An algorithm for finding a minimal solution to fuzzy relation inequalities with addition-min composition

Xue-Ping Wang (✉ xpwang1@hotmail.com)

Sichuan Normal University <https://orcid.org/0000-0003-4376-9859>

Xiao Mi

Sichuan Normal University

Research Article

Keywords: Fuzzy relational inequality, Addition-min composition, Minimal solution, Algorithm

Posted Date: March 8th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-581893/v2>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Noname manuscript No. (will be inserted by the editor)
--

An algorithm for finding a minimal solution to fuzzy relation inequalities with addition-min composition

Xue-ping Wang · Xiao Mi

Received: date / Accepted: date

Abstract We first show a sufficient and necessary condition that a solution of fuzzy relation inequalities with addition-min composition is a minimal one. We then prove that for every solution of the fuzzy relation inequalities there exists a minimal solution that is less than or equal to the solution in a very different way. We finally give an algorithm to find a minimal solution for a given solution, which is illustrated by numerical examples.

Keywords Fuzzy relational inequality; Addition-min composition; Minimal solution; Algorithm

1 Introduction

The fuzzy relation equation was first introduced by Sanchez [9]. Since then, fuzzy relation equations or inequalities with different kinds of composition operators have been investigated (see, e.g., [3]). In general, the solution set of fuzzy relation equations or inequalities with sup-* composition is completely characterized by the greatest solution and a finite number of minimal solutions, where * is a continuous conjunctive operator (see, e.g., [3, 8]), so minimal solutions play a significant role in obtaining the solution set. It is well known that the data transmission mechanism in BitTorrent-like peer-to-peer file-sharing systems is reduced to the system of fuzzy relation inequations with

Xue-ping Wang (Corresponding author)
School of Mathematical Sciences, Sichuan Normal University, Chengdu 610066, P. R. China
Tel.: +86-28-84761393
Fax: +86-28-84761393
E-mail: xpwang1@hotmail.com

Xiao Mi
School of Mathematical Sciences, Sichuan Normal University, Chengdu 610066, P. R. China
E-mail: 727563547@qq.com

addition-min composition as follows:

$$\begin{cases} a_{11} \wedge x_1 + a_{12} \wedge x_2 + \cdots + a_{1n} \wedge x_n \geq b_1, \\ a_{21} \wedge x_1 + a_{22} \wedge x_2 + \cdots + a_{2n} \wedge x_n \geq b_2, \\ \cdots \\ a_{m1} \wedge x_1 + a_{m2} \wedge x_2 + \cdots + a_{mn} \wedge x_n \geq b_m, \end{cases} \quad (1)$$

where $a_{ij}, x_j \in [0, 1]$, $b_i > 0$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$), $a_{ij} \wedge x_j = \min\{a_{ij}, x_j\}$, and the operation “+” is the ordinary addition [6, 10].

Li and Yang [6] showed that the solution sets of system (1) depend on all minimal solutions, and they provided an algorithm to find a minimal solution. After the first paper, the resolution of system (1) and the related optimization problem attract many researchers’ attention. For example, Yang [10] defined the pseudominimal indexes of system (1) and gave an algorithm (called the *PMI* algorithm) to get the set of pseudominimal indexes. Moreover, by computing the pseudominimal indexes of system (1), he proved that every optimal solution of the optimizations with system (1) as constraints is a minimal solution of system (1) and proposed an algorithm to find the optimal solution. Guu and Wu [5] proved the existence of an uncountably infinite number of minimal solutions for system (1); therefore, from the computational viewpoint, Yang’s algorithm based on the *PMI* algorithm may be inefficient. Almost at the same time, Yang et al. [13] also verified that system (1) has an infinite number of minimal solutions if system (1) has more than one minimal solution and suggested an algorithm for obtaining the variable-ordering solutions of system (1), where a variable-ordering solution is a minimal solution as well. Yang [11] was also aware of the existence of an infinite number of minimal solutions and so he turned to discussing the uniqueness of minimal solutions and suggested an algorithm for finding minimal solutions by using the critical set of system (1). In particular, by applying Zorn’s lemma, Cao et al. [1] identified that every solution of system (1) is between a minimal solution and the greatest solution, so the solution set is completely determined by the greatest solution and their minimal solutions (see [1, Theorem 5]); that is, by using Zorn’s lemma, they proved that for every solution of system (1) there exists a minimal solution that is less than or equal to the solution. Recently, Li and Wang [7] pointed out that neither Yang’s algorithm (see [6, p. 455]) nor Yang’s *PMI* algorithm in [10] cannot find a minimal solution of a given solution. Furthermore, on the basis of both Lemma 4 and Theorem 4 in [6], they first directly got that for every solution of system (1) there exists a minimal solution that is less than or equal to the solution, and they then developed an algorithm to calculate one minimal solution for a given solution. Because of the important role of minimal solutions as illustrated above in solving both system (1) and the optimizations with system (1) as constraints (see, e.g., [4, 12] for an extended bibliography), this article further focuses on minimal solutions of system (1). In particular, we try to develop an algorithm for finding more than one minimal solution for a given solution of system (1) as possible.

The rest of this article is organized as follows. In Section 2, we recall some necessary definitions and previous results. In Section 3, we first present a suf-

efficient and necessary condition that a solution of system (1) is a minimal one. We then prove that for every solution of system (1) there exists a minimal solution that is less than or equal to the solution in a new way, and we finally give an algorithm to find a minimal solution for a given solution with computational complexity $O(n^3)$ or $O(m^3)$. A conclusion is drawn in Section 4.

2 Preliminaries

In this section, we present some concepts and existing results which will be used throughout the paper.

Let $I = \{1, 2, \dots, m\}$, and $J = \{1, 2, \dots, n\}$. For $x^1 = (x_1^1, x_2^1, \dots, x_n^1)$, $x^2 = (x_1^2, x_2^2, \dots, x_n^2) \in [0, 1]^n$, we say $x^1 \leq x^2$ if and only if $x_j^1 \leq x_j^2$ for all $j \in J$, and $x^1 < x^2$ if and only if $x^1 \leq x^2$ and there exists a $j \in J$ such that $x_j^1 < x_j^2$. Then system (1) can be tersely described as follows:

$$A \odot x^T \geq b^T$$

where $A = (a_{ij})_{m \times n}$, $x = (x_1, x_2, \dots, x_n)$, $b = (b_1, b_2, \dots, b_m)$ and

$$(a_{i1}, a_{i2}, \dots, a_{in}) \odot (x_1, x_2, \dots, x_n)^T = a_{i1} \wedge x_1 + a_{i2} \wedge x_2 + \dots + a_{in} \wedge x_n.$$

System (1) is solvable if there exists an $x \in [0, 1]^n$ such that $A \odot x^T \geq b^T$. Denote the solution set of system (1) by $X(A, b) = \{x \in [0, 1]^n \mid \sum_{j \in J} a_{ij} \wedge x_j \geq b_i, i \in I\}$. We need the following definition and lemmas.

Definition 1 ([3]) An $\hat{x} \in X(A, b)$ is called the greatest solution if $x \leq \hat{x}$ for all $x \in X(A, b)$. An $\check{x} \in X(A, b)$ is called a minimal solution if $x \leq \check{x}$ for any $x \in X(A, b)$, implies $x = \check{x}$. Denote the set of all minimal solutions of system (1) by $\check{X}(A, b)$.

Lemma 1 ([6]) (i) $X(A, b) \neq \emptyset$ if and only if $\sum_{j \in J} a_{ij} \geq b_i$ for any $i \in I$. (ii) If $x \in X(A, b)$ satisfies $x \leq x^*$, then $x^* \in X(A, b)$.

Lemma 2 ([6]) If $X(A, b) \neq \emptyset$, then $(1, 1, \dots, 1)$ is the greatest solution of system (1).

Lemma 3 ([6]) Let $x = (x_1, x_2, \dots, x_n) \in X(A, b)$. Then

- (1) $x \neq 0$;
- (2) For any $j \in J$, $x_j \geq b_i - \sum_{k \in J - \{j\}} a_{ik} \wedge x_k \geq b_i - \sum_{k \in J - \{j\}} a_{ik}$ for any $i \in I$;
- (3) For any $j \in J$, $a_{ij} \geq b_i - \sum_{k \in J - \{j\}} a_{ik} \wedge x_k$ for any $i \in I$.

Remark 1 In system (1), for any $i \in I, j \in J$, denote

$$\dot{a}_{ij} = \max\{0, b_i - \sum_{k \in J - \{j\}} a_{ik}\},$$

$\dot{a}_j = \max\{\dot{a}_{ij} \mid i \in I\}$ and $\dot{a} = (\dot{a}_1, \dot{a}_2, \dots, \dot{a}_n)$. Then it follows from Lemma 3 that if $x = (x_1, x_2, \dots, x_n) \in X(A, b)$ then $x_j \geq \dot{a}_j$ for any $j \in J$.

Lemma 4 ([10]) *System (1) has the unique minimal solution if and only if $\check{a} \in X(A, b)$. In particular, when system (1) has the unique minimal solution, $\check{a}$ is the unique minimal solution.*

3 The algorithm of minimal solutions

In this section, we shall show a sufficient and necessary condition that a solution of system (1) is a minimal one, and prove that every solution of system (1) has a minimal solution. Then we shall give an algorithm for finding a minimal solution of system (1) with computational complexity $O(n^3)$ or $O(m^3)$.

If $x = (x_1, x_2, \dots, x_n) \in X(A, b)$, then for any $j \in J$, let

$$H_j(x) = \{x_{*j} \in [0, 1] \mid \sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge x_{*j} \geq b_i \text{ for any } i \in I\}$$

and

$$\gamma_j(x) = \min\{x_{*j} \mid x_{*j} \in H_j(x)\} \text{ for any } j \in J.$$

Remark 2 Let $x = (x_1, x_2, \dots, x_n) \in X(A, b)$. Then it is easy to see that $x_j \in H_j(x)$, $H_j(x) = [\max\{\max\{b_i - \sum_{k \in J - \{j\}} a_{ik} \wedge x_k \mid i \in I\}, 0\}, 1]$ and $\max\{\max\{b_i - \sum_{k \in J - \{j\}} a_{ik} \wedge x_k \mid i \in I\}, 0\} = \check{a}_j$ for any $j \in J$.

Proposition 1 If $x = (x_1, x_2, \dots, x_n) \in X(A, b)$, then $\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + [(a_{ij} \wedge x_j) \wedge (a_{ij} \wedge u)] = (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge x_j) \wedge (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge u)$.

Proof There are two cases as follows:

Case (1). If $u \leq x_j$, then $a_{ij} \wedge u \leq a_{ij} \wedge x_j$. Thus, $\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + [(a_{ij} \wedge x_j) \wedge (a_{ij} \wedge u)] = \sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge u = (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge x_j) \wedge (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge u)$.

Case (2). Similar to Case (1), one can prove that $\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + [(a_{ij} \wedge x_j) \wedge (a_{ij} \wedge u)] = (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge x_j) \wedge (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge u)$ if $u > x_j$.

Theorem 1 If $x = (x_1, x_2, \dots, x_n) \in X(A, b)$, then $x \in \check{X}(A, b)$ if and only if $x_j = \gamma_j(x)$ for any $j \in J$.

Proof If $x \in \check{X}(A, b)$, then by Remark 2, $x_j \in H_j(x)$ for any $j \in J$. Thus, $x_j \geq \gamma_j(x)$ and by Proposition 1, for any $x_{*j} \in H_j(x)$, we have

$$\begin{aligned} & \sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge (x_j \wedge x_{*j}) \\ &= \sum_{k \in J - \{j\}} a_{ik} \wedge x_k + [(a_{ij} \wedge x_j) \wedge (a_{ij} \wedge x_{*j})] \\ &= (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge x_j) \wedge (\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge x_{*j}) \\ &\geq b_i \wedge b_i \\ &= b_i. \end{aligned}$$

Therefore, for any $j \in J$, $x_j \wedge x_{*j} \in H_j(x)$. Furthermore, define

$$x_p = (x_{1p}, x_{2p}, \dots, x_{np})$$

with

$$x_{kp} = \begin{cases} x_j \wedge x_{*j}, & k = j, \\ x_k, & k \neq j. \end{cases}$$

Then clearly $x_p \leq x$ and $x_p \in X(A, b)$. Thus $x_p = x$ since $x \in \check{X}(A, b)$. Hence $x_j = x_j \wedge x_{*j}$, i.e., $x_j \leq x_{*j}$. Therefore, by the arbitrariness of x_{*j} , $x_j \leq \gamma_j(x)$. It follows that $x_j = \gamma_j(x)$ for any $j \in J$.

Conversely, suppose that $x_j = \gamma_j(x)$ for any $j \in J$. Let

$$x_t = (x_{1t}, x_{2t}, \dots, x_{nt}) \in X(A, b)$$

be such that $x_t \leq x$. Define $x_q = (x_{1q}, x_{2q}, \dots, x_{nq})$ with

$$x_{kq} = \begin{cases} x_{jt}, & k = j, \\ x_k, & k \neq j. \end{cases}$$

Then clearly $x_t \leq x_q$. Thus by Lemma 1, $x_q \in X(A, b)$ since $x_t \in X(A, b)$. Thus, by Remark 2, $x_{jt} \in H_j(x)$ which means that $x_{jt} \geq x_j$. The arbitrariness of j implies that $x_t \geq x$. Therefore, $x_t = x$, i.e., $x \in \check{X}(A, b)$.

Theorem 2 Let $x = (x_1, x_2, \dots, x_n) \in X(A, b)$, and define

$$x_l = (x_{1l}, x_{2l}, \dots, x_{nl})$$

with

$$x_{kl} = \begin{cases} \gamma_j(x), & k = j, \\ x_k, & k \neq j. \end{cases}$$

Then $x_l \in X(A, b)$ and $x_l \leq x$.

Proof By Proposition 1, for any $i \in I$, we have the following:

$$\begin{aligned} \sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge \gamma_j(x) &= \sum_{k \in J - \{j\}} a_{ik} \wedge x_k + a_{ij} \wedge \left(\min_{x_{*j} \in H_j(x)} x_{*j} \right) \\ &= \sum_{k \in J - \{j\}} a_{ik} \wedge x_k + \min_{x_{*j} \in H_j(x)} (a_{ij} \wedge x_{*j}) \\ &= \min_{x_{*j} \in H_j(x)} \left[\sum_{k \in J - \{j\}} a_{ik} \wedge x_k + (a_{ij} \wedge x_{*j}) \right] \\ &\geq \min_{x_{*j} \in H_j(x)} b_i \\ &= b_i. \end{aligned}$$

Therefore, $x_l \in X(A, b)$. Clearly $x_l \leq x$.

For any $j \in J$, denote

$$S^{(12 \cdots j)} = \begin{cases} (\gamma_1(x), x_2, \dots, x_n), & j = 1, \\ (\gamma_1(x), \dots, \gamma_j(S^{(12 \cdots (j-1))}), x_{j+1}, \dots, x_n), & j \geq 2. \end{cases}$$

Then we have the following theorem.

Theorem 3 *If $x \in X(A, b)$, then there exists an $x_* \in \check{X}(A, b)$ such that $x_* \leq x$.*

Proof Let $x = (x_1, x_2, \dots, x_n) \in X(A, b)$. Then we distinguish two cases as follow:

Case 1. If $\check{a} \in X(A, b)$, then by Lemma 4, $x_* = \check{a} \leq x$.

Case 2. If $\check{a} \notin X(A, b)$, then $x > \check{a}$ by Remark 1. Because $x \in X(A, b)$, by Theorem 2, $S^{(1)} = (\gamma_1(x), x_2, \dots, x_n) \in X(A, b)$. Again, by Theorem 2, we have $S^{(12)} = (\gamma_1(x), \gamma_2(S^{(1)}), x_3, \dots, x_n) \in X(A, b)$, and so on, we finally have $S^{(12 \cdots n)} = (\gamma_1(x), \gamma_2(S^{(1)}), \dots, \gamma_n(S^{(12 \cdots (n-1))})) \in X(A, b)$ and $S^{(12 \cdots n)} \leq x$. Now, suppose that $x_\nu = (x_{1\nu}, x_{2\nu}, \dots, x_{n\nu}) \in X(A, b)$ satisfies $x_\nu \leq S^{(12 \cdots n)}$. Then for any $j \in J$, $x_{j\nu} \leq \gamma_j(S^{(12 \cdots (j-1))})$. Let $x_{*1} = (x_{1\nu}, x_2, \dots, x_n)$. Then $x_\nu \leq x_{*1}$. Thus, from Lemma 1, $x_{*1} \in X(A, b)$ which means $x_{1\nu} \geq \gamma_1(x)$ by Remark 2. Hence $x_{1\nu} = \gamma_1(x)$ since $x_\nu \leq S^{(12 \cdots n)}$, i.e. $x_\nu = (\gamma_1(x), x_{2\nu}, \dots, x_{n\nu})$. Let $x_{*2} = (\gamma_1(x), x_{2\nu}, x_3, \dots, x_n)$. Then $x_\nu \leq x_{*2}$. Thus, by Lemma 1, $x_{*2} \in X(A, b)$. Due to Remark 2, we get $x_{2\nu} \geq \gamma_2(S^{(1)})$. Hence, $x_{2\nu} = \gamma_2(S^{(1)})$ since $x_\nu \leq S^{(12 \cdots n)}$, i.e., $x_\nu = (\gamma_1(x), \gamma_2(S^{(1)}), \dots, x_{n\nu}) \in X(A, b)$. Continuing as above, we have $x_{j\nu} = \gamma_j(S^{(12 \cdots (j-1))})$, $j = 3, \dots, n$. Therefore,

$$x_\nu = (\gamma_1(x), \gamma_2(S^{(1)}), \dots, \gamma_n(S^{(12 \cdots (n-1))})) = S^{(12 \cdots n)},$$

i.e., $x_* = S^{(12 \cdots n)} \in \check{X}(A, b)$ and $x_* \leq x$.

To the best of our knowledge, up to now, there are two articles that prove Theorem 3. One of them is [1] in which the authors proved it by using Zorn's lemma (see [1, Theorem 5]). The other one is [7] in which the authors proved it by using Lemma 4 and Theorem 4 of [6] (see [7, Theorem 3.1]). It is evident that the proof of Theorem 3 is different from both of them and it is not related to Lemma 4 and Theorem 4 of [6].

Using Lemma 2 and Theorem 3, the following theorem holds.

Theorem 4 ([6]) $X(A, b) = \bigcup_{\check{x} \in \check{X}(A, b)} \{x \in X \mid \check{x} \leq x \leq (1, 1, \dots, 1)\}$.

From the proof of Theorem 3, for any $x \in X(A, b)$ we can propose an algorithm to find a minimal solution x_* such that $x_* \leq x$ as below.

Algorithm 1 *Input* $x = (x_1, x_2, \dots, x_n) \in X(A, b)$, $j = 0$ and $x_* = 0$. *Output* x_* .

Step 1. Compute $\check{a}$. If $\check{a} \in X(A, b)$, then $x_* := \check{a}$, go to step 8.

Step 2. $j := j + 1$.

Step 3. Compute $H_j(S^{(12 \cdots (j-1))}) = \{x_{*j} \in [0, 1] \mid a_{i1} \wedge \gamma_1(x) + \dots + a_{i(j-1)} \wedge \gamma_{j-1}(S^{(12 \cdots (j-2))}) + a_{ij} \wedge x_{*j} + \sum_{k=j+1}^n a_{ik} \wedge x_k \geq b_i \text{ for any } i \in I\}$ where

1 $S^{(0)} = x$.

2 *Step 4. Compute*

$$3 \quad \gamma_j(S^{(12 \dots (j-1))}) = \min\{x_{*j} \mid x_{*j} \in H_j(S^{(12 \dots (j-1))})\}.$$

4 *Step 5. $S^{(12 \dots j)} := (\gamma_1(x), \gamma_2(S^{(1)}), \dots, \gamma_j(S^{(12 \dots (j-1))}), x_{j+1}, \dots, x_n)$.*

5 *Step 6. If $j < n$, then go to step 2.*

6 *Step 7. $x_* := S^{(12 \dots j)}$.*

7 *Step 8. Output x_* .*

8 *Remark 3 (i) In Algorithm 1, it is clear that we can consider all the permutations of the set $\{1, 2, \dots, n\}$ instead of the n -tuple $(1, 2, \dots, n)$, and by repeating the procedure as presented in Algorithm 1 a minimal solution can be deduced. In general, two distinct permutations of the set $\{1, 2, \dots, n\}$ will not produce two different minimal solutions for a same given solution. (ii) Algorithm 1 are different from Algorithm 3.2 of [7] completely.*

9 The following two examples illustrate Algorithm 1 and Remark 3, respectively.

10 *Example 1 Consider a minimal solution of the following inequalities in [6].*

$$11 \quad \begin{pmatrix} 0.8 & 0.3 & 0.5 & 0.2 & 0.1 & 0.3 \\ 0.5 & 0.2 & 0.3 & 0.4 & 0.7 & 0.2 \\ 0.3 & 0.6 & 0.2 & 0.7 & 0.3 & 0.4 \\ 0.3 & 0.9 & 0.8 & 0.5 & 0.8 & 0.1 \\ 0.9 & 0.5 & 0.1 & 0.7 & 0.8 & 0.2 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geq \begin{pmatrix} 1.5 \\ 2.0 \\ 1.8 \\ 2.9 \\ 2.6 \end{pmatrix} \quad (2)$$

12 It is obvious that $x = (0.8, 0.6, 0.7, 0.6, 0.8, 0.4)$ is a solution. Applying Algorithm 1, we shall find a minimal solution of the solution as follows:

13 Step 1. Compute $\check{a} = (0.3, 0.4, 0.3, 0.1, 0.4, 0)$. Clearly $\check{a}$ is not a solution of (2).

14 Step 2. $j := 0 + 1$.

15 Step 3. Compute $H_1(x) = \{x \mid 0.4 \leq x\}$.

16 Step 4. Compute $\gamma_1(x) = 0.4$.

17 Step 5. $S^{(1)} := (0.4, 0.6, 0.7, 0.6, 0.8, 0.4)$.

18 Step 6. $j = 1 < 6$, $j := 1 + 1$.

19 Step 7. Compute $H_2(S^{(1)}) = \{x \mid 0.5 \leq x\}$.

20 Step 8. Compute $\gamma_2(S^{(1)}) = 0.5$.

21 Step 9. $S^{(12)} := (0.4, 0.5, 0.7, 0.6, 0.8, 0.4)$.

22 Step 10. $j = 2 < 6$, $j := 2 + 1$.

23 Step 11. Compute $H_3(S^{(12)}) = \{x \mid 0.7 \leq x\}$.

24 Step 12. Compute $\gamma_3(S^{(12)}) = 0.7$.

25 Step 13. $S^{(123)} := (0.4, 0.5, 0.7, 0.6, 0.8, 0.4)$.

26 Step 14. $j = 3 < 6$, $j := 3 + 1$.

27 Step 15. Compute $H_4(S^{(123)}) = \{x \mid 0.6 \leq x\}$.

28 Step 16. Compute $\gamma_4(S^{(123)}) = 0.6$.

Step 17. $S^{(1234)} := (0.4, 0.5, 0.7, 0.6, 0.8, 0.4)$.
 Step 18. $j = 4 < 6$, $j := 4 + 1$.
 Step 19. Compute $H_5(S^{(1234)}) = \{x \mid 0.8 \leq x\}$.
 Step 20. Compute $\gamma_5(S^{(1234)}) = 0.8$.
 Step 21. $S^{(12345)} := (0.4, 0.5, 0.7, 0.6, 0.8, 0.4)$.
 Step 22. $j = 5 < 6$, $j := 5 + 1$.
 Step 23. Compute $H_6(S^{(12345)}) = \{x \mid 0.2 \leq x\}$.
 Step 24. Compute $\gamma_6(S^{(12345)}) = 0.2$.
 Step 25. $S^{(123456)} := (0.4, 0.5, 0.7, 0.6, 0.8, 0.2)$.
 Step 26. $j = 6 \not< 6$.
 Step 27. $x_* := S^{(123456)}$.
 Step 28. Output $x_* = (0.4, 0.5, 0.7, 0.6, 0.8, 0.2)$.
 Therefore, x_* is a minimal solution of the inequalities (2) with $x_* \leq x$.

Example 2 Consider a minimal solution of the following inequalities in [10].

$$\begin{cases} 0.4 \wedge x_1 + 0.8 \wedge x_2 + 0.7 \wedge x_3 \geq 1.7, \\ 0.4 \wedge x_1 + 0.6 \wedge x_2 + 0.5 \wedge x_3 \geq 1.4, \\ 0.7 \wedge x_1 + 0.5 \wedge x_2 + 0.8 \wedge x_3 \geq 1.5. \end{cases} \quad (3)$$

Clearly $x = (0.6, 0.7, 0.9)$ is a solution. Using Algorithm 1, we can obtain two minimal solutions $S^{(213)}$ and $S^{(321)}$ with $S^{(213)} \leq x$ and $S^{(321)} \leq x$, respectively.

Compute $\check{a} = (0.3, 0.6, 0.5)$. Clearly $\check{a}$ is not a solution of inequalities (3).

Compute

$$S^{(2)} = (0.6, \gamma_2(x), 0.9) = (0.6, 0.7, 0.9),$$

$$S^{(21)} = (\gamma_1(S^{(2)}), 0.7, 0.9) = (0.3, 0.7, 0.9)$$

and

$$S^{(213)} = (0.3, 0.7, \gamma_3(S^{(21)})) = (0.3, 0.7, 0.7).$$

Therefore, $S^{(213)}$ is a minimal solution of inequalities (3) with $S^{(213)} \leq x$.

Similarly, compute

$$S^{(3)} = (0.6, 0.7, \gamma_3(x)) = (0.6, 0.7, 0.6),$$

$$S^{(32)} = (0.6, \gamma_2(S^{(3)}), 0.6) = (0.6, 0.7, 0.6)$$

and

$$S^{(321)} = (\gamma_1(S^{(32)}), 0.7, 0.6) = (0.4, 0.7, 0.6).$$

So $S^{(321)}$ is a minimal solution of inequalities (3) with $S^{(321)} \leq x$.

Obviously $S^{(213)} \neq S^{(321)}$. However,

$$S^{(31)} = (\gamma_1(S^{(3)}), 0.7, 0.6) = (0.4, 0.7, 0.6)$$

and

$$S^{(312)} = (0.4, \gamma_2(S^{(31)}), 0.6) = (0.4, 0.7, 0.6).$$

So $S^{(312)}$ is a minimal solution of inequalities (3) with $S^{(312)} \leq x$, and $S^{(321)} = S^{(312)}$.

Theorem 5 *Algorithm 1 terminates after $O(n^3)$ or $O(m^3)$ operations.*

Proof The amount of computation of Step 1 is $[n + (m - 1)]n + 2mn$. For any $j \in J$, the amount of computation of Step 3 is $2mn - m - 1$, and the amount of computation of Step 4 is 1; so from Steps 2 to 6, the amount of computation is $[(2mn - m - 1) + 1]n$. Therefore, the amount of computation of Algorithm 1 is $(n^2 + mn - n + 2mn) + (2mn^2 - mn) = 2mn^2 + 2mn + n^2 - n$. Consequently, the computational complexity of Algorithm 1 is $O(n^3)$ or $O(m^3)$.

4 Conclusions

We mainly supplied a sufficient and necessary condition that a solution of system (1) is a minimal one, and proved that for every solution of system (1) there exists a minimal solution that is less than or equal to the solution in a new way. We finally provided an algorithm to find a minimal solution for a given one whose computational complexity is $O(n^3)$ or $O(m^3)$. It is worth pointing out that the idea of Algorithm 1 comes from [2], and by comparing with Algorithm 3.2 of [7], Algorithm 1 sometimes may find more than one minimal solution for a given solution by changing the permutations of the set $\{1, 2, \dots, n\}$ as shown in Example 2 when system (1) has more than one minimal solution. Unfortunately, there is no algorithm to find the whole minimal solution set including its structure since the number of minimal solutions for system (1) is uncountably infinite (cf. [5, 13]).

Compliance with Ethical Standards

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Funding This paper is supported by the National Natural Science Foundation of China (No.12071325).

Conflict of interest All the authors in the paper have no conflict of interest.

Informed consent Informed consent was obtained from all individual participants included in the study.

Author contributions

Dr. Xue-ping Wang wrote the main manuscript text. Miss Xiao Mi supplied Proposition 3.1, Theorem 3.1 and Examples 3.1 and 3.2.

References

1. B. Y. Cao, X. P. Yang, X. G. Zhou, Properties of fuzzy relation inequalities with addition-min composition, Fuzzy Information and Engineering and Decision (B. Y. Cao Ed.), Advances in Intelligent Systems and Computing, Springer, 646, 2018, pp. 177-185.

2. A. Di Nola, On solving relational equations in Brouwerian lattices, *Fuzzy Sets and Systems* 34 (1990) 365-376.
3. A. Di Nola, S. Sessa, W. Pedrycz and E. Sanchez, *Fuzzy Relation Equations and Their Applications to Knowledge Engineering*. Dordrecht, Boston/London: Kluwer Academic Publishers, 1989.
4. F. F. Guo, J. Shen, A smoothing approach for minimizing a linear function subject to fuzzy relation inequalities with addition-min composition, *Int. J. Fuzzy Syst.* 21 (2019) 281-290.
5. S. M. Guu, Y. K. Wu, A linear programming approach for minimizing a linear function subject to fuzzy relational inequalities with addition-min composition, *IEEE Transactions on Fuzzy Systems* 25 (2017) 985-992.
6. J. X. Li, S. J. Yang, Fuzzy relation inequalities about the data transmission mechanism in BitTorrent-like Peer-to-Peer file sharing systems, in: *Proceedings of the 2012 9th International Conference on Fuzzy systems and Knowledge Discover*, IEEE: FSKD 2012, pp. 452-456.
7. Meng Li, Xue-ping Wang, Remarks on minimal solutions of fuzzy relation inequalities with addition-min composition, *Fuzzy Sets and Systems* 410 (2021) 19-26.
8. K. Peeva, Y. Kyosev, *Fuzzy Relational Calculus: Theory, Applications and Software*. Singapore: World Scientific Publishing Company, 2004.
9. E. Sanchez, Resolution of composite fuzzy relation equations, *Information and Control* 30 (1976) 38-48.
10. S. J. Yang, An algorithm for minizing a linear objective function subject to the fuzzy relation inequalities with addition-min composition, *Fuzzy Sets and Systems* 255 (2014) 41-51.
11. S. J. Yang, Some results of fuzzy relation inequalities with addition-min composition, *IEEE Transactions on Fuzzy Systems* 26 (2018) 239-245.
12. X. P. Yang, Optimal-vector-based algorithm for solving min-max programming subject to addition-min fuzzy relation inequality, *IEEE Transactions on Fuzzy Systems* 25 (2017) 1127-1140.
13. X. P. Yang, H. T. Lin, X. G. Zhou, B. Y. Cao, Addition-min fuzzy relation inequalities with application in BitTorrent-like Peer-to-Peer file sharing system, *Fuzzy Sets and Systems* 343 (2018) 126-140.