that $0 \in (x + x') \cap (x' + x)$ (we denote x' by $-x$),
- (iv) if $z \in x + y$, then $y \in z - x$ and $x \in z - y$.

For every $x, y \in H$, we have $-(-x) = x$ and $-(x + y) = -y - x$. A *canonical hypergroup* is a commutative polygroup.

Definition 2.4. [7] A *Krasner hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

- (i) $(R, +)$ is a canonical hypergroup,
- (ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$,

- (iii) The multiplacation is distributive with resect to the hyperoperation $+$.

Definition 2.5. [1] Let $(R, \oplus)$ and $(\Gamma, +)$ be two canonical hypergroup. Then, R is called a Γ -hyperring, if following conditions are satisfies for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$,

- (i) $x\alpha y \in R$,
- (ii) $(x \oplus y)\alpha z = x\alpha z \oplus y\alpha z$, $x\alpha(y \oplus z) = x\alpha y \oplus x\alpha z$,
- (iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.6. [18] Let $(R, +, \Gamma)$ be a Γ -hyperring and $(M, \oplus)$ be a canonical hypergroup. Then, M is called *left Γ -hypermodule* over R if there exists a scaler operation $\alpha : R \times \Gamma \times M \longrightarrow M$ such that for every $x, y \in M$ and $\alpha, \beta \in \Gamma$ and $r_1, r_2 \in R$,

- (i) $r_1\alpha(x \oplus y) = r_1\alpha x \oplus r_1\alpha y$,
- (ii) $(r_1 + r_2)\alpha x \subseteq r_1\alpha x \oplus r_2\alpha x$,
- (iii) $(r_1\alpha r_2)\beta x = r_1\alpha(r_2\beta x)$.

EXAMPLE 2. Let M be an R -module over an unitary ring R and I be a normal subsemigroup of multiplicative semigroup $(R, \cdot)$ and $(\Gamma, +)$ be a subgroup of $(R, +)$. We define an equivalence relation $\equiv$ on M as follows:

$$m_1 \equiv m_2 \iff \exists n \in I : m_1 = n \cdot m_2.$$

Let $(M/\equiv) = \{\overline{m} : m \in M\}$ be the set of equivalence classes and we define a hyperoperation $\oplus$ as follows:

$$\overline{m}_1 \oplus \overline{m}_2 = \{\overline{m} \in M/\equiv : \overline{m} \subseteq \overline{m}_1 + \overline{m}_2\}$$

Thus $(M, \oplus)$ become a canonical hypergroup. In a same way, we can define the quotient hyperring $R/\equiv$ of R by I . Hence $M/\equiv$ is a Γ -hypermodule by scaler operation form $(R/\equiv) \times (M/\equiv) \longrightarrow M/\equiv$ defined by

$$\overline{a} \cdot \overline{m} = \overline{am},$$

for every $a \in I$ and $m \in M$.

EXAMPLE 3. Let R be a Γ -hyperring. Then, $R \oplus k\mathbb{Z} = \{(r, kn) : r \in R, n \in \mathbb{Z}\}$ is a left Γ -hypermodule, where hyperaddition $\oplus$ defined as follows:

$$(r_1, kn_1) \oplus (r_2, kn_2) := \{(r, n) : r \in r_1 + r_2, n = kn_1 + kn_2\},$$

and scalar operation $R \times \Gamma \times M \longrightarrow M$ is defined:

$$(r, \alpha, (r_1, kn)) := (r\alpha r_1, kn).$$

EXAMPLE 4. Let R be a Γ -hyperring. Then, $R \oplus R$ is a Γ -hypermodule by following hyperaddition and scalar operation:

$$(r_1, r_2) \oplus (r_3, r_4) := \{(s_1, s_2) : s_1 \in r_1 + r_3, s_2 \in r_2 + r_4\},$$

$$r \odot_{\alpha} (r_1, r_2) := (r\alpha r_1, r\alpha r_2),$$

where $(r_1, r_2), (r_3, r_4) \in R \oplus R$ and $\alpha \in \Gamma$.

A nonempty subset M_1 of a Γ -hypermodule M is called a Γ -*subhypermodule* if it is a Γ -hypermodule.

Throughout this paper, R and M are Γ -hyperring and Γ -hypermodule, respectively, unless otherwise specified. A Γ -subhypermodule N of M is called *normal* if $x + N - x \subseteq N$, for all $x \in M$.

Definition 2.7. [18] A fuzzy set μ of M is called a *fuzzy Γ -subhypermodule* of M if the following conditions hold:

- (i) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y} \mu(z)$,
- (ii) $\mu(x) \leq \mu(-x)$,
- (iii) $\mu(x) \leq \mu(r\alpha x)$,

for every $x, y \in M$ and $\alpha \in \Gamma$.

EXAMPLE 5. Let $R \oplus \mathbb{Z}$ be the Γ -hypermodule defined in the Example 4, when $k = 1$ and $\mu(r, n) = 1/2$, where n is even and $\mu(r, n) = 1/3$, where n is odd. Then, μ is a fuzzy Γ -subhypermodule.

Definition 2.8. Let M_1 and M_2 be Γ -hypermodules, μ_1 and μ_2 be fuzzy Γ -subhypermodules on M_1 and M_2 , respectively. Then, $\varphi : M_1 \longrightarrow M_2$ is called *homomorphism* if

- (i) $\varphi(x + y) = \varphi(x) + \varphi(y)$,
- (ii) $\varphi(r\alpha x) = r\alpha\varphi(x)$,

$$(iii) \quad \varphi(0) = 0,$$

$$(iv) \quad \mu_2 \circ \varphi = \mu_1,$$

where $\alpha \in \Gamma$, $x, y \in M_1$ and $r \in R$.

EXAMPLE 6. Let $R \oplus \mathbb{Z}$ and $R \oplus 3\mathbb{Z}$ be Γ -hypermodules defined in the Example 4 and μ_1, μ_2 be Γ -subhypermodule on $R \oplus \mathbb{Z}$ and $R \oplus 3\mathbb{Z}$, respectively, such that $\mu_1(r, n)_1 = 1/2$, where n is even and $\mu_1(r, n) = 1/3$, where n is odd. Also, $\mu_2(r, 3n) = 1/2$, where $3n$ is an even number and $\mu_2(r, 3n) = 1/3$, where $3n$ is an odd number. Then, $\varphi : R \oplus \mathbb{Z} \longrightarrow R \oplus 3\mathbb{Z}$, defined by $\varphi(r, n) = (r, 3n)$ is a homomorphism.

3 Quotient Γ -hypermodules induced by fuzzy Γ -subhypermodules

In this paragraph, we analyze the equivalence relation μ^* defined by X. Ma et. al. [18] and J. Zhan et. al.[25]. Also, we define the relation $\mu^*[N]$ on Γ -hypermodules(hypermodules) as a generalization of μ^* . Also, we show that all quotient Γ -hypermodules and quotient hypermodules considered in [18, 25] are Γ -modules and modules, respectively and the relation μ^* is a strongly regular relation.

A fuzzy Γ -subhypermodule μ of M is called *normal* if $\mu(z) \leq \inf_{z \in x+y-x} \mu(z)$, for all $x, y \in M$.

Let μ be a fuzzy Γ -subhypermodule of M and equivalence relation μ^* defined as follows:

$$x \equiv y(mod(\mu)) \iff \exists r \in x - y : \mu(r) = \mu(0),$$

denote this relation by μ^* . The equivalence class containing the element $x \in M$ denoted by $\mu^*[x]$ and the set of all equivalence classes denoted by M/μ^* . On the other hand, since reproduction axioms hold for the additive law in M , $r \in x - y$ implies that $y \in x - r$. Hence

$$\mu^*[x] = \{y \in M : \exists r \in M, r \in x - y, \mu(r) = \mu(0)\} = \bigcup_{r \in M, \mu(r)=0} x - r.$$

The relation μ^* define by J. Zhan and et. al. [25] on hypermodules, when μ is normal. They prove that when μ is normal the relation μ^* is equivalence. But in the Proposition 3.1, we prove that μ^* is an equivalence relation, when the fuzzy hypermodule μ is not normal.

Proposition 3.1. *Let M be a hypermodule and μ be a fuzzy subhypermodule. Then, the relation μ^* is equivalence.*

Proof. Since for every $x \in M$, $0 \in x - x$, then the relation μ^* is reflexive. If $x\mu^*y$, then there exists $r \in x - y$ such that $\mu(r) = \mu(0)$. It follows that $y\mu^*x$. Since $-r \in y - x$ and $\mu(-r) = \mu(r)$. Thus, μ^* is symmetric. To prove that μ^* is transitive, let $x\mu^*y$ and $y\mu^*z$. Then, there exist $r_1 \in x - y$ and $r_2 \in y - z$ such that $\mu(r_1) = \mu(r_2) = 0$. Hence, $r_1 \in (x - z) - r_2$ and there exists $t \in x - z$ such that $r_1 \in t - r_2$. Since, $t \in r_1 + r_2$ and $\min\{\mu(r_1), \mu(r_2)\} \leq \mu(t)$. This implies that $\mu(t) = 0$. Consequently, $t \in x - z$ and $\mu(t) = 0$ and so $x\mu^*z$. This completes the proof. $\square$

Let M be a Γ -hypermodule and ρ be an equivalence relation on M . Then, for every nonempty subset A and B

$$A\bar{\rho}B \iff \forall a \in A, \exists b \in B : a\rho b, \quad \forall b \in B, \exists a \in A : a\rho b.$$

$$A\bar{\bar{\rho}}B \iff \forall a \in A, b \in B, \quad a\rho b.$$

An equivalence relation ρ on Γ -hypermodule M is called *regular*, when

$$x_1\rho x_2 \implies \forall x \in M, r \in R, \alpha \in \Gamma, \text{ implies that } (x_1+x)\bar{\rho}(x_2+x), (r\alpha x_1)\rho(r\alpha x_2).$$

Also, ρ is called *strongly regular*,

$$x_1\rho x_2 \implies \forall x \in M, r \in R, \alpha \in \Gamma, (x_1+x)\bar{\bar{\rho}}(x_2+x), (r\alpha x_1)\rho(r\alpha x_2).$$

Proposition 3.2. *Let M be a Γ -hypermodule and ρ be a regular relation on M . Then, M/ρ is a Γ -hypermodule (Γ -module) with respect the following relation if and only if the relation ρ is a regular (strongly regular):*

$$\rho(x) \oplus \rho(y) = \{\rho(z) : z \in x + y\},$$

$$r \odot_{\alpha} \rho(x) = \rho(r\alpha x),$$

where $x, y \in M$ and $\alpha \in \Gamma$.

Proof. Suppose that ρ is a regular relation and $\rho(x_1) = \rho(x_2)$ and $\rho(y_1) = \rho(y_2)$. Thus, $(x_1 + y_1)\bar{\rho}(x_2 + y_1)$ and $(x_2 + y_1)\bar{\rho}(x_2 + y_2)$. Since ρ is regular, for every $z_1 \in x_1 + y_1$ there exists $z_2 \in x_2 + y_2$ such that $\rho(z_1) = \rho(z_2)$. Hence $\rho(x_1) \oplus \rho(y_1) \subseteq \rho(x_2) \oplus \rho(y_2)$. In a same way, we can see that $\rho(x_2) \oplus \rho(y_2) \subseteq \rho(x_1) \oplus \rho(y_1)$. The remaining axioms are easy to verify and we conclude that M/ρ is a Γ -hypermodule. Let ρ be a strongly regular and $\rho(x), \rho(y) \in M/\rho$ and $z_1, z_2 \in x + y$. Since ρ is strongly regular implies that $\rho(z_1) = \rho(z_2)$. Thus, the hyperoperation $\oplus$ is an operation. Hence M is a Γ -module. $\square$

Let M be a Γ -hypermodule and N be a Γ -subhypermodule and μ be a fuzzy Γ -subhypermodule. Then, we define the relation

$$x \equiv y(\text{mod} N) \iff \exists n \in N, \quad x \in y + n, \quad \mu(n) = \mu(0).$$

This relation is denoted by $\mu^*[N]$.

Proposition 3.3. *Let M be a Γ -hypermodule and μ be a fuzzy Γ -subhypermodule. Then, $\mu^*[N]$ is an equivalence relation.*

Proof. Since $x = x + 0$ and $0 \in N$, implies that $x\mu^*[N]x$. Then, $\mu^*[N]$ is reflexive. Let $x\mu^*[N]y$. Then, there exists $n \in N$ such that $x \in y + n$ and $\mu(n) = \mu(0)$ which implies that $y \in x - n$. Since, N is a Γ -subhypermodule and $\mu(n) = \mu(-n)$ implies that $y\mu^*[N]x$ and so $\mu^*[N]$ is symmetric. Let $x\mu^*[N]y$ and $y\mu^*[N]z$. Then, there exist $n_1, n_2 \in N$ such that $x \in y + n_1$ and $y \in z + n_2$ and $\mu(n_1) = \mu(n_2) = \mu(0)$. So, $x \in y + n_1 \subseteq z + n_1 + n_2$. Thus, there exist $n \in n_1 + n_2 \subseteq N$ such that $x \in z + n$. Also, $n \in n_1 + n_2$ implies that $\min\{\mu(n_1), \mu(n_2)\} \leq \mu(n)$. Hence $\mu(n) = \mu(0)$ and $x\mu^*[N]z$. Therefore, the relation $\mu^*[N]$ is an equivalence. $\square$

Theorem 3.4. *Let M be a Γ -hypermodule, N be a Γ -subhypermodule and μ be a fuzzy Γ -hypermodule. Then, the set of all classes $M/\mu_N^* = \{\mu_N^*(x) : x \in M\}$ is a Γ -hypermodule as follows:*

$$\mu_N^*(x) \oplus \mu_N^*(y) = \{\mu_N^*(z) : z \in x + y\}.$$

$$r \odot \mu_N^*(x) = \mu_N^*(rx).$$

Proof. The proof is similar to 3.14. $\square$

Theorem 3.5. *Let M be a Γ -hypermodule, μ be a fuzzy Γ -subhypermodule and N be a Γ -subhypermodule of M . Then, M/μ_N^* is a fuzzy Γ -subhypermodule.*

Proof. By Theorem 3.4, M/μ_N^* is a Γ -hypermodule. Assume that $\hat{\mu}$ is a fuzzy set on M/μ_N^* defined by $\hat{\mu}(\mu_N^*(x)) = \mu(x)$, where $x \in M$. Let $\mu_N^*(x) = \mu_N^*(y)$. Then, there exists $n \in N$ such that $x \in y + n$. Thus, $\min\{\mu(y), \mu(n)\} \leq \mu(x)$. Since, $\mu(n) = \mu(0)$, implies that $\mu(y) \leq \mu(x)$. In a same way, $\mu(x) \leq \mu(y)$. Then, $\mu(x) = \mu(y)$ and $\hat{\mu}$ is well-defined. It is easy to see that the other properties holds. $\square$

Proposition 3.6. *Let M be a Γ -hypermodule, N_1 and N_2 be Γ -subhypermodule, $\mu_{N_1}^*, \mu_{N_2}^*, \mu_{N_1 \cap N_2}^*, \mu_{N_1 + N_2}^*$ be equivalence relations on $N_1, N_2, N_1 \cap N_2, N_1 + N_2$, respectively. Then,*

$$(i) \mu_{N_1 \cap N_2}^* \subseteq \mu_{N_1}^* \cap \mu_{N_2}^*,$$

$$(ii) \mu_{N_1}^* \cup \mu_{N_2}^* \subseteq \mu_{N_1 + N_2}^*.$$

Proof. The proof is straightforward. $\square$

Definition 3.7. Let M_1 and M_2 be Γ -hypermodules and μ fuzzy Γ -subhypermodule on M_1 . Then, the *kernel* of φ is the set $\ker \varphi = \{x \in M_1 : \varphi(x) = 0, \mu(x) = \mu(0)\}$.

EXAMPLE 7. The kernel of homomorphism defined in the Example 6, is $\{(0, 0)\}$.

Proposition 3.8. Let M_1 and M_2 be Γ -hypermodules and μ fuzzy Γ -hypermodule on M_1 . Then, the kernel of φ is a Γ -subhypermodule of M .

Proof. Since $0 \in \ker \varphi$, implies that kernel of φ is a nonempty set. Let $x, y \in \ker \varphi$ and $r \in R$. This implies that $\varphi(x + y) = \varphi(x) + \varphi(y) = 0$ and $0 = \mu(x) \leq \mu(r\alpha x)$. Hence $\mu(r\alpha x) = 0$ and $x + y \subseteq \ker \varphi$. Therefore, $\ker \varphi$ is a Γ -subhypermodule of M . $\square$

Theorem 3.9. Let $\varphi : M_1 \longrightarrow M_2$ be a homomorphism with the kernel K and μ be a fuzzy Γ -hypermodule on M_1 . Then, $M_1/\mu_{1K}^* \simeq \text{Im} \varphi$.

Proof. Suppose that $\psi : M_1/\mu_K^* \longrightarrow \text{Im} \varphi$ defined by $\psi(\mu_K^*(x)) = \varphi(x)$, for $x \in M_1$. Let $\mu_K^*(x) = \mu_K^*(y)$. Then, there exists $k \in K$ such that $x \in y + k$. Thus, $\varphi(x) \in \varphi(y + k) = \varphi(y) + \varphi(k) = \{\varphi(y)\}$. Consequently, $\varphi(x) = \varphi(y)$. Clearly, ψ is onto. To show that ψ is one to one, suppose that $\varphi(x) = \varphi(y)$. Then, $0 \in \varphi(x - y)$, and so there exists $z \in x - y$ with $\varphi(z) = 0$. Thus, $x \in y + z$ and $\mu_2 \circ \varphi(z) = \mu_1(z)$ and $\mu_2(0) = \mu_1(z)$. Also, $\mu_2 \circ \varphi(0) = \mu_1(0)$ implies that $\mu_2(0) = \mu_1(0)$. Hence $\mu_1(z) = \mu_1(0)$ and $z \in \ker \varphi$. Then, $\mu_K^*(x) = \mu_K^*(y)$ and ψ is one to one. By Proposition 3.21, M_1/μ_K^* is a fuzzy Γ -subhypermodule. Obviously, $\mu_2 \circ \psi = \hat{\mu}_1$. This completes the proof. $\square$

Theorem 3.10. Let N_1 and N_2 be Γ -subhypermodules of M . Then,

$$N_1/\mu_{N_1 \cap N_2}^* \simeq (N_1 + N_2)/\mu_{N_2}^*.$$

Proof. Suppose that $\varphi : N_1 \longrightarrow (N_1 + N_2)/\mu_{N_2}^*$ defined by $\varphi(n_1) = \mu_{N_2}^*(n_1)$. Since $n_1 \in N_1 + N_2$, implies that φ is well-defined. Let $\mu_{N_2}^*(x) \in (N_1 + N_2)/\mu_{N_2}^*$. Then, there exist $n_1 \in N_1$ and $n_2 \in N_2$ such that $x \in n_1 + n_2$. Thus, $\mu_{N_2}^*(x) = \mu_{N_2}^*(n_1)$ and $\varphi(\mu_{N_2}^*(n_1)) = \mu_{N_2}^*(x)$. Hence the map φ is onto. Also,

$$x \in \mu_{N_2}^*(n_1) \iff \exists n_2 \in N_2, x \in n_1 + n_2.$$

Since $n_2 \in x - n_1$, implies that $n_2 \in N_1 \cap N_2$. By Theorem 3.9

$N_1/\mu_{N_1 \cap K_2}^* \simeq (N_1 + N_2)/\mu_{N_2}^*$. This completes the proof. $\square$

Let M be a Γ -hypermodule and N_1 and N_2 be Γ -subhypermodules such that $N_1 \subseteq N_2$. Then,

$$\mu_{N_2}^*/\mu_{N_1}^* = \{(\mu_{N_1}^*(x), \mu_{N_1}^*(y)) : \mu_{N_2}^*(x) = \mu_{N_2}^*(y)\}.$$

Then, $\mu_{N_2}^*/\mu_{N_1}^*$ is a regular relation on $[M : \mu_{N_1}^*]$

Theorem 3.11. *Let M be a Γ -hypermodule and N_1, N_2 be Γ -subsemihypergroup of M such that $N_1 \subseteq N_2$. Then,*

$$(M/\mu_{N_1}^*)/(\mu_{N_2}^*/\mu_{N_1}^*) \cong M/\mu_{N_2}^*.$$

Proof. Suppose that $\varphi : M/\mu_{N_1}^* \longrightarrow M/\mu_{N_2}^*$ defined by $\varphi(\mu_{N_1}^*(m)) = \mu_{N_2}^*(m)$. Let $\mu_{N_1}^*(m_1) = \mu_{N_1}^*(m_2)$. Then, there exist $n_1 \in N_1$ such that $m_1 \in m_2 + n_1$. Since $N_1 \subseteq N_2$, $\mu_{N_2}^*(m_1) = \mu_{N_2}^*(m_2)$. Hence φ is well-defined. Obviously, φ is a homomorphism. Let K be a kernel of φ . Then,

$$\ker \varphi = \mu_{N_2}^*/\mu_{N_1}^*.$$

Thus, by Theorem 3.9, proof complete. $\square$

Proposition 3.12. *Let M be a Γ -hypermodule and μ be a fuzzy Γ -subhypermodule. Then, following sets are equal.*

$$\Delta_1 = \{\mu^*(z) : z \in x + y\}, \quad \Delta_2 = \{\mu^*(z) : z \in \mu^*(x) + \mu^*(y)\},$$

$$\Delta_3 = \{\mu^*(z) : \mu^*(z) \subseteq \mu^*(x) + \mu^*(y)\}.$$

Proof. Obviously, $\Delta_1 \subseteq \Delta_2$. Let $\mu^*(z) \in \Delta_2$. Then, $z \in \mu^*(x) + \mu^*(y)$ and for some $x_1 \in \mu^*(x)$ and $y_1 \in \mu^*(y)$ such that $z \in x_1 + y_1$. This implies that there are $r_1, r_2 \in M$ such that $z \in x + y - (r_1 + r_2)$, $\mu(r_1) = \mu(r_2) = \mu(0)$. Then, for some $z_1 \in x + y$ and $r_3 \in r_1 + r_2$ such that $z \in z_1 - r_3$. Hence $\min\{\mu(r_1), \mu(r_2)\} \leq \mu(r_3)$. This implies that $\mu(r_3) = \mu(0)$ and $\mu^*(z) = \mu^*(z_1)$. Hence $\Delta_2 \subseteq \Delta_1$. Let $\mu^*(z) \in \Delta_3$. Then, $\mu^*(z) \subseteq \mu^*(x) + \mu^*(y)$. This implies that for some $r_1, r_2 \in M$ such that $z \in x + y - (r_1 + r_2)$ where $\mu(r_1) = \mu(r_2) = \mu(0)$. Hence for some $t \in x + y$ and $r_3 \in r_1 + r_2$, $z \in t - r_3$. On the other hands $\min\{\mu(r_1), \mu(r_2)\} \leq \mu(r_3)$ implies that $\mu(r_3) = \mu(0)$. Then, $\mu^*(z) = \mu^*(t)$. Hence, $\Delta_3 \subseteq \Delta_1$. Assume that $\mu^*(z) \in \Delta_2$. Then, $z \in \mu^*(x) + \mu^*(y)$ and for some $t \in x + y$ and $r \in r_1 + r_2$, $z \in t - r$, where $\mu(r_1) = \mu(r_2) = \mu(0)$. Let $w \in \mu^*(z)$. Then, for some $r_1 \in M$, $w \in z - r_1$. This implies that $w \in t - r - r_1 \subseteq x + y - (r + r_1) = (x - r) + (y - r_1)$. Hence there are $z_1 \in x - r$ and $z_2 \in y - r_1$ such that $w \in z_1 + z_2 \subseteq \mu^*(x) + \mu^*(y)$. Then, $\mu^*(z) \subseteq \mu^*(x) + \mu^*(y)$. Therefore, $\Delta_2 \subseteq \Delta_3 \subseteq \Delta_1$. This complete the proof. $\square$

Proposition 3.13. [18] *Let M be a Γ -hypermodule. Then, M/μ^* is a Γ -hypermodule by following hyperoperation and scaler operation:*

$$\mu^*[x] \oplus \mu^*[y] = \{\mu^*(z) : z \in \mu^*[x] + \mu^*[y]\}, \quad r \odot_\alpha \mu^*[x] = \mu^*[r\alpha x],$$

where $\mu^*[x], \mu^*[y] \in M/\mu^*$ and $r \in R$.

By Proposition 3.12, the hyperoperation defined in Theorem 3.14, and the hyperoperation defined in Proposition 3.13 are equals. Hence we have the following theorem.

Theorem 3.14. *Let M be a Γ -hypermodule and μ be a fuzzy Γ -subhypermodule. Then, M/μ^* is a Γ -hypermodule with respect the following hyperoperation and scaler operation:*

$$\mu^*(x) \oplus \mu^*(y) = \{\mu^*(z) : z \in x + y\}.$$

$$r \odot_\alpha \mu^*(x) = \mu^*(r\alpha x),$$

where $r \in R$ and $\mu^*(x), \mu^*(y) \in M/\mu^*$.

Proof. Obviously, the hyperoperation $\oplus$ is well-defined. Let $\mu^*(x), \mu^*(y), \mu^*(z) \in M/\mu^*$ and $\mu^*(u) \in (\mu^*(x) \oplus \mu^*(y)) \oplus \mu^*(z)$. Then, for some $\mu^*(w) \in \mu^*(x) \oplus \mu^*(y)$, we have $\mu^*(u) \in \mu^*(w) \oplus \mu^*(z)$. This implies that $\mu^*(u) = \mu^*(u_1)$, where $u_1 \in w + z$ and $\mu^*(w) = \mu^*(w_1)$, where $w_1 \in x + y$. Hence $u \in (x + y) + z = x + (y + z)$. So, $u \in x + t$ for some $t \in y + z$. Since $\mu^*(t) \in \mu^*(x) \oplus \mu^*(y)$, we have $\mu^*(u) = \mu^*(x) \oplus \mu^*(t) \subseteq \mu^*(x) \oplus (\mu^*(x) \oplus \mu^*(z))$. This means that $\mu^*(u) \in \mu^*(x) \oplus (\mu^*(y) \oplus \mu^*(z))$. Hence $(\mu^*(x) \oplus \mu^*(y)) \oplus \mu^*(z) \subseteq \mu^*(x) \oplus (\mu^*(y) \oplus \mu^*(z))$. Similarly, we can see $\mu^*(x) \oplus (\mu^*(y) \oplus \mu^*(z)) \subseteq (\mu^*(x) \oplus \mu^*(y)) \oplus \mu^*(z)$. Thus, the hyperoperation $\oplus$ is associative. Consider the element $\mu^*(0)$. Then, for every $\mu^*(x)$, we have

$$\mu^*(x) \oplus \mu^*(0) = \{\mu^*(z) : z \in x + 0\} = \mu^*(x).$$

In a same way, we can see $\mu^*(0) \oplus \mu^*(x) = \mu^*(x)$. Since for every $x \in M$, we get $0 \in x - x$. This implies that, $\mu^*(0) \in \mu^*(x) \oplus \mu^*(-x)$. Let $\mu^*(x), \mu^*(y), \mu^*(z) \in M/\mu^*$ and $\mu^*(x) \in \mu^*(y) \oplus \mu^*(z)$. This implies that $x \in y + z$. By the reversible property in M , we have $y \in x - z$ and $z \in x - y$. Thus, $\mu^*(y) \in \mu^*(x) \oplus \mu^*(-z)$ and $\mu^*(z) \in \mu^*(x) \oplus \mu^*(-y)$. Since H is commutative, it is obvious that M/μ^* is also commutative. Thus, M/μ^* is a canonical hypergroup. We can obtain the other properties. $\square$

Definition 3.15. Let M be a Γ -subhypermodule and N be a nonempty subset of M . Then, we define $\Omega_N = \{x \in M : \inf \mu(n)_{n \in N} \leq \inf \mu(z)_{z \in x - x}\}$.

When $N = \{0\}$, we denote $\Omega_{\{0\}}$ by Ω_0 .

Proposition 3.16. *Let M be a Γ -hypermodule and N_1, N_2 be nonempty subsets of M such that $N_1 \subseteq N_2$. Then, $\Omega_{N_1} \subseteq \Omega_{N_2}$.*

Proof. The proof is straightforward. $\square$

Theorem 3.17. *Let M be a Γ -hypermodule and N be a nonempty subset of M . Then, Ω_N is a subcanonical hypergroup of M and contain N .*

Proof. Suppose that $x = 0$. Then, $\inf \mu(z)_{z \in x-x} = \mu(0)$. Since for every $n \in N$, $\mu(n) \leq \mu(0)$, we have $\inf \mu(n)_{n \in N} \leq \mu(0)$. Hence Ω_N is a nonempty set. Let $x, y \in \Omega_N$. Then, $\inf \mu(n)_{n \in N} \leq \inf \mu(z)_{z \in x-x}$ and $\inf \mu(n)_{n \in N} \leq \inf \mu(z)_{z \in y-y}$. Hence for every $w \in x+y$ and $z \in w-w$ there exist $z_1 \in x-x$ and $z_2 \in y-y$ such that $z \in z_1 + z_2$. Thus, $\min\{\mu(z_1), \mu(z_2)\} \leq \mu(z)$ and so $\inf \mu(n)_{n \in N} \leq \mu(z)$. Since z is an arbitrary element, $\inf \mu(n)_{n \in N} \leq \inf \mu(z)_{z \in w-w}$. Thus, $x + y \subseteq \Omega_N$ and Ω_N is a subcanonical hypergroup. Let $n \in N$. Then, for every $z \in n - n$, $\min\{\mu(n), \mu(-n)\} \leq \mu(z)$. This implies that $\inf \mu(n)_{n \in N} \leq \inf \mu(z)_{z \in n-n}$. Hence $N \subseteq \Omega_N$. This implies that proof. $\square$

Proposition 3.18. *Let M be a Γ -hypermodule. Then, the hyperoperation defined on M/μ^* is an operation on Ω_0 and Ω_0 containing all subgroup of M .*

Proof. Suppose that $x, y \in \Omega_0$ and $a, b \in x + y$. Then, $\inf_{z \in x-x} \mu(z) = \mu(0)$ and $\inf_{z \in y-y} \mu(z) = \mu(0)$. Let $z \in a - b \subseteq (x - x) + (y - y)$. Then, there are $z_1 \in x - x$ and $z_2 \in y - y$ such that $z \in z_1 + z_2$. This implies that $\min\{\mu(z_1), \mu(z_2)\} \leq \mu(z)$ and $\mu(0) \leq \mu(z)$. Also, $z \in a - b$ implies that $a \in z + b$ and $\min\{\mu(z), \mu(b)\} \leq \mu(a)$. Since $0 \in z - z$, it follows that $\min\{\mu(z), \mu(-z)\} \leq \mu(0)$. Hence $\mu(z) \leq \mu(0)$. Then, $\mu(b) \leq \mu(a)$. In a same way, it is proved that $\mu(b) \leq \mu(a)$. Hence $\mu^*(x) \oplus \mu^*(y)$ is a singleton set for all $x, y \in \Omega_0$. Let M_1 is any subgroup of M and $x \in M_1$. Then, $\inf \mu(z)_{z \in x-x} = \mu(0)$. Thus $M_1 \subseteq \Omega_0$. $\square$

Corollary 3.19. *Let M be a Γ -hypermodule. Then, $(M, +)$ is a group if and only if $\Omega_0 = M$.*

Now, we determine some necessary and sufficient conditions so that a fuzzy Γ -subhypermodule is normal.

Theorem 3.20. *Let M be a Γ -hypermodule and μ be a fuzzy Γ -subhypermodule of M . Then, μ is normal if and only if $\Omega_0 = M$.*

Proof. Suppose that μ is normal. Then, for every $y \in M$, we have $\mu(y) \leq \inf \mu(z)_{z \in x+y-x}$. Thus, for $y = 0$, we get $\mu(0) \leq \inf \mu(z)_{z \in x-x}$. Since for every $z \in x - x$, $\mu(z) \leq \mu(0)$, hence $\inf \mu(z)_{z \in x-x} = \mu(0)$. Thus, $\Omega_0 = M$. Conversely, assume that $\Omega_0 = M$. Then, for every $x \in M$, $\mu(0) = \inf \mu(z)_{z \in x-x}$. This implies that

$$\inf \mu(z)_{z \in x+y-x} = \inf \mu(z)_{z \in z_1+y, z_1 \in x-x} \geq \min\{\mu(z_1), \mu(y)\} \geq \mu(y)$$

Thus, μ is normal. This completes the proof. $\square$

Corollary 3.21. Let M be a Γ -hypermodule and μ be a fuzzy Γ -subhypermodule. Then, $(M/\mu^*, +)$ is an abelian group if and only if the fuzzy Γ -subhypermodule μ is normal.

Let μ be a normal fuzzy Γ -subhypermodule on M . Then, by Corollary 3.21, the relation μ^* defined by J. Zhan and et. al [25], is a strong regular equivalence relation on canonical hypergroup $(M, +)$. Hence the quotient set $(M/\mu^*, \oplus)$ is an abelian group. Similarly, the relation μ^* defined by X. Ma [18] is strongly regular relation. Similarly, we can see that the relation μ^* defined by J. Zhan [25] is a strongly regular relation.

Corollary 3.22. Let μ_1 and μ_2 be fuzzy Γ -subhypermodules such that $\mu_1 \leq \mu_2$ and μ_1 be normal. Then, μ_2 is also normal.

Corollary 3.23. Let M be a Γ -hypermodule, μ be a normal fuzzy Γ -hypermodule. Then, for every Γ -subhypermodule N , $\Omega_N = M$.

4 Quotient soft Γ -hypermodules

The concept of Γ -hypermodules introduce by Zhan et.al.[24]. They, construct three isomorphism theorems of Γ -hypermodules. They, construct quotient Γ -hypermodules by normal Γ -subhypermodules. In this section, we consider the equivalence relation defined by Zhan et.al. [24], is a strongly regular relation. Hence, all quotient Γ -subhypermodules are only Γ -modules. Also, all quotient soft Γ -hypermodules defined by Ma et.al. [18], are soft Γ -modules.

Let N be a Γ -subhypermodule of M . Then, we define the relation on M as follows:

$$x \equiv y \iff (x - y) \cap N \neq \emptyset.$$

This relation denoted by xN^*y . In this section all result are valid when M is a hypermodule.

In Lemma 4.1, we prove that the relation N^* defined on Γ -hypermodules defined by X. Ma [18] is an equivalence relation but N is not normal. Also, we prove that when N is normal the relation N^* is strongly regular and the hyperaddition defined on quotient hypermodule is an addition. Then, these quotient Γ -hypermodules are Γ -modules.

Lemma 4.1. *Let N be a Γ -subhypermodule of M . Then, the relation N^* is an equivalence.*

Proof. Since $0 \in (x - x) \cap N$ for all $x \in M$, it follows xN^*x , ie., N^* is reflexive. Let xN^*y , then there exists $z \in (x - y) \cap N$, which means $-z \in y - x$ and $-z \in N$, which means that yN^*x and so N^* is symmetric. Let xN^*y and yN^*z , where $x, y, z \in M$. Then, there exist $a \in (x - y) \cap N$ and $b \in (y - z) \cap N$. So, $x \in a + y$ and $y \in b + z$, which implies that $x \in a + b + z$. Hence there exist $c \in a + b$ such that $x \in c + z$. Since $c \in a + b \subseteq N$ and $c \in x - z$, it follows that $(x - z) \cap N \neq \emptyset$. Therefore, N^* is transitive. $\square$

Definition 4.2. Let M be a Γ -hypermodule and N be a Γ -subhypermodule of M . Then, we define *normal closure* by

$$Cl(N) = \{m \in M : \exists n \in N, m - m \subseteq N\}.$$

Proposition 4.3. *Let M be a Γ -hypermodule and N be a Γ -subhypermodule of M . Then, $Cl(N)$ is a Γ -subhypermodule of M .*

Proof. Since $0 \in Cl(N)$, implies that $Cl(N)$ is a nonempty subset. Let $a, b \in Cl(N)$ and $r \in R$. Then, for every $x \in a - b$ and $t \in x - x$, we have

$$t - t \subseteq (a - b) - (a - b) = (a - a) - (b - b) \subseteq N,$$

$$rat - rat = r\alpha(t - t) = r\alpha((a - a) - (b - b)) \subseteq N.$$

This implies that $Cl(N)$ is a Γ -subhypermodule. $\square$

Proposition 4.4. *Let M be a Γ -hypermodule and N be a Γ -subhypermodule. Then, N is normal if and only if $Cl(N) = M$.*

Proof. Suppose that N is normal and $m \in M$. Then, $m - m \subseteq m + N - m \subseteq N$. Hence $Cl(N) = M$. Conversely, assume that $Cl(N) = M$. Thus, for every $m \in M$, we have $m + N - m = m - m + N \subseteq N + N \subseteq N$. $\square$

Proposition 4.5. *Let M be a Γ -hypermodule and N_1, N_2 be Γ -subhypermodules of M . Then, $Cl(N_1) \subseteq Cl(N_2)$.*

Proof. The proof is straightforward. $\square$

Proposition 4.6. *Let M be a Γ -hypermodule and N be a normal Γ -subhypermodule. Then, Ω_N is also normal.*

Proof. By Proposition 3.17, Ω_N contain N and by Proposition 4.5, $Cl(N) \subseteq Cl(\Omega_N)$. Since N is normal, by Proposition 4.4, $Cl(N) = M$. This implies that $Cl(\Omega_N) = M$. Therefore, Ω_N is normal. $\square$

Definition 4.7. Let M be a Γ -hypermodule. Then, we define

$$S(M) = \left\{ x \in M : \exists n \in \mathbb{N}, x_i \in M, x \in \sum_{i=1}^n (x_i - x_i) \right\}.$$

Proposition 4.8. *Let M be a Γ -hypermodule. Then, $S(M)$ is a normal Γ -subhypermodule. Also, a Γ -subhypermodule N is normal if and only if contain $S(M)$.*

Proof. Suppose that $x, y \in S(M)$, $r \in R$ and $\alpha \in \Gamma$. Then, $x \in \sum_{i=1}^n (x_i - x_i)$ and $y \in \sum_{j=1}^m (y_j - y_j)$, for some $x_i, y_j \in M$. Thus, $x - y \subseteq \sum_{i=1}^n (x_i - x_i) + \sum_{j=1}^m (y_j - y_j) \subseteq S(M)$ and $r\alpha x \in \sum_{i=1}^n (r\alpha x_i - r\alpha x_i) \in S(M)$. This implies that $S(M)$ is closed with respect the hyperoperation and scalar operation. Thus, $S(M)$ is a Γ -subhypermodule. Assume that Γ -subhypermodule N contain $S(M)$ and $x \in M$. Then, $x + N - x = x - x + N \subseteq S(M) + N \subseteq N$. This implies that N is normal. Conversely, let N is normal Γ -hypermodule. Then, for every $x \in M$, $x + N - x \subseteq N$. Thus, $x - x \subseteq x - x + N = x + N - x \subseteq N$. Since N is closed with respect the hyperoperation, we have $S(M) \subseteq N$. $\square$

Corollary 4.9. Let M be a Γ -hypermodule. Then, $S(M)$ is a smallest normal Γ -hypermodule of M .

Proposition 4.10. *Let $(M, +)$ be a Γ -hypermodule. Then, $Cl(\{0\})$ is a subgroup of $(M, +)$. In other word, the hyperoperation $+$ defined on $Cl(\{0\})$ is an operation.*

Proof. By the Proposition 4.3, $Cl(\{0\})$ is a Γ -subhypermodule of M . We prove that the hyperoperation $+$ is a operation on $Cl(\{0\})$. Let $x, y \in Cl(\{0\})$ and $a, b \in x + y$. Then, $a - b \subseteq (x + y) - (x + y) = (x - x) + (y - y) = 0 + 0$. Thus, $a = b$. Hence for every $x, y \in Cl(\{0\})$, $x + y$ has only one element. This completes the proof. $\square$

Proposition 4.11. *Let N_1 and N_2 be Γ -subhypermodules of M and $N_1 \subseteq N_2$ such that N_1 is normal. Then, N_2 is also normal.*

Proof. Since $N_1 \subseteq N_2$, we have $Cl(N_1) \subseteq Cl(N_2)$. By Proposition 4.4, $Cl(N_1) = M$. Hence $Cl(N_2) = M$ and N_2 is normal. $\square$

Corollary 4.12. Let N_1 and N_2 be Γ -subhypermodules of M . Then,

- (i) if $N_1 \cap N_2$ are normal, then N_1 and N_2 are also normal,
- (ii) if N_1 is normal, the $N_1 + N_2$ is also normal.

Proof. By the Proposition 4.11, the proof is straightforward. $\square$

Corollary 4.13. Let M be a Γ -hypermodule such that $\{0\}$ is normal. Then, every Γ -subhypermodules are normal.

Theorem 4.14. *Let M be a Γ -hypermodule and N be a normal Γ -subhypermodule of M . Then, M/N^* is a Γ -module.*

Proof. Obviously, the zero element quotient Γ -hypermodule is N . By Proposition 4.4, $Cl(N) = M$. Therefore, by Proposition 4.10, $Cl(N)$ is a Γ -module. This completes the proof. $\square$

REMARK 1. Let M be a Γ -hypermodule (hypermodule) and N be Γ -subhypermodule (hypermodule) of M . Then, M/N^* is also a Γ -hypermodule(hypermodule). Also, when N is a normal Γ -subhypermodules (hypermodule)of M by Theorem 4.14, M/N^* is an abelian group and is not canonical hypergroup. Hence the additive hyperoperation defined on quotient soft Γ -hypermodules(hypermodule) considered in [18], are just operation.

Conflict of interest

The authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants performed by any of the authors.

Informed consent

Informed consent was obtained from all individual participants included in the study.

Authorship contributions

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Sohrab Ostadhadi-Dehkordi, Thomas Vougiouklis and Bijan Davvaz. The first draft of the manuscript was written by Sohrab Ostadhadi-Dehkordi and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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