Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability

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Abstract This article introduces the concept of weak sharp minima (WSM) for convex interval-valued functions (IVFs). To identify a set of WSM of a convex IVF, we provide its primal and dual characterizations. The primal characterization is given in terms of gH -directional derivatives. On the other hand, to derive dual characterizations, we propose the notions of the support function of a subset of $I(\mathbb{R})^n$ and gH -subdifferentiability for convex IVFs. Further, we develop the required gH -subdifferential calculus for convex IVFs. Thereafter, by using the proposed gH -subdifferential calculus, we provide dual characterizations for the set of WSM of convex IVFs.

Keywords Interval-valued function · gH -directional derivative · gH -subgradient · Interval optimization · Weak sharp minima

1 Introduction

Due to the presence of uncertainty, deterministic optimization fails to represent many real-life optimization problems. In such cases, we need to proceed with the tools of uncertain optimization. If the uncertainty is given by a random variable, then these optimization problems come under the umbrella of stochastic optimization. On the other, if the uncertainty is given by a membership function, then these optimization problems are solved with the techniques of fuzzy optimization. Also, it is seen that the uncertainty of many practical problems is expressed using closed and bounded intervals. Thus, interval optimization is an indispensable way to deal with the uncertainty present in many real-life problems.

In 1966, Moore [28] introduced interval analysis to investigate interval-valued functions (IVFs). In [28], Moore extensively gave arithmetic of intervals. Subsequently, there was a need to improve this arithmetic [18], especially the subtraction. Due to this, Hukuhara [18] presented a notion of the difference between intervals, which is known as Hukuhara difference (H -difference). Stefanini and Bede [34] proposed an extended version of H -difference, known as generalized Hukuhara difference (gH -difference), which has been comprehensively adopted in interval analysis.

We know that the solution concepts of optimization problems depend widely on ordering the range set of the objective function. Unlike real numbers, the set of intervals is not linearly ordered. Thus, to introduce a solution concept for optimization problems under

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interval uncertainties, many partial ordering relations of intervals were proposed in the literature (see [3, 21, 24, 31, 39, 40], and the references therein). With the help of the existing ordering concepts of a pair of intervals, many theories and methods have been developed regarding solutions of optimization problems with IVFs or of interval optimization problems (IOPs) [20, 22, 29, 32, 37, 41]. Inuiguchi et al. [20] proposed treatment of optima for IVFs by minimax regret criteria. Chanas and Kutcha [8] gave a solution concept based on a preference relation of intervals. A robust efficient solution for interval linear programming was given in [19]. Chen et al. [9] reported a solution concept by midpoint deterministic approach. Wu [38] defined type I and type II LU -optimal solution concepts similar to Pareto optimality. A survey on the different ordering of intervals and related optimality concepts can be found in [14, 23] and from their references.

The major developments on IOPs started after a rich calculus of IVFs was ready to be used. Hukuhara [18] laid the foundation to develop the calculus of IVFs by introducing the concept of H -differentiability of IVFs. However, this definition of H -differentiability is restrictive [7]. To overcome the deficiencies of H -differentiability, Stefanini and Bede [34] proposed the notion of gH -differentiability for IVFs. Later, by using gH -differentiability, many concepts on calculus have been developed, for instance, see [7, 13, 26]. In 2007, Wu [38] proposed two solution concepts by considering two partial ordering concepts on the set of all closed intervals and derived KKT optimality conditions for IOPs using H -derivative. Subsequently, Wu investigated KKT optimality conditions for multi-objective IOPs [41]. In 2012, Bhurjee et al. [3] developed a methodology to study the existence of the solution of general IOPs by expressing IVFs in the parametric form. Chalco-cano et al. [6] derived KKT optimality conditions for IOPs using gH -derivative and explained the advantages of using gH -derivative instead of H -derivative. In 2016, Singh et al. [33] proposed the concept of Pareto optimal solution for the interval-valued multi-objective programming problems. Many other researchers have also proposed optimality conditions and solution concepts for IOPs, see for instance [1, 12, 15, 16, 35] and the references therein.

1.1 Motivation and work done

To analyse optimization problems with multiple minima points, Burke and Ferris [5] introduced the notion of WSM in conventional optimization. It is known that WSM plays an important role in the sensitivity analysis and convergence analysis of conventional optimization problems [5, 11, 42]. It is also seen that many algorithms exhibit finite termination at WSM [5, 11, 27, 36, 42]. Motivated by these properties and wide applications of WSM in conventional optimization, in this article, we attempt to propose and mathematically characterize the notion of WSM for convex IVFs. To give characterizations of WSM for convex IVFs, we defined gH -subdifferentiability for convex IVFs and support function of a subset of $I(\mathbb{R})^n$. Some required fundamental characteristics of gH -subdifferential set are proposed. Few related results on the support function of a nonempty subset of $I(\mathbb{R})^n$ are also derived.

1.2 Delineation

The article is presented in the following manner. In Section 2, basic terminologies and definitions on intervals and IVFs are provided. In Section 3, we propose the concept of the support function of a subset of $I(\mathbb{R})^n$; alongside, few necessary results on extended support function are also given. Next, we derive the idea of gH -subdifferentiability for convex IVFs, in Section 4 that are required in the subsequent sections. The concept of WSM for convex IVFs is presented in Section 5; further, we give primal and dual characterizations of WSM. Lastly, the conclusion and future scopes are given in Section 6.

2 Preliminaries and Terminologies

In this article, the following notations are used throughout.

- $\mathbb{R}$ denotes the set of real numbers
- $\mathbb{R}^+$ denotes the set of nonnegative real numbers
- $\|\cdot\|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$
- $I(\mathbb{R})$ represents the set of all closed and bounded intervals
- Bold capital letters refer to the elements of $I(\mathbb{R})$
- $\overline{I(\mathbb{R})} = I(\mathbb{R}) \cup \{-\infty, +\infty\}$
- $\mathbb{B} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denotes the closed unit ball in $\mathbb{R}^n$.

2.1 Fundamental Operations on Intervals

Arithmetic operations of two intervals $\mathbf{A} = [\underline{a}, \bar{a}]$ and $\mathbf{B} = [\underline{b}, \bar{b}]$ are defined by

$$\mathbf{A} \oplus \mathbf{B} = [\underline{a} + \underline{b}, \bar{a} + \bar{b}], \quad \mathbf{A} \ominus \mathbf{B} = [\underline{a} - \bar{b}, \bar{a} - \underline{b}],$$

and

$$\lambda \odot \mathbf{A} = \mathbf{A} \odot \lambda = \begin{cases} [\lambda \underline{a}, \lambda \bar{a}], & \text{if } \lambda \geq 0 \\ [\lambda \bar{a}, \lambda \underline{a}], & \text{if } \lambda < 0, \end{cases}$$

where λ is a real constant.

The norm [28] of an interval $\mathbf{A} = [\underline{a}, \bar{a}]$ in $I(\mathbb{R})$ is defined by $\|\mathbf{A}\|_{I(\mathbb{R})} = \max\{|\underline{a}|, |\bar{a}|\}$.

The norm of an interval vector $\hat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \in I(\mathbb{R})^n$ is given by (see [28])

$$\|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} = \sum_{i=1}^n \|\mathbf{A}_i\|_{I(\mathbb{R})}.$$

It is to note that a real number p can be represented by the interval $[p, p]$.

Definition 1 (*gH -difference of intervals* [34]). Let $\mathbf{A}$ and $\mathbf{B}$ be two elements in $I(\mathbb{R})$. The gH -difference between $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \ominus_{gH} \mathbf{B}$, is defined by the interval $\mathbf{C}$ such that

$$\mathbf{A} = \mathbf{B} \oplus \mathbf{C} \quad \text{or} \quad \mathbf{B} = \mathbf{A} \ominus \mathbf{C}.$$

It is to be noted that for $\mathbf{A} = [\underline{a}, \bar{a}]$ and $\mathbf{B} = [\underline{b}, \bar{b}]$,

$$\mathbf{A} \ominus_{gH} \mathbf{B} = [\min\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\}, \max\{\underline{a} - \bar{b}, \bar{a} - \underline{b}\}],$$

and $\mathbf{A} \ominus_{gH} \mathbf{A} = \mathbf{0}$.

Definition 2 (*Algebraic operations on $I(\mathbb{R})^n$*). Let $\hat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n)$ and $\hat{\mathbf{B}} = (\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_n)$ be two elements in $I(\mathbb{R})^n$. An algebraic operation $\star$ between $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, denoted by $\hat{\mathbf{A}} \star \hat{\mathbf{B}}$, is defined by

$$\hat{\mathbf{A}} \star \hat{\mathbf{B}} = (\mathbf{A}_1 \star \mathbf{B}_1, \mathbf{A}_2 \star \mathbf{B}_2, \dots, \mathbf{A}_n \star \mathbf{B}_n),$$

where $\star \in \{\oplus, \ominus, \ominus_{gH}\}$.

Definition 3 (*Special product*). For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and a vector of intervals $\hat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \in I(\mathbb{R})^n$ with $\mathbf{A}_i = [\underline{a}_i, \bar{a}_i]$ for each $i = 1, 2, \dots, n$, the special product between x and $\hat{\mathbf{A}}$, denoted by $x^\top \odot \hat{\mathbf{A}}$, is given by

$$x^\top \odot \hat{\mathbf{A}} = \left[\min \left\{ \sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \bar{a}_i \right\}, \max \left\{ \sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \bar{a}_i \right\} \right].$$

Remark 1 It is to notice that if all the components of $\hat{\mathbf{A}}$ are degenerate intervals, i.e., $\hat{\mathbf{A}} \in \mathbb{R}^n$, then the special product $x^\top \odot \hat{\mathbf{A}}$ reduces to the standard inner product of $x \in \mathbb{R}^n$ with $\hat{\mathbf{A}}$.

Definition 4 (*Dominance of intervals* [40]). Let $\mathbf{A} = [\underline{a}, \bar{a}]$ and $\mathbf{B} = [\underline{b}, \bar{b}]$ be two elements in $I(\mathbb{R})$.

- (i) $\mathbf{B}$ is said to be dominated by $\mathbf{A}$ if $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$, and then we write $\mathbf{A} \preceq \mathbf{B}$;
- (ii) $\mathbf{B}$ is said to be strictly dominated by $\mathbf{A}$ if $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$, and then we write $\mathbf{A} \prec \mathbf{B}$. Equivalently, $\mathbf{A} \prec \mathbf{B}$ if and only if any of the following holds:
 $\underline{a} < \underline{b}$ and $\bar{a} \leq \bar{b}$ or $\underline{a} \leq \underline{b}$ and $\bar{a} < \bar{b}$ or $\underline{a} < \underline{b}$ and $\bar{a} < \bar{b}$;
- (iii) if neither $\mathbf{A} \preceq \mathbf{B}$ nor $\mathbf{B} \preceq \mathbf{A}$, we say that none of $\mathbf{A}$ and $\mathbf{B}$ dominates the other, or $\mathbf{A}$ and $\mathbf{B}$ are not comparable. Equivalently, $\mathbf{A}$ and $\mathbf{B}$ are not comparable if either $\underline{a} < \underline{b}$ and $\bar{a} > \bar{b}$ or $\underline{a} > \underline{b}$ and $\bar{a} < \bar{b}$.

2.2 Calculus of IVFs

Throughout this subsection, $\mathbf{F}$ is an IVF defined on a nonempty subset X of $\mathbb{R}^n$.

Definition 5 (*gH-continuity* [12]). Let $\mathbf{F}$ be an IVF and let $\bar{x}$ be a point of X and $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$. The function $\mathbf{F}$ is said to be *gH-continuous* at $\bar{x}$ if

$$\lim_{\|h\| \rightarrow 0} (\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{0}.$$

Definition 6 (*gH-derivative* [34]). The *gH-derivative* of an IVF $\mathbf{F} : \mathbb{R} \rightarrow I(\mathbb{R})$ at $\bar{x} \in \mathbb{R}$ is defined by

$$\mathbf{F}'(\bar{x}) = \lim_{h \rightarrow 0} \frac{\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})}{d}, \text{ provided the limit exists.}$$

Remark 2 (See [34]). Let $\mathbf{F} = [\underline{F}, \bar{F}]$ be an IVF on X , where $\underline{F}$ and $\bar{F}$ are real-valued functions defined on X . Then, the *gH-derivative* of $\mathbf{F}$ at $\bar{x} \in X$ exists if the derivatives of $\underline{F}$ and $\bar{F}$ at $\bar{x}$ exist and

$$\mathbf{F}'(\bar{x}) = \left[\min \left\{ \underline{F}'(\bar{x}), \bar{F}'(\bar{x}) \right\}, \max \left\{ \underline{F}'(\bar{x}), \bar{F}'(\bar{x}) \right\} \right].$$

Definition 7 (*gH-partial derivative* [12]). Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)^\top$ be a point of X . For a given $i \in \{1, 2, \dots, n\}$, we define a function $\mathbf{G}_i$ by $\mathbf{G}_i(x_i) = \mathbf{F}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n)$. If *gH-derivative* of $\mathbf{G}_i$ exists at $\bar{x}_i$, then we say that $\mathbf{F}$ has the *i*th *gH-partial derivative* at $\bar{x}$. We denote the *i*th *gH-partial derivative* of $\mathbf{F}$ at $\bar{x}$ by $D_i \mathbf{F}(\bar{x})$, i.e., $D_i \mathbf{F}(\bar{x}) = \mathbf{G}_i'(\bar{x}_i)^\top$.

Definition 8 (*gH-gradient* [12]). The *gH-gradient* of $\mathbf{F}$ at a point $\bar{x} \in X$, denoted by $\nabla \mathbf{F}(\bar{x}) \in I(\mathbb{R})^n$, is defined by

$$\nabla \mathbf{F}(\bar{x}) = (D_1 \mathbf{F}_1(\bar{x}), D_2 \mathbf{F}_2(\bar{x}), \dots, D_n \mathbf{F}_n(\bar{x}))^\top.$$

Lemma 1 Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are in $I(\mathbb{R})$. Then, for any real number r ,

- (i) $[r, r] \preceq \mathbf{A}$ and $\mathbf{A} \preceq \mathbf{B} \ominus_{gH} \mathbf{C} \implies \mathbf{C} \oplus [r, r] \preceq \mathbf{B}$ and
- (ii) $((1 - \lambda) \odot \mathbf{A} \oplus \lambda \odot \mathbf{B}) \ominus_{gH} \mathbf{A} = \lambda \odot (\mathbf{B} \ominus_{gH} \mathbf{A})$ for any $\lambda \in [0, 1]$.

Proof See Appendix A.

Definition 9 (*Convex IVF* [38]). Let X be a nonempty convex subset of $\mathbb{R}^n$. An IVF $\mathbf{F} : X \rightarrow I(\mathbb{R})$ is said to be convex on X if for any x_1 and x_2 in X ,

$$\mathbf{F}(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 \odot \mathbf{F}(x_1) \oplus \lambda_2 \odot \mathbf{F}(x_2) \text{ for all } \lambda_1, \lambda_2 \in [0, 1] \text{ with } \lambda_1 + \lambda_2 = 1.$$

Lemma 2 (See [38]). Let X be a nonempty convex subset of $\mathbb{R}^n$, and $\mathbf{F} = [\underline{F}, \bar{F}]$ be an IVF on X , where $\underline{F}$ and $\bar{F}$ are real-valued functions defined on X . Then, $\mathbf{F}$ is convex on X if and only if $\underline{F}$ and $\bar{F}$ are convex on X .

Definition 10 (*gH-directional derivative* [13]). Let $\mathbf{F}$ be an IVF on X . Let $\bar{x} \in X$ and $d \in \mathbb{R}^n$. If the limit

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}(\bar{x}))$$

exists, then the limit is said to be *gH-directional derivative* of $\mathbf{F}$ at $\bar{x}$ in the direction d , and it is denoted by $\mathbf{F}_{\mathcal{D}}(\bar{x})(d)$.

Definition 11 (*gH -differentiability [12]*). An IVF $\mathbf{F}$ is said to be gH -differentiable at $\bar{x} \in X$ if there exist two IVFs $\mathbf{E}(\mathbf{F}(\bar{x}); h)$ and $\mathbf{L}_{\bar{x}} : \mathbb{R}^n \rightarrow I(\mathbb{R})$ such that

$$\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) = \mathbf{L}_{\bar{x}}(h) \oplus \|h\| \odot \mathbf{E}(\mathbf{F}(\bar{x}); h)$$

for $\|h\| < \delta$ for some $\delta > 0$, where $\lim_{\|h\| \rightarrow 0} \mathbf{E}(\mathbf{F}(\bar{x}); h) = \mathbf{0}$ and $\mathbf{L}_{\bar{x}}$ is such a function that satisfies

- (i) $\mathbf{L}_{\bar{x}}(x + y) = \mathbf{L}_{\bar{x}}(x) \oplus \mathbf{L}_{\bar{x}}(y)$ for all $x, y \in X$, and
- (ii) $\mathbf{L}_{\bar{x}}(cx) = c \odot \mathbf{L}_{\bar{x}}(x)$ for all $c \in \mathbb{R}$ and $x \in X$.

Theorem 1 (See [12]). Let $\mathbf{F} : X \rightarrow I(\mathbb{R})$ be gH -differential at $\bar{x}$. Then, $\mathbf{L}_{\bar{x}}$ exists for every $h \in \mathbb{R}^n$ and

$$\mathbf{L}_{\bar{x}} = \sum_{i=1}^n h_i \odot D_i \mathbf{F}(\bar{x}),$$

where $\sum_{i=1}^n h_i \odot D_i \mathbf{F}(\bar{x}) = h_1 \odot D_1 \mathbf{F}(\bar{x}) \oplus h_2 \odot D_2 \mathbf{F}(\bar{x}) \oplus \cdots \oplus h_n \odot D_n \mathbf{F}(\bar{x})$.

Remark 3 (See [12]). Let $\mathbf{F} : X \rightarrow I(\mathbb{R})$ be gH -differentiable at $\bar{x} \in X$. Then, there exists a nonzero λ and $\delta > 0$ such that

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{L}_{\bar{x}}(h)$$

for all $h \in \mathbb{R}^n$ with $|\lambda| \|h\| < \delta$, where $\mathbf{L}_{\bar{x}}$ is an IVF, defined in Definition 11 of gH -differentiability.

Lemma 3 Let $\mathbf{F} : X \rightarrow I(\mathbb{R})$ be a gH -differentiable at $\bar{x} \in X$. Then, $\mathbf{F}$ has gH -directional derivative at $\bar{x}$ for every direction $h \in \mathbb{R}^n$ and

$$\mathbf{F}_{\mathcal{D}}(\bar{x})(h) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n,$$

where $\mathbf{L}_{\bar{x}}$ is as defined in Definition 11.

Proof Since $\mathbf{F}$ is gH -differentiable at $\bar{x}$, by Remark 3, we have

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n \\ \implies & \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n. \end{aligned}$$

Hence, by Definition 10, we conclude that $\mathbf{F}$ has gH -directional derivative at $\bar{x}$ and

$$\mathbf{F}_{\mathcal{D}}(\bar{x})(h) = \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n.$$

Definition 12 (*Proper IVF*). An extended IVF $\mathbf{F} : X \rightarrow \overline{I(\mathbb{R})}$ is called a proper IVF if there exists $\bar{x} \in X$ such that $\mathbf{F}(\bar{x}) \prec [+ \infty, + \infty]$ and $[- \infty, - \infty] \prec \mathbf{F}(x)$ for all $x \in X$.

Definition 13 (*Effective domain of IVF*). The effective domain of an extended IVF $\mathbf{F} : X \rightarrow \overline{I(\mathbb{R})}$ is the collection of all such points at which $\mathbf{F}$ is finite. It is denoted by $\text{dom}(\mathbf{F})$, i.e.,

$$\text{dom}(\mathbf{F}) = \left\{ x \in X : \mathbf{F}(x) \prec [+ \infty, + \infty] \right\}.$$

Theorem 2 Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\mathbf{F} : X \rightarrow I(\mathbb{R})$ be a convex IVF with $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}$ and $\overline{F}$ are real-valued functions defined on X . Then, at any $\bar{x} \in X$, gH -directional derivative $\mathbf{F}_{\mathcal{D}}(\bar{x})(d)$ exists and

$$\mathbf{F}_{\mathcal{D}}(\bar{x})(d) = \left[\min \left\{ \underline{F}_{\mathcal{D}}(\bar{x})(d), \overline{F}_{\mathcal{D}}(\bar{x})(d) \right\}, \max \left\{ \underline{F}_{\mathcal{D}}(\bar{x})(d), \overline{F}_{\mathcal{D}}(\bar{x})(d) \right\} \right].$$

Proof Similar to the proof of Theorem 3.1 in [13].

Definition 14 (*gH-Lipschitz continuous IVF* [13]). An IVF $\mathbf{F}$ is said to be *gH-Lipschitz continuous* on X if there exists $M > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq M\|x - y\| \text{ for all } x, y \in X.$$

The constant M is called a Lipschitz constant.

Definition 15 (*Supremum of a subset of $\overline{I(\mathbb{R})}$* [25]). Let $\mathbf{S}$ be a nonempty subset of $\overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is said to be an upper bound of $\mathbf{S}$ if $\mathbf{B} \preceq \bar{\mathbf{A}}$ for all $\mathbf{B}$ in $\mathbf{S}$. An upper bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called a supremum of $\mathbf{S}$, denoted by $\sup \mathbf{S}$, if for all upper bounds $\mathbf{C}$ of $\mathbf{S}$ in $I(\mathbb{R})$, $\bar{\mathbf{A}} \preceq \mathbf{C}$. Moreover, if the supremum of the set $\mathbf{S}$ belongs to the set itself, then it is called maximum of $\mathbf{S}$, denoted by $\max \mathbf{S}$.

Remark 4 (See [25]). Let Λ be an index set, and $\lambda \in \Lambda$. For any subset $\mathbf{S} = [a_\lambda, b_\lambda]$ of $\overline{I(\mathbb{R})}$, we have $\sup \mathbf{S} = \left[\sup_{\lambda \in \Lambda} a_\lambda, \sup_{\lambda \in \Lambda} b_\lambda \right]$.

Lemma 4 (See [25]). Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two proper extended IVFs defined on $\mathcal{S}$, which is a nonempty subset of X . Then,

- (i) $\inf_{x \in \mathcal{S}} \mathbf{F}_1(x) \oplus \inf_{x \in \mathcal{S}} \mathbf{F}_2(x) \preceq \inf_{x \in \mathcal{S}} (\mathbf{F}_1(x) \oplus \mathbf{F}_2(x))$ and
- (ii) $\sup_{x \in \mathcal{S}} (\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)) \preceq \sup_{x \in \mathcal{S}} \mathbf{F}_1(x) \oplus \sup_{x \in \mathcal{S}} \mathbf{F}_2(x)$.

Definition 16 (*Lower limit and gH-lower semicontinuity of an extended IVF* [25]). The lower limit of an extended IVF $\mathbf{F}$ at $\bar{x} \in X$, denoted by $\liminf_{x \rightarrow \bar{x}} \mathbf{F}(x)$, is defined by

$$\liminf_{x \rightarrow \bar{x}} \mathbf{F}(x) = \lim_{\delta \downarrow 0} (\inf \{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) = \sup_{\delta > 0} (\inf \{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}),$$

where $B_\delta(\bar{x})$ is an open ball with radius δ centered at $\bar{x}$. $\mathbf{F}$ is called *gH-lower semicontinuous* (*gH-lsc*) at a point $\bar{x}$ if $\mathbf{F}(\bar{x}) \preceq \liminf_{x \rightarrow \bar{x}} \mathbf{F}(x)$. Further, $\mathbf{F}$ is called *gH-lsc* on X if $\mathbf{F}$ is *gH-lsc* at every $\bar{x} \in X$.

Remark 5 By Note 5 of [25], we see that $\mathbf{F}$ is *gH-lsc* at $\bar{x} \in X$ if and only if $\underline{\mathbf{F}}$ and $\overline{\mathbf{F}}$ both are *lsc* at $\bar{x}$.

Lemma 5 Let $\mathbf{F} : \mathbb{R}^n \rightarrow \overline{I(\mathbb{R})}$ be a proper convex IVF. Then, for all $x, y \in \text{dom}(\mathbf{F})$, we have

$$\mathbf{F}_{\mathcal{D}}(x)(y - x) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x).$$

Proof By Definition 10 of *gH-directional derivative*, we have

$$\mathbf{F}_{\mathcal{D}}(x)(d) = \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \odot (\mathbf{F}(x + \lambda d) \ominus_{gH} \mathbf{F}(x)). \quad (1)$$

By taking $d = y - x$ in (1), we get

$$\begin{aligned} \mathbf{F}_{\mathcal{D}}(x)(d) &= \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \odot (\mathbf{F}(x + \lambda(y - x)) \ominus_{gH} \mathbf{F}(x)) \\ &= \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \odot (\mathbf{F}((1 - \lambda)x + \lambda(y)) \ominus_{gH} \mathbf{F}(x)) \\ &\preceq \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \odot \{((1 - \lambda) \odot \mathbf{F}(x) \oplus \lambda \odot \mathbf{F}(y)) \ominus_{gH} \mathbf{F}(x)\} \\ &= \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \odot \lambda (\mathbf{F}(y) \ominus_{gH} \mathbf{F}(x)) \text{ by (ii) of Lemma 1} \\ &= \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x). \end{aligned}$$

Definition 17 (Convergence of a sequence in $I(\mathbb{R})^n$). A sequence $\widehat{\mathbf{G}} : \mathbb{N} \rightarrow I(\mathbb{R})^n$ is said to be convergent if there exists a $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that

$$\|\widehat{\mathbf{G}}_k \ominus_{gH} \widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

where $\widehat{\mathbf{G}}(k) = \widehat{\mathbf{G}}_k$, $k \in \mathbb{N}$.

Remark 6 It is noteworthy that if a sequence $\{\widehat{\mathbf{G}}_k\}$ in $I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \dots, \mathbf{G}_{kn}) \in I(\mathbb{R})^n$ with $\mathbf{G}_{ki} = [g_{ki}, \bar{g}_{ki}]$, converges to $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n) \in I(\mathbb{R})^n$ with $\mathbf{G}_i = [g_i, \bar{g}_i]$, then according to Definition 2 and norm on $I(\mathbb{R})^n$, the corresponding sequence $\{\mathbf{G}_{ki}\}$ in $I(\mathbb{R})$ converges to $\mathbf{G}_i \in I(\mathbb{R})$ for each $i = 1, 2, \dots, n$. Also, by Definition 17, the sequences g_{ki} and $\bar{g}_{ki}$ in $\mathbb{R}$ converge to g_i and $\bar{g}_i$ in $\mathbb{R}$, respectively, for each $i = 1, 2, \dots, n$.

2.3 Results from convex analysis

Apart from the results of interval analysis, we use the following results from classical convex analysis throughout the article.

Definition 18 (*Projection* [30]). Let A be a nonempty closed set in $\mathbb{R}^n$. Then, the projection of a point $x \in \mathbb{R}^n$ onto the set A is denoted by $P(x \mid A)$, and is defined by

$$P(x \mid A) = \{y \in A : \|x - y\| = \inf\{\|x - u\| : u \in A\}\}.$$

Definition 19 (*Polar cone* [30]). Let A be a nonempty set in $\mathbb{R}^n$. Then, the polar cone of the set A is

$$A^\circ = \{x^* \in \mathbb{R}^n : \langle x^*, x \rangle \leq 0 \text{ for all } x \in A\}.$$

Definition 20 (*Tangent cone* [30]). Let A be a nonempty closed convex set in $\mathbb{R}^n$. Then, the tangent cone to the set A at $x \in A$ is defined by

$$T_A(x) = \text{cl} \left(\bigcup_{t>0} \frac{A - x}{t} \right).$$

Definition 21 (*Normal cone* [30]). The normal cone to a nonempty set A in $\mathbb{R}^n$ at x is polar of the tangent cone at x to the A , i.e., $N_A(x) = T_A(x)^\circ$. Therefore,

$$N_A(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \leq 0, \text{ for any } y \in A\}.$$

Lemma 6 [17] Consider a convex set $S \subseteq \mathbb{R}^n$. Then, $\bar{x}$ is an element in the closure of S if and only if $\langle x, \bar{x} \rangle \leq \psi_S^*(x)$ for all $x \in \mathbb{R}^n$, where ψ_S^* is the support function of S , i.e., $\psi_S^*(x) = \sup_{s \in S} \langle x, s \rangle$.

Lemma 7 [4] Let C be a nonempty closed convex subset of $\mathbb{R}^n$.

- (i) For all $y \in \mathbb{R}^n$, $\text{dist}(y, C) = \sup_{x \in C} \text{dist}(y, x + T_C(x))$, where the distance function is given by $\text{dist}(y, C) = \inf_{\bar{x} \in C} \|y - \bar{x}\|$.
- (ii) Define $\rho(x) = \text{dist}(x, C)$. Then, for all $x \in C$ and $d \in \mathbb{R}^n$,

$$\rho_{\mathcal{D}}(x)(d) = \text{dist}(d, T_C(x)) = \psi_{\mathbb{B} \cap N_C(x)}^*(d).$$

Moreover, if $d \in N_C(x)$, then $\rho_{\mathcal{D}}(x)(d) = \text{dist}(d, T_C(x)) = \psi_{\mathbb{B} \cap N_C(x)}^*(d) = \|d\|$.

3 Support function in $I(\mathbb{R})^n$

In this section, we attempt to extend the conventional notion of support functions for subsets of $I(\mathbb{R})^n$. The derived concepts of support function are used later in Section 5 to derive dual characterizations of WSM for convex IVFs.

Definition 22 (*Support function of a subset of $I(\mathbb{R})^n$*). Let $\mathbf{S}$ be a nonempty subset of $I(\mathbb{R})^n$. Then, the support function of $\mathbf{S}$ at $x \in \mathbb{R}^n$, denoted by $\psi_{\mathbf{S}}^*(x)$, is defined by

$$\psi_{\mathbf{S}}^*(x) = \sup_{\hat{\mathbf{A}} \in \mathbf{S}} x^\top \odot \hat{\mathbf{A}}.$$

Lemma 8 Let $\mathbf{S}_1, \mathbf{S}_2$ be two nonempty subsets of $I(\mathbb{R})^n$ such that $\mathbf{S}_1 \subseteq \mathbf{S}_2$. Then, for any $x \in X \subseteq \mathbb{R}^n$,

$$\psi_{\mathbf{S}_1}^*(x) \preceq \psi_{\mathbf{S}_2}^*(x).$$

Proof For any $\hat{\mathbf{B}} = ([b_1, \bar{b}_1], [b_2, \bar{b}_2], \dots, [b_n, \bar{b}_n]) \in \mathbf{S}_2$ and $x \in X$, we have $x^\top \odot \hat{\mathbf{B}} \preceq \psi_{\mathbf{S}_2}^*(x)$. Given $\mathbf{S}_1 \subseteq \mathbf{S}_2$, i.e., for any $\hat{\mathbf{D}} = ([d_1, \bar{d}_1], [d_2, \bar{d}_2], \dots, [d_n, \bar{d}_n]) \in \mathbf{S}_1$, we have $\hat{\mathbf{D}} \in \mathbf{S}_2$. Therefore,

$$x^\top \odot \hat{\mathbf{D}} \preceq \psi_{\mathbf{S}_2}^*(x).$$

Since $\hat{\mathbf{D}}$ is arbitrary, we get

$$\psi_{\mathbf{S}_1}^*(x) = \sup_{\hat{\mathbf{E}} \in \mathbf{S}_1} x^\top \odot \hat{\mathbf{E}} \preceq \psi_{\mathbf{S}_2}^*(x).$$

Theorem 3 Let K be a nonempty closed convex cone in $X \subseteq \mathbb{R}^n$. Let $\mathbf{P}$ and $\mathbf{Q}$ be two nonempty subsets of $I(\mathbb{R})^n$. Then,

$$\psi_{\mathbf{P}}^*(x) \preceq \psi_{\mathbf{Q}}^*(x) \text{ for all } x \in K \text{ if and only if } \psi_{\mathbf{P}}^*(x) \preceq \psi_{\mathbf{Q} \oplus K^\circ}^*(x) \text{ for all } x \in X,$$

where K° is the polar cone of K .

Proof Let $\psi_{\mathbf{P}}^*(x) \preceq \psi_{\mathbf{Q}}^*(x)$ for all $x \in K$. Consider $x \in K$. Clearly $\mathbf{Q} \subseteq \mathbf{Q} \oplus K^\circ$. Then, by Lemma 8, we have

$$\begin{aligned} \psi_{\mathbf{Q}}^*(x) &\preceq \psi_{\mathbf{Q} \oplus K^\circ}^*(x) \\ &\preceq \psi_{\mathbf{Q}}^*(x) \oplus \psi_{K^\circ}^*(x) \text{ by (ii) of Lemma 4 and Definition 22} \\ &\preceq \psi_{\mathbf{Q}}^*(x) \text{ because } \psi_{K^\circ}^*(x) = 0. \end{aligned}$$

Therefore,

$$\psi_{\mathbf{Q}}^*(x) = \psi_{\mathbf{Q} \oplus K^\circ}^*(x) \text{ for all } x \in K. \quad (2)$$

Also, by hypothesis, we have $\psi_{\mathbf{P}}^*(x) \preceq \psi_{\mathbf{Q}}^*(x)$ for all $x \in K$, and hence

$$\psi_{\mathbf{P}}^*(x) \preceq \psi_{\mathbf{Q} \oplus K^\circ}^*(x) \text{ for all } x \in K. \quad (3)$$

Suppose now if $x \notin K$, then there exists $z \in K^\circ$ such that $\langle z, x \rangle > 0$. Thus, for any $\hat{\mathbf{A}} \in \mathbf{Q}$ and $\lambda \geq 0$, $\hat{\mathbf{A}} \oplus \lambda z \in \mathbf{Q} \oplus K^\circ$. Also, $x^\top \odot (\hat{\mathbf{A}} \oplus \lambda z)$

$$\begin{aligned} &= \left[\min \left\{ \sum_{i=1}^n x_i(\underline{a}_i + \lambda z_i), \sum_{i=1}^n x_i(\bar{a}_i + \lambda z_i) \right\}, \max \left\{ \sum_{i=1}^n x_i(\underline{a}_i + \lambda z_i), \sum_{i=1}^n x_i(\bar{a}_i + \lambda z_i) \right\} \right] \\ &= \left[\min \left\{ \sum_{i=1}^n x_i \underline{a}_i + \lambda x^\top z, \sum_{i=1}^n x_i \bar{a}_i + \lambda x^\top z \right\}, \max \left\{ \sum_{i=1}^n x_i \underline{a}_i + \lambda x^\top z, \sum_{i=1}^n x_i \bar{a}_i + \lambda x^\top z \right\} \right]. \end{aligned}$$

Note that as $\lambda \rightarrow +\infty$, $\lambda x^\top z \rightarrow +\infty$, and therefore $x^\top \odot (\hat{\mathbf{A}} \oplus \lambda z) \rightarrow +\infty$, which implies

$$\psi_{\mathbf{Q} \oplus K^\circ}^*(x) = [+ \infty, + \infty].$$

Thus,

$$\psi_P^*(x) \preceq \psi_{Q \oplus K^\circ}^*(x) \text{ for all } x \in X \setminus K. \quad (4)$$

Therefore, from (3) and (4), we have

$$\psi_P^*(x) \preceq \psi_{Q \oplus K^\circ}^*(x) \text{ for all } x \in X.$$

Proof of the converse part follows from (2). This completes the proof.

Lemma 9 *Let P be nonempty a subset of $\mathbb{R}^n$ and Q be a nonempty closed convex subset of $I(\mathbb{R})^n$. Then, for any $x \in \mathbb{R}^n$,*

$$\psi_P^*(x) \preceq \psi_Q^*(x) \text{ if and only if } P \subseteq Q.$$

Proof Let $\psi_P^*(x) \preceq \psi_Q^*(x)$ for $x \in \mathbb{R}^n$. Therefore, for any $p \in P$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \langle x, p \rangle &\preceq \sup_{\hat{Q}_i \in \mathbf{Q}} x^\top \odot \hat{Q}_i, \text{ where } \hat{Q}_i = \left([q_{i1}, \bar{q}_{i1}], [q_{i2}, \bar{q}_{i2}], \dots, [q_{in}, \bar{q}_{in}] \right) \\ \implies \langle x, p \rangle &\preceq \sup_{\hat{Q}_i \in \mathbf{Q}} \left[\min \left\{ \sum_{j=1}^n x_j q_{ij}, \sum_{j=1}^n x_j \bar{q}_{ij} \right\}, \max \left\{ \sum_{j=1}^n x_j q_{ij}, \sum_{j=1}^n x_j \bar{q}_{ij} \right\} \right]. \end{aligned}$$

We now consider the following two possible cases.

- Case 1. Let $\sum_{j=1}^n x_j q_{ij} \leq \sum_{j=1}^n x_j \bar{q}_{ij}$. In this case, we have

$$\langle x, p \rangle \preceq \sup_{\hat{Q}_i \in \mathbf{Q}} \left[\sum_{j=1}^n x_j q_{ij}, \sum_{j=1}^n x_j \bar{q}_{ij} \right]. \quad (5)$$

Next, define two sets S_1 and S_2 such that $S_1 = \{Q_1, Q_2, \dots, Q_n, \dots\}$ and $S_2 = \{\bar{Q}_1, \bar{Q}_2, \dots, \bar{Q}_n, \dots\}$, where $Q_i = (q_{i1}, q_{i2}, \dots, q_{in}) \in \mathbb{R}^n$ and $\bar{Q}_i = (\bar{q}_{i1}, \bar{q}_{i2}, \dots, \bar{q}_{in}) \in \mathbb{R}^n$.

Therefore, (5) along with Remark 4 gives,

$$\langle x, p \rangle \leq \sup_{Q_i \in S_1} \langle x, Q_i \rangle \quad (6)$$

$$\text{and } \langle x, p \rangle \leq \sup_{\bar{Q}_i \in S_2} \langle x, \bar{Q}_i \rangle. \quad (7)$$

Thus, from (6) and Lemma 6, we have $p \in S_1$, i.e., $p = Q_m$ for some m .

To show that $p \in Q$, we have to show that $p = \bar{Q}_m$ as well.

Note that

$$\begin{aligned} \langle x, p \rangle &= \langle x, Q_m \rangle \leq \langle x, \bar{Q}_m \rangle \text{ for all } x \in \mathbb{R}^n \text{ because } \sum_{j=1}^n x_j q_{mj} \leq \sum_{j=1}^n x_j \bar{q}_{mj} \\ \implies \langle x, p \rangle &\leq \sup \langle x, \bar{Q}_m \rangle \text{ for all } x \in \mathbb{R}^n \\ \implies \langle x, p \rangle &\leq \psi_{S'}^*(x) \text{ for all } x \in \mathbb{R}^n, \text{ where } S' \text{ is the singleton set } \{\bar{Q}_m\}. \end{aligned} \quad (8)$$

Thus, from equation (8) and Lemma 6, we have $p \in S'$, i.e., $p = \bar{Q}_m$.

Hence, $p \in Q$. Since p is arbitrary, $P \subseteq Q$.

- Case 2. Let $\sum_{j=1}^n x_j \bar{q}_{ij} \leq \sum_{j=1}^n x_j q_{ij}$. By following similar steps as in Case 1, in this case also, we get $P \subseteq Q$.

Proof of the converse part follows from Lemma 8.

Lemma 10 For $x \in \mathbb{R}^n$ and $\hat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \in \mathbf{S} \subseteq \mathbb{R}^n$, we have

$$x^\top \odot \hat{\mathbf{A}} \preceq \|x\| \odot \left[\|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} \right].$$

Proof Note that $x^\top \odot \hat{\mathbf{A}}$

$$\begin{aligned} &= \left[\min \left\{ \sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \bar{a}_i \right\}, \max \left\{ \sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \bar{a}_i \right\} \right] \\ &\preceq \left[\min \left\{ \sum_{i=1}^n |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}, \sum_{i=1}^n |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})} \right\}, \max \left\{ \sum_{i=1}^n |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})}, \sum_{i=1}^n |x_i| \|\mathbf{A}_i\|_{I(\mathbb{R})} \right\} \right] \\ &= \left[\min \left\{ \|x\| \|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|x\| \|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} \right\}, \max \left\{ \|x\| \|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|x\| \|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} \right\} \right] \\ &\preceq \|x\| \odot \left[\|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} \right]. \end{aligned}$$

Lemma 11 The support function of a nonempty set $\mathbf{S} \subseteq I(\mathbb{R})^n$ is finite everywhere if and only if $\mathbf{S}$ is bounded.

Proof Suppose that $\mathbf{S}$ is bounded, i.e., we have $M > 0$ such that $\|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} \leq M$ for all $\hat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \in \mathbf{S}$ with $\mathbf{A}_i = [\underline{a}_i, \bar{a}_i]$ for each $i = 1, 2, \dots, n$. By Lemma 10 and $\|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} \leq M$, for any $x \in \mathbb{R}^n$, we have

$$x^\top \odot \hat{\mathbf{A}} \preceq \|x\| \odot \left[\|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n}, \|\hat{\mathbf{A}}\|_{I(\mathbb{R})^n} \right] \preceq \|x\| \odot [M, M] \preceq \|x\| M.$$

Since $\hat{\mathbf{A}} \in \mathbf{S}$ is arbitrary chosen, therefore

$$\psi_{\mathbf{S}}^*(x) = \sup_{\hat{\mathbf{A}} \in \mathbf{S}} x^\top \odot \hat{\mathbf{A}} \preceq \|x\| M.$$

Hence, $\psi_{\mathbf{S}}^*(x)$ is finite everywhere.

Conversely, let $\psi_{\mathbf{S}}^*(x)$ is finite for every $x \in \mathbb{R}^n$. Therefore, there exists an $M > 0$ such that $\psi_{\mathbf{S}}^*(x) \preceq M$, which implies that for any $x \in \mathbb{R}^n$ and $\hat{\mathbf{A}} \in \mathbf{S}$, we have

$$\begin{aligned} x^\top \odot \hat{\mathbf{A}} &= \left[\min \left\{ \sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \bar{a}_i \right\}, \max \left\{ \sum_{i=1}^n x_i \underline{a}_i, \sum_{i=1}^n x_i \bar{a}_i \right\} \right] \preceq M \\ \implies \sum_{i=1}^n x_i \underline{a}_i &\leq M \text{ and } \sum_{i=1}^n x_i \bar{a}_i \leq M. \end{aligned}$$

Take $\sum_{i=1}^n x_i \underline{a}_i \leq M$, then by Remark 1, we have

$$\langle x, \underline{a} \rangle \leq M, \text{ where } \underline{a} = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n) \in \mathbb{R}^n. \quad (9)$$

If $\underline{a} \neq 0$, choose $x = \frac{\underline{a}}{\|\underline{a}\|}$, then (9) gives

$$\begin{aligned} &\left\langle \frac{\underline{a}}{\|\underline{a}\|}, \underline{a} \right\rangle \leq M \\ \implies \|\underline{a}\| &\leq M, \text{ where } \underline{a} = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n) \in \mathbb{R}^n \\ \implies |\underline{a}_i| &\leq M \text{ for each } i = 1, 2, \dots, n. \end{aligned}$$

Similarly, when we take $\sum_{i=1}^n x_i \bar{a}_i \leq M$, we get $|\bar{a}_i| \leq M$ for each $i = 1, 2, \dots, n$. Therefore, we have

$$\begin{aligned} \mathbf{A}_i &= [\underline{a}_i, \bar{a}_i] \preceq M \text{ for each } i = 1, 2, \dots, n \\ \implies \hat{\mathbf{A}} &\preceq M. \end{aligned}$$

Since $\hat{\mathbf{A}} \in \mathbf{S}$ was arbitrary chosen, therefore we have $\hat{\mathbf{A}} \preceq M$ for all $\hat{\mathbf{A}} \in \mathbf{S}$. Hence, $\mathbf{S}$ is bounded.

4 gH -subdifferentiability of convex IVFs

In this section we develop gH -subdifferential calculus for convex IVFs that are used later to find dual characterization of WSM for convex IVFs.

Definition 23 (gH -subdifferentiability). Let $\mathbf{F} : X \subseteq \mathbb{R}^n \rightarrow \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{dom}(\mathbf{F})$. Then, gH -subdifferential of $\mathbf{F}$ at $\bar{x}$, denoted by $\partial \mathbf{F}(\bar{x})$ is defined by

$$\partial \mathbf{F}(\bar{x}) = \left\{ \widehat{\mathbf{G}} \in I(\mathbb{R})^n : (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in X \right\}. \quad (10)$$

The elements of (10) are known as gH -subgradients of $\mathbf{F}$ at $\bar{x}$. Further, if $\partial \mathbf{F}(\bar{x}) \neq \emptyset$, we say that $\mathbf{F}$ is gH -subdifferentiable at $\bar{x}$.

Example 1 Consider $\mathbf{F} : \mathbb{R} \rightarrow I(\mathbb{R})$ be a convex IVF such that $\mathbf{F}(x) = |x| \odot \mathbf{A}$, where $\mathbf{0} \preceq \mathbf{A}$. Let us check gH -subdifferentiability of $\mathbf{F}$ at 0.

$$\begin{aligned} \partial \mathbf{F}(0) &= \{ \mathbf{G} \in I(\mathbb{R}) : (x - 0) \odot \mathbf{G} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(0) \text{ for all } x \in \mathbb{R} \} \\ &= \{ \mathbf{G} \in I(\mathbb{R}) : x \odot \mathbf{G} \preceq |x| \odot \mathbf{A} \text{ for all } x \in \mathbb{R} \} \end{aligned} \quad (11)$$

- Case 1. $x \leq 0$. In this case, for all $x \in \mathbb{R}$, (11) gives,

$$x \odot \mathbf{G} \preceq (-x) \odot \mathbf{A} \implies (-1) \odot \mathbf{A} \preceq \mathbf{G}.$$

- Case 2. $x > 0$. In this case, for all $x \in \mathbb{R}$, (11) gives,

$$x \odot \mathbf{G} \preceq x \odot \mathbf{A} \implies \mathbf{G} \preceq \mathbf{A}.$$

Hence, from Case 1 and Case 2, we have $\partial \mathbf{F}(0) = \{ \mathbf{G} \in I(\mathbb{R}) : (-1) \odot \mathbf{A} \preceq \mathbf{G} \preceq \mathbf{A} \}$.

Fig. 1 The IVF $\mathbf{F}$ of Example 1

In Fig. 1, the IVF $\mathbf{F}$, with $\mathbf{A} = [\frac{1}{4}, 1]$, is drawn by the gray shaded region between two red dashed lines, and its possible two gH -subgradients $\mathbf{G}_1$ and $\mathbf{G}_2$ at 0 are shown by blue and green shaded regions, respectively.

Lemma 12 Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\mathbf{F} : X \rightarrow \overline{I(\mathbb{R})}$ be a proper convex IVF. Then, for any $\bar{x} \in \text{dom}(\mathbf{F})$ and $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, the gH -subdifferential set of $\mathbf{F}$ at $\bar{x}$ is

$$\partial \mathbf{F}(\bar{x}) = \left\{ \widehat{\mathbf{G}} \in I(\mathbb{R})^n : h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_{\mathcal{D}}(\bar{x})(h) \right\},$$

where $\mathbf{F}_{\mathcal{D}}(\bar{x})(h)$ is gH -directional derivative of $\mathbf{F}$ at $\bar{x}$ in the direction of h .

Proof Suppose $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$. Then, by Definition 23, we have

$$(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in X. \quad (12)$$

By taking $x = \bar{x} + \lambda h$ with $\lambda > 0$ and $h \in \mathbb{R}^n$ in (12), we get

$$\begin{aligned} h^\top \odot \widehat{\mathbf{G}} &\preceq \frac{\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})}{\lambda} \\ \implies h^\top \odot \widehat{\mathbf{G}} &\preceq \lim_{\lambda \rightarrow 0} \frac{\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})}{\lambda} \\ \implies h^\top \odot \widehat{\mathbf{G}} &\preceq \mathbf{F}_{\mathcal{D}}(\bar{x})(h). \end{aligned}$$

Next, if we take any $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that $h^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_{\mathcal{D}}(\bar{x})(h)$ for all $h \in \mathbb{R}^n$. Then, by a similar reasoning as above it can be seen that $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$.

Theorem 4 Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\mathbf{F}: X \rightarrow \overline{I(\mathbb{R})}$ be a proper convex IVF with $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}, \overline{F}: X \rightarrow \mathbb{R}$ are extended real-valued functions. Then, for any $\bar{x} \in \text{dom}(\mathbf{F})$, $\partial \mathbf{F}(\bar{x})$ is closed and convex.

Proof We first prove the closedness of $\partial \mathbf{F}(\bar{x})$. Let $\{\widehat{\mathbf{G}}_k\}$ be a sequence in $\partial \mathbf{F}(\bar{x})$, which converges to $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{k1}, \mathbf{G}_{k2}, \dots, \mathbf{G}_{kn})$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n)$. Since $\widehat{\mathbf{G}}_k \in \partial \mathbf{F}(\bar{x})$, for all $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$\begin{aligned} h^\top \odot \widehat{\mathbf{G}}_k &\preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}), \\ \implies \min \left\{ \sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki} \right\} &\leq \min \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\} \\ \text{and } \max \left\{ \sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki} \right\} &\leq \max \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\}. \end{aligned} \quad (13)$$

Since the sequence $\{\widehat{\mathbf{G}}_k\}$ converges to $\widehat{\mathbf{G}}$, in view of Remark 6, the sequences $\{\underline{g}_{ki}\}$ and $\{\overline{g}_{ki}\}$ converge to $\underline{g}_i$ and $\overline{g}_i$, respectively, for each $i = 1, 2, \dots, n$. Thus,

$$\sum_{i=1}^n h_i \underline{g}_{ki} \rightarrow \sum_{i=1}^n h_i \underline{g}_i \text{ and } \sum_{i=1}^n h_i \overline{g}_{ki} \rightarrow \sum_{i=1}^n h_i \overline{g}_i \text{ as } k \rightarrow \infty. \quad (14)$$

Therefore, in view of (13) and (14), we have

$$\begin{aligned} \left(\min \left\{ \sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki} \right\} \right) &\rightarrow \left(\min \left\{ \sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i \right\} \right) \\ &\leq \min \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\} \end{aligned}$$

and

$$\begin{aligned} \left(\max \left\{ \sum_{i=1}^n h_i \underline{g}_{ki}, \sum_{i=1}^n h_i \overline{g}_{ki} \right\} \right) &\rightarrow \left(\max \left\{ \sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i \right\} \right) \\ &\leq \max \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} \left[\min \left\{ \sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i \right\}, \max \left\{ \sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \overline{g}_i \right\} \right] &\preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \\ \implies h^\top \odot \widehat{\mathbf{G}} &\preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } h \in X. \end{aligned}$$

Therefore, $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$, and hence $\partial \mathbf{F}(\bar{x})$ is closed.

To prove the convexity of $\partial \mathbf{F}(\bar{x})$, let $\widehat{\mathbf{H}} = (\mathbf{H}_1, \mathbf{H}_2, \dots, \mathbf{H}_n)$ and $\widehat{\mathbf{K}} = (\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_n)$ be any two elements of $\partial \mathbf{F}(\bar{x})$ with $\mathbf{H}_i = [\underline{h}_i, \bar{h}_i]$ and $\mathbf{K}_i = [\underline{k}_i, \bar{k}_i]$ for each $i = 1, 2, \dots, n$. Then, for all $\lambda_1, \lambda_2 \geq 0$, with $\lambda_1 + \lambda_2 = 1$ and for any $d \in \mathbb{R}^n$, we have

$$d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) = \left[\min \left\{ \sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \bar{h}_i + \lambda_2 \bar{k}_i) \right\}, \right. \\ \left. \max \left\{ \sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \bar{h}_i + \lambda_2 \bar{k}_i) \right\} \right].$$

- Case 1. Let $\min \left\{ \sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \bar{h}_i + \lambda_2 \bar{k}_i) \right\} = \sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i)$. Then,

$$\begin{aligned} d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) &= \left[\sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \bar{h}_i + \lambda_2 \bar{k}_i) \right] \\ &= \left[\sum_{i=1}^n \lambda_1 d_i \underline{h}_i, \sum_{i=1}^n \lambda_1 d_i \bar{h}_i \right] \oplus \left[\sum_{i=1}^n \lambda_2 d_i \underline{k}_i, \sum_{i=1}^n \lambda_2 d_i \bar{k}_i \right] \\ &= \lambda_1 \odot d^\top \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot d^\top \odot \widehat{\mathbf{K}} \\ &\preceq \lambda_1 \odot \mathbf{F}_{\mathcal{D}}(\bar{x})(d) \oplus \lambda_2 \odot \mathbf{F}_{\mathcal{D}}(\bar{x})(d) \text{ by Lemma 12} \\ &= \mathbf{F}_{\mathcal{D}}(\bar{x})(d) \text{ for any } d \in \mathbb{R}^n. \end{aligned}$$

Hence, $d^\top \odot (\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) \preceq \mathbf{F}_{\mathcal{D}}(\bar{x})(d)$ for any $d \in \mathbb{R}^n$. Therefore, by Lemma 12, $\lambda_1 \odot \widehat{\mathbf{H}} \oplus \lambda_2 \odot \widehat{\mathbf{K}} \in \partial \mathbf{F}(\bar{x})$.

- Case 2. Let $\min \left\{ \sum_{i=1}^n d_i(\lambda_1 \underline{h}_i + \lambda_2 \underline{k}_i), \sum_{i=1}^n d_i(\lambda_1 \bar{h}_i + \lambda_2 \bar{k}_i) \right\} = \sum_{i=1}^n d_i(\lambda_1 \bar{h}_i + \lambda_2 \bar{k}_i)$. Proof contains similar steps as in Case 1.

Thus, for any $\bar{x} \in \text{dom}(\mathbf{F})$, $\partial \mathbf{F}(\bar{x})$ is convex.

Theorem 5 Let X be a nonempty convex subset of $\mathbb{R}^n$ and let $\mathbf{F} : X \rightarrow I(\mathbb{R})$ be a gH -differentiable convex IVF at $\bar{x} \in X$. Then,

$$\partial \mathbf{F}(\bar{x}) = \{\nabla \mathbf{F}(\bar{x})\}.$$

Proof Let $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$. Since $\mathbf{F}$ is gH -differentiable at $\bar{x}$, with the help of Lemma 3 and Lemma 12, we get

$$\begin{aligned} h^\top \odot \widehat{\mathbf{G}} &\preceq \mathbf{L}_{\bar{x}}(h) \text{ for all } h \in \mathbb{R}^n \\ \implies h^\top \odot \widehat{\mathbf{G}} &\preceq \sum_{i=1}^n h_i \odot D_i \mathbf{F}(\bar{x}) \text{ by Theorem 1.} \end{aligned} \quad (15)$$

Replacing h by $-h$ in (15), we obtain

$$\begin{aligned} (-h)^\top \odot \widehat{\mathbf{G}} &\preceq \sum_{i=1}^n (-h_i) \odot D_i \mathbf{F}(\bar{x}) \\ \implies \sum_{i=1}^n h_i \odot D_i \mathbf{F}(\bar{x}) &\preceq h^\top \odot \widehat{\mathbf{G}} \text{ for all } h \in \mathbb{R}^n. \end{aligned} \quad (16)$$

Thus, (15) and (16), simultaneously give

$$\sum_{i=1}^n h_i \odot D_i \mathbf{F}(\bar{x}) = h^\top \odot \widehat{\mathbf{G}} \text{ for all } h \in \mathbb{R}^n. \quad (17)$$

Therefore, for each $i \in \{1, 2, \dots, n\}$, by choosing $h = e_i$ in (17), we have $D_i \mathbf{F}(\bar{x}) = \mathbf{G}_i$. Hence, $\nabla \mathbf{F}(\bar{x}) = \widehat{\mathbf{G}}$. Since $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$ is arbitrary, $\partial \mathbf{F}(\bar{x}) = \{\nabla \mathbf{F}(\bar{x})\}$.

Lemma 13 Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\mathbf{F}: X \rightarrow \overline{I(\mathbb{R})}$ be a proper convex IVF with $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}, \overline{F}: X \rightarrow \mathbb{R}$ are extended real-valued functions. Then, the subdifferential set of $\mathbf{F}$ at $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$ can be obtained by the subdifferential sets of $\underline{F}$ and $\overline{F}$ at $\bar{x}$ and vice-versa.

Proof Since $\mathbf{F}$ is proper convex, with the help of Lemma 2, we note that $\underline{F}$ and $\overline{F}$ are also convex. Therefore, by the property of real-valued proper convex functions, the subdifferential sets of $\underline{F}$ and $\overline{F}$ at $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$ are nonempty (see [2]). Let $\underline{g} = (g_1, g_2, \dots, g_n) \in \partial \underline{F}(\bar{x})$ and $\overline{g} = (\overline{g}_1, \overline{g}_2, \dots, \overline{g}_n) \in \partial \overline{F}(\bar{x})$. Then, by Definition 23 of gH -subdifferentiability, for any $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$h^\top \odot \underline{g} \leq \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}) \text{ and } h^\top \odot \overline{g} \leq \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}). \quad (18)$$

Note that $\mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x})$

$$\begin{aligned} &= \left[\min \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\}, \max \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\} \right] \\ \implies &\left[\min \left\{ h^\top \odot \underline{g}, h^\top \odot \overline{g} \right\}, \max \left\{ h^\top \odot \underline{g}, h^\top \odot \overline{g} \right\} \right] \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ by (18)} \\ \implies &h^\top \odot \hat{\mathbf{G}} \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}), \text{ where } \hat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n) \text{ with } \mathbf{G}_i = [g_i, \overline{g}_i] \\ \implies &\hat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x}). \end{aligned}$$

Thus, for any $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\overline{g} \in \partial \overline{F}(\bar{x})$, we have the corresponding $\hat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$.

To prove the converse part, for any $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, take $\hat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n) \in \partial \mathbf{F}(\bar{x})$ with $\mathbf{G}_i = [g_i, \overline{g}_i]$, $i = 1, 2, \dots, n$. Then, by Definition 23 of gH -subdifferentiability, we have

$$\begin{aligned} &h^\top \odot \hat{\mathbf{G}} \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X \\ \implies &\left[\min \left\{ \sum_{i=1}^n h_i g_i, \sum_{i=1}^n h_i \overline{g}_i \right\}, \max \left\{ \sum_{i=1}^n h_i g_i, \sum_{i=1}^n h_i \overline{g}_i \right\} \right] \preceq \mathbf{F}(\bar{x} + h) \ominus_{gH} \mathbf{F}(\bar{x}). \end{aligned}$$

Therefore,

$$\min \left\{ \sum_{i=1}^n h_i g_i, \sum_{i=1}^n h_i \overline{g}_i \right\} \leq \min \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\} \quad (19)$$

and

$$\max \left\{ \sum_{i=1}^n h_i g_i, \sum_{i=1}^n h_i \overline{g}_i \right\} \leq \max \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\}. \quad (20)$$

We now consider the following two possible cases.

- Case 1. Let $\min \left\{ \sum_{i=1}^n h_i g_i, \sum_{i=1}^n h_i \overline{g}_i \right\} = \sum_{i=1}^n h_i g_i$ and $\min \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\} = \underline{F}(\bar{x} + h) - \underline{F}(\bar{x})$. In this case, by (19) and (20), we have

$$\begin{aligned} &\sum_{i=1}^n h_i g_i \leq \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}) \text{ and } \sum_{i=1}^n h_i \overline{g}_i \leq \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \\ \implies &h^\top \odot \underline{g} \leq \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}) \text{ and } h^\top \odot \overline{g} \leq \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}), \\ &\text{where } \underline{g} = (g_1, g_2, \dots, g_n) \in \mathbb{R}^n \text{ and } \overline{g} = (\overline{g}_1, \overline{g}_2, \dots, \overline{g}_n) \in \mathbb{R}^n. \end{aligned}$$

Thus, we get $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\overline{g} \in \partial \overline{F}(\bar{x})$, which are required.

- Case 2. Let $\min \left\{ \sum_{i=1}^n h_i g_i, \sum_{i=1}^n h_i \overline{g}_i \right\} = \sum_{i=1}^n h_i \overline{g}_i$ and $\min \left\{ \underline{F}(\bar{x} + h) - \underline{F}(\bar{x}), \overline{F}(\bar{x} + h) - \overline{F}(\bar{x}) \right\} = \overline{F}(\bar{x} + h) - \overline{F}(\bar{x})$. Proof contains similar steps as in Case 1.

From Case 1 and Case 2, it is clear that for any $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$, we can obtain the subgradients of $\bar{F}$ and $\underline{F}$ at $\bar{x}$. This completes the proof for the converse part.

Remark 7 By Lemma 13, it is easy to note that for any proper convex IVF $\mathbf{F}(x) = [\underline{F}(x), \bar{F}(x)]$ and $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, $\partial \mathbf{F}(\bar{x})$ is nonempty.

Theorem 6 Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\mathbf{F}: X \rightarrow \overline{I(\mathbb{R})}$ be a proper convex IVF with $\mathbf{F}(x) = [\underline{F}(x), \bar{F}(x)]$, where $\underline{F}, \bar{F}: X \rightarrow \overline{\mathbb{R}}$ are extended real-valued functions. Then, at any $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$,

$$\mathbf{F}_{\mathcal{D}}(\bar{x})(h) = \psi_{\partial \mathbf{F}(\bar{x})}^*(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X,$$

where $\mathbf{F}_{\mathcal{D}}(\bar{x})(h)$ is gH -directional derivative of $\mathbf{F}$ at $\bar{x}$ in the direction of h .

Proof Note that $\underline{F}(x)$ and $\bar{F}(x)$ are proper convex, and therefore $\partial \underline{F}(\bar{x})$ and $\partial \bar{F}(\bar{x})$ are nonempty. Let $\underline{g} = (\underline{g}_1, \underline{g}_2, \dots, \underline{g}_n) \in \partial \underline{F}(\bar{x})$ and $\bar{g} = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n) \in \partial \bar{F}(\bar{x})$ for $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$. By the property of real-valued convex functions (see [10]), we have

$$\underline{F}_{\mathcal{D}}(\bar{x})(h) = \psi_{\partial \underline{F}(\bar{x})}^*(h) \text{ and } \bar{F}_{\mathcal{D}}(\bar{x})(h) = \psi_{\partial \bar{F}(\bar{x})}^*(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X.$$

Due to Theorem 2, we get

$$\begin{aligned} \mathbf{F}_{\mathcal{D}}(\bar{x})(h) &= \left[\min \left\{ \underline{F}_{\mathcal{D}}(\bar{x})(h), \bar{F}_{\mathcal{D}}(\bar{x})(h) \right\}, \max \left\{ \underline{F}_{\mathcal{D}}(\bar{x})(h), \bar{F}_{\mathcal{D}}(\bar{x})(h) \right\} \right] \\ &= \left[\min \left\{ \psi_{\partial \underline{F}(\bar{x})}^*(h), \psi_{\partial \bar{F}(\bar{x})}^*(h) \right\}, \max \left\{ \psi_{\partial \underline{F}(\bar{x})}^*(h), \psi_{\partial \bar{F}(\bar{x})}^*(h) \right\} \right]. \end{aligned} \quad (21)$$

We now consider the following two possible cases.

- Case 1. Let $\psi_{\partial \underline{F}(\bar{x})}^*(h) \leq \psi_{\partial \bar{F}(\bar{x})}^*(h)$. In this case, by (21), we get

$$\begin{aligned} \mathbf{F}_{\mathcal{D}}(\bar{x})(h) &= \left[\psi_{\partial \underline{F}(\bar{x})}^*(h), \psi_{\partial \bar{F}(\bar{x})}^*(h) \right] = \left[\sup_{\underline{g} \in \partial \underline{F}(\bar{x})} h^\top \odot \underline{g}, \sup_{\bar{g} \in \partial \bar{F}(\bar{x})} h^\top \odot \bar{g} \right] \\ &= \sup_{\underline{g} \in \partial \underline{F}, \bar{g} \in \partial \bar{F}(\bar{x})} \left[h^\top \odot \underline{g}, h^\top \odot \bar{g} \right] = \sup_{\underline{g} \in \partial \underline{F}, \bar{g} \in \partial \bar{F}(\bar{x})} \left[\sum_{i=1}^n h_i \underline{g}_i, \sum_{i=1}^n h_i \bar{g}_i \right]. \end{aligned} \quad (22)$$

We have seen in Lemma 13 that corresponding to every $\underline{g} \in \partial \underline{F}(\bar{x})$ and $\bar{g} \in \partial \bar{F}(\bar{x})$, we get $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$ and vice-versa. Thus, for $\underline{g} = (\underline{g}_1, \underline{g}_2, \dots, \underline{g}_n) \in \partial \underline{F}(\bar{x})$ and $\bar{g} = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n) \in \partial \bar{F}(\bar{x})$, by (22), we obtain

$$\begin{aligned} \mathbf{F}_{\mathcal{D}}(\bar{x})(h) &= \sup_{\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})} h^\top \odot \widehat{\mathbf{G}}, \text{ where } \widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n) \text{ with } \mathbf{G}_i = [\underline{g}_i, \bar{g}_i] \\ &= \psi_{\partial \mathbf{F}(\bar{x})}^*(h). \end{aligned}$$

- Case 2. Let $\psi_{\partial \bar{F}(\bar{x})}^*(h) \leq \psi_{\partial \underline{F}(\bar{x})}^*(h)$. Proof contains similar steps as in Case 1.

Hence, from Case 1 and Case 2, we have

$$\mathbf{F}_{\mathcal{D}}(\bar{x})(h) = \psi_{\partial \mathbf{F}(\bar{x})}^*(h) \text{ for all } h \in \mathbb{R}^n \text{ such that } \bar{x} + h \in X.$$

Theorem 7 Let $\mathbf{F}: X \rightarrow \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$. Then, the gH -subdifferential set of $\mathbf{F}$ at $\bar{x}$ is bounded.

Proof Note that by Theorem 2, for $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$, the directional derivative of $\mathbf{F}$ at $\bar{x}$ exists everywhere. Thus, for all $h \in \mathbb{R}^n$ such that $\bar{x} + h \in X$, we have

$$\begin{aligned} \mathbf{F}_{\mathcal{D}}(\bar{x})(h) &\text{ is finite} \\ \implies \psi_{\partial \mathbf{F}(\bar{x})}^*(h) &\text{ is finite by Theorem 6} \\ \implies \partial \mathbf{F}(\bar{x}) &\text{ is bounded by Lemma 11.} \end{aligned}$$

Hence, the gH -subdifferential set of $\mathbf{F}$ at $\bar{x} \in \text{int}(\text{dom}(\mathbf{F}))$ is bounded, i.e., for every $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{x})$, there exists an $M > 0$ such that $\|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \leq M$.

Lemma 14 Let $\mathbf{F}$ be an IVF on a nonempty set $X \subseteq \mathbb{R}^n$ such that

$$\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq c\|x - y\| \text{ for all } x, y \in X,$$

where $c \in \mathbb{R}$. Then,

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq c\|x - y\| \text{ for all } x, y \in X.$$

Proof We have $\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq c\|x - y\|$ for all $x, y \in X$, which implies that

$$\underline{F}(x) - \underline{F}(y) \leq c\|x - y\| \text{ and } \overline{F}(x) - \overline{F}(y) \leq c\|x - y\|. \quad (23)$$

Interchanging x and y in (23), we obtain

$$\underline{F}(y) - \underline{F}(x) \leq c\|x - y\| \text{ and } \overline{F}(y) - \overline{F}(x) \leq c\|x - y\|. \quad (24)$$

With the help of (23) and (24), we get

$$|\underline{F}(x) - \underline{F}(y)| \leq c\|x - y\| \text{ and } |\overline{F}(x) - \overline{F}(y)| \leq c\|x - y\| \text{ for all } x, y \in X,$$

which implies

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq c\|x - y\| \text{ for all } x, y \in X.$$

Theorem 8 Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\mathbf{F}$ be a convex IVF on X such that $\mathbf{F}$ has gH -subgradient at every $x \in X$. Then, $\mathbf{F}$ is gH -Lipschitz continuous on X .

Proof Since $\mathbf{F}$ has gH -subgradient at every $x \in X$, then there exists a $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ such that

$$\begin{aligned} & (y - x)^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ for all } y \in X \\ \implies & (-1) \odot \left((x - y)^\top \odot \widehat{\mathbf{G}} \right) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \\ \implies & \mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq (x - y)^\top \odot \widehat{\mathbf{G}} \\ \implies & \mathbf{F}(x) \ominus_{gH} \mathbf{F}(y) \preceq \|x - y\| \odot \left[\|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n}, \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \right] \text{ by Lemma 10} \\ \implies & \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \|x - y\| \text{ by Lemma 14} \\ \implies & \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq M\|x - y\|, \text{ where } \|\widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} \leq M \text{ by Theorem 7.} \end{aligned}$$

Thus, $\mathbf{F}$ is gH -Lipschitz continuous on X .

5 Weak sharp minima and its characterizations

In this section, we present the main results—primal and dual characterizations of WSM for a gH -lsc and convex IVF.

Definition 24 (*WSM for an IVF*). Let $\mathbf{F} : \mathbb{R}^n \rightarrow \overline{I(\mathbb{R})}$ be a gH -lsc and convex IVF. Let $\bar{S}$ and S be two nonempty closed convex sets such that $\bar{S} \subseteq S \subseteq \mathbb{R}^n$. Further, let $\text{dom}(\mathbf{F}) \cap S \neq \emptyset$. Then, the set $\bar{S}$ is said to be a set of WSM of $\mathbf{F}$ over the set S with modulus $\alpha > 0$ if

$$\mathbf{F}(\bar{x}) \oplus \alpha \text{dist}(x, \bar{S}) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S.$$

Remark 8 Let $\mathbf{F} : \mathbb{R}^n \rightarrow \overline{I(\mathbb{R})}$ be a gH -lsc and convex IVF with $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$ for all $x \in \mathbb{R}^n$, where $\underline{F}, \overline{F} : \mathbb{R}^n \rightarrow \mathbb{R}$ be two extended real-valued functions. Then, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$ if and only if $\bar{S}$ is a set of WSM of $\underline{F}$ and $\overline{F}$ over S with modulus $\alpha > 0$. The reason is as follows. By Remark 5 and Lemma 2, it is easy to see that the functions $\underline{F}$ and $\overline{F}$ are lsc and convex. Let $\bar{S}$ be a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$. Then,

$$\begin{aligned} & \mathbf{F}(\bar{x}) \oplus \alpha \text{dist}(x, \bar{S}) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S \\ \iff & [\underline{F}(\bar{x}) + \alpha \text{dist}(x, \bar{S}), \overline{F}(\bar{x}) + \alpha \text{dist}(x, \bar{S})] \preceq [\underline{F}(x), \overline{F}(x)] \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S \\ \iff & \underline{F}(\bar{x}) + \alpha \text{dist}(x, \bar{S}) \leq \underline{F}(x) \text{ and } \overline{F}(\bar{x}) + \alpha \text{dist}(x, \bar{S}) \leq \overline{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S \\ \iff & \bar{S} \text{ is a set of WSM of both } \underline{F} \text{ and } \overline{F} \text{ over } S \text{ with modulus } \alpha > 0. \end{aligned}$$

Example 2 Let $\mathbf{F} : \mathbb{R}^2 \rightarrow \overline{I(\mathbb{R})}$ be an IVF defined by

$$\mathbf{F}(x) = [5 - x_1x_2 - x_1, 10 - x_1^2x_2 - x_2^2x_1].$$

Let $S = [-a, 0] \times [-a, 0] \subseteq \mathbb{R}^2$ and $\bar{S} = \{0\} \times [-a, 0]$, where $a > 0$. Thus, $\bar{S} \subseteq S \subseteq \mathbb{R}^n$. Clearly, the functions $\underline{F}$ and $\overline{F}$ are $5 - x_1x_2 - x_1$ and $10 - x_1^2x_2 - x_2^2x_1$, respectively. Note that for any $\alpha > 0$,

$$\underline{F}(\bar{x}) + \alpha \text{dist}(x, \bar{S}) \leq \underline{F}(x) \text{ and } \overline{F}(\bar{x}) + \alpha \text{dist}(x, \bar{S}) \leq \overline{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S.$$

Thus, $\bar{S} = \{0\} \times [-a, 0]$ is a set of WSM of both $\underline{F}$ and $\overline{F}$ over S with modulus α , for any $\alpha > 0$. Therefore, by Remark 8, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$.

Theorem 9 (Primal characterization). *Let $\mathbf{F}$, S , and $\bar{S}$ be as in Definition 24. Further, define an IVF $\mathbf{F}_o : \mathbb{R}^n \rightarrow \overline{I(\mathbb{R})}$ by*

$$\mathbf{F}_o = \begin{cases} \mathbf{F}(x), & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the set $\bar{S}$ is a set of WSM of $\mathbf{F}$ over the set S with modulus $\alpha > 0$ if and only if

$$\alpha \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in \mathbb{R}^n. \quad (25)$$

Proof Suppose $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$. Then, by Definition 24, for any $x \in \bar{S}$, $d \in \mathbb{R}^n$, and $t > 0$, we have

$$\begin{aligned} & \mathbf{F}_o(x) \oplus \alpha \text{dist}(x + td, \bar{S}) \preceq \mathbf{F}_o(x + td) \\ \implies & \alpha \text{dist}(x + td, \bar{S}) \preceq \mathbf{F}_o(x + td) \ominus_{gH} \mathbf{F}_o(x) \\ \implies & \frac{\alpha}{t} (\text{dist}(x + td, \bar{S}) - \text{dist}(x, \bar{S})) \preceq \frac{1}{t} \odot (\mathbf{F}_o(x + td) \ominus_{gH} \mathbf{F}_o(x)) \\ \implies & \lim_{t \rightarrow 0} \frac{\alpha}{t} (\text{dist}(x + td, \bar{S}) - \text{dist}(x, \bar{S})) \preceq \lim_{t \rightarrow 0} \frac{1}{t} \odot (\mathbf{F}_o(x + td) \ominus_{gH} \mathbf{F}_o(x)) \\ \implies & \alpha \lim_{t \rightarrow 0} \frac{1}{t} (\text{dist}(x + td, \bar{S}) - \text{dist}(x, \bar{S})) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ by Definition 10} \\ \implies & \alpha \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ by part (ii) of Lemma 7.} \end{aligned}$$

Thus,

$$\alpha \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in \mathbb{R}^n.$$

For the converse part, let $y \in S$ and $x \in \bar{S}$. Therefore, from Lemma 5, we get

$$\begin{aligned} & \mathbf{F}_{o\mathcal{D}}(x)(y - x) \preceq \mathbf{F}_o(y) \ominus_{gH} \mathbf{F}_o(x) \\ \implies & \mathbf{F}_o(x) \oplus \mathbf{F}_{o\mathcal{D}}(x)(y - x) \preceq \mathbf{F}_o(y) \\ \implies & \mathbf{F}_o(x) \oplus \alpha \text{dist}(y - x, T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y) \text{ by (25)} \\ \implies & \mathbf{F}_o(x) \oplus \alpha \text{dist}(y, x + T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y). \end{aligned}$$

Since $x \in \bar{S}$ is arbitrary, we have

$$\begin{aligned} & \mathbf{F}_o(x) \oplus \alpha \sup_{x \in \bar{S}} \text{dist}(y, x + T_{\bar{S}}(x)) \preceq \mathbf{F}_o(y) \\ \implies & \mathbf{F}_o(x) \oplus \alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}_o(y) \text{ for all } x \in \bar{S} \text{ and } \alpha > 0 \text{ by part (i) of Lemma 7.} \end{aligned}$$

Hence, $\bar{S}$ is the set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$, and the proof is complete.

Theorem 10 (Dual characterizations). *Let $\mathbf{F}$, S , and $\bar{S}$ be as in Definition 24. Define an IVF $\mathbf{F}_o : \mathbb{R}^n \rightarrow \overline{I(\mathbb{R})}$ by*

$$\mathbf{F}_o(x) = \begin{cases} \mathbf{F}(x), & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, for any $\alpha > 0$, the following statements are equivalent.

- (a) The set $\bar{S}$ is a set of WSM of $\mathbf{F}$ over the set S with modulus α .
 (b) The normal cone inclusion holds. That is,

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \partial\mathbf{F}_o(x) \text{ for all } x \in \bar{S}.$$

- (c) For all $x \in \bar{S}$ and $d \in T_S(x)$,

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathcal{D}}(x)(d).$$

- (d) The following inclusion holds,

$$\alpha\mathbb{B} \cap \left(\bigcup_{x \in \bar{S}} N_{\bar{S}}(x) \right) \subseteq \bigcup_{x \in \bar{S}} \partial\mathbf{F}_o(x).$$

- (e) For all $x \in \bar{S}$ and $d \in T_S(x) \cap N_{\bar{S}}(x)$,

$$\alpha\|d\| \preceq \mathbf{F}_{\mathcal{D}}(x)(d).$$

- (f) For all $y \in S$,

$$\alpha \operatorname{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathcal{D}}(p)(y - p),$$

where $p \in P(y \mid \bar{S})$.

Proof (a) $\iff$ (b). Let $x \in \bar{S}$. By hypothesis, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S . Therefore, by Theorem 9, we get

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n,$$

which along with Theorem 6 imply

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \psi_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}^*(d) \text{ for all } d \in \mathbb{R}^n. \quad (26)$$

Notice that for all $x \in \bar{S}$ and $d \in \mathbb{R}^n$, we have

$$\begin{aligned} \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) &= \alpha \psi_{\mathbb{B} \cap N_{\bar{S}}(x)}^*(d) \text{ by (ii) of Lemma 7} \\ &= \alpha \sup \langle z, d \rangle, \text{ where } z \in \mathbb{B} \cap N_{\bar{S}}(x) \\ &= \sup \langle \alpha z, d \rangle, \text{ where } z \in \mathbb{B} \cap N_{\bar{S}}(x) \text{ and } \alpha > 0 \\ &= \sup \langle z, d \rangle, \text{ where } z \in \alpha\mathbb{B} \cap N_{\bar{S}}(x) \\ &= \psi_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}^*(d) \text{ for all } d \in \mathbb{R}^n. \end{aligned}$$

That is,

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) = \psi_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}^*(d) \text{ for all } d \in \mathbb{R}^n. \quad (27)$$

Thus, by (26) and (27), we get

$$\psi_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}^*(d) \preceq \psi_{\partial\mathbf{F}_o(x)}^*(d). \quad (28)$$

Next, with the help of Lemma 9, we get the desired result

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \partial\mathbf{F}_o(x) \text{ for all } x \in \bar{S}. \quad (29)$$

Conversely, we have

$$\alpha\mathbb{B} \cap N_{\bar{S}}(x) \subseteq \partial\mathbf{F}_o(x) \text{ for all } x \in \bar{S} \quad (30)$$

$$\implies \psi_{\alpha\mathbb{B} \cap N_{\bar{S}}(x)}^*(d) \preceq \psi_{\partial\mathbf{F}_o(x)}^*(d) \text{ for all } d \in \mathbb{R}^n \text{ by Lemma 8}$$

$$\implies \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \psi_{\partial\mathbf{F}_o(x)}^*(d) \text{ for all } d \in \mathbb{R}^n \text{ by (27)}. \quad (31)$$

Also, by Theorem 6, we have

$$\psi_{\partial\mathbf{F}_o(x)}^*(d) = \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n.$$

Thus, by (31), we get

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ for all } d \in \mathbb{R}^n.$$

Therefore, by Theorem 9, $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S with modulus α .

(a) $\iff$ (c). Let the statement (a) holds. Let $x \in \bar{S}$. Therefore, by Theorem 9, we have

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ for all } d \in T_S(x).$$

Note that for $x \in \bar{S}$, $\mathbf{F}_o(x) = \mathbf{F}(x)$. Thus,

$$\mathbf{F}_{o\mathcal{D}}(x)(d) = \mathbf{F}_{\mathcal{D}}(x)(d) \text{ for } x \in \bar{S} \text{ and } d \in T_S(x). \quad (32)$$

By (32) and Theorem 9, we get

$$\alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathcal{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ and } x \in \bar{S}.$$

Conversely, we are given that

$$\begin{aligned} & \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathcal{D}}(x)(d) \text{ for all } x \in \bar{S} \text{ and } d \in T_S(x) \\ \implies & \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \psi_{\partial \mathbf{F}(x)}^*(d) \text{ for all } d \in T_S(x) \text{ by Theorem 6.} \end{aligned} \quad (33)$$

Note that for $x \in \bar{S}$, we have

$$\psi_{\partial \mathbf{F}(x)}^*(d) = \psi_{\partial \mathbf{F}_o(x)}^*(d) \text{ for all } d \in \mathbb{R}^n. \quad (34)$$

In view of (33) and (34), we have

$$\begin{aligned} & \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \psi_{\partial \mathbf{F}_o(x)}^*(d) \text{ for all } d \in T_S(x) \\ \implies & \alpha \operatorname{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{o\mathcal{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ by Theorem 6.} \end{aligned}$$

Hence, by Theorem 9, $\bar{S}$ is the set of WSM of $\mathbf{F}$ over S with modulus $\alpha > 0$.

(b) $\iff$ (d). If the statement (b) holds, then obviously the statement (d) also holds.

Conversely, let the statement (d) holds. Let $x \in \bar{S}$ and $\hat{G} \in \alpha \mathbb{B} \cap N_{\bar{S}}(x)$. Therefore, there exists a $\bar{y} \in \bar{S}$ such that $\hat{G} \in \partial \mathbf{F}_o(\bar{y})$. Thus, by Definition 23, we get

$$(z - \bar{y})^\top \odot \hat{G} \preceq \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(\bar{y}) \text{ for all } z \in \mathbb{R}^n. \quad (35)$$

In particular, for any $z \in \bar{S}$, $\mathbf{F}_o(z) = \mathbf{F}_o(\bar{y})$. Thus, (35) reduces to

$$(z - \bar{y})^\top \odot \hat{G} \preceq \mathbf{0} \text{ for all } z \in \bar{S}.$$

Since $\hat{G} \in \mathbb{R}^n$, by using Remark 1, $(z - \bar{y})^\top \odot \hat{G} = \langle \hat{G}, z - \bar{y} \rangle \leq 0$ for all $z \in \bar{S}$. Therefore,

$$\begin{aligned} & \langle \hat{G}, z \rangle \leq \langle \hat{G}, \bar{y} \rangle \text{ for all } z \in \bar{S} \\ \implies & \sup_{z \in \bar{S}} \langle \hat{G}, z \rangle \leq \langle \hat{G}, \bar{y} \rangle \\ \implies & \psi_{\bar{S}}^*(\hat{G}) = \langle \hat{G}, \bar{y} \rangle \text{ because } \bar{y} \in \bar{S}. \end{aligned} \quad (36)$$

Since $\hat{G} \in N_{\bar{S}}(x)$, by Definition 21, we have

$$\begin{aligned} & \langle \hat{G}, z - x \rangle \leq 0 \text{ for all } z \in \bar{S} \\ \implies & \psi_{\bar{S}}^*(\hat{G}) = \langle \hat{G}, x \rangle. \end{aligned} \quad (37)$$

Combining (36) and (37), we get

$$\langle \hat{G}, x \rangle = \langle \hat{G}, \bar{y} \rangle. \quad (38)$$

Note that

$$\begin{aligned}
(z - x)^\top \odot \widehat{G} &= \left\langle \widehat{G}, z - x \right\rangle \text{ for all } z \in \mathbb{R}^n \\
&= \left\langle \widehat{G}, z - \bar{y} \right\rangle \text{ for all } z \in \mathbb{R}^n \text{ by (38)} \\
&= (z - \bar{y})^\top \odot \widehat{G} \text{ for all } z \in \mathbb{R}^n \text{ by Remark 1} \\
&\preceq \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(\bar{y}) \text{ for all } z \in \mathbb{R}^n \text{ by (35)} \\
&= \mathbf{F}_o(z) \ominus_{gH} \mathbf{F}_o(x) \text{ for all } z \in \mathbb{R}^n \text{ because } \mathbf{F}(x) = \mathbf{F}(\bar{y}).
\end{aligned}$$

Hence, $\widehat{G} \in \partial \mathbf{F}_o(x)$. Since $x \in \bar{S}$ is arbitrary, the statement (b) holds.
(c) $\implies$ (e). From the statement (c), we have

$$\begin{aligned}
&\alpha \text{dist}(d, T_{\bar{S}}(x)) \preceq \mathbf{F}_{\mathcal{D}}(x)(d) \text{ for all } d \in T_S(x) \text{ and } x \in \bar{S} \\
\implies &\alpha \|d\| \preceq \mathbf{F}_{\mathcal{D}}(x)(d) \text{ for all } d \in T_S(x) \cap N_{\bar{S}}(x) \text{ by (ii) of Lemma 7.}
\end{aligned}$$

Hence, the statement (e) holds.

(e) $\implies$ (a). Let $y \in S$. Set $x = P(y \mid \bar{S})$, then $(y - x) \in T_S(x) \cap N_{\bar{S}}(x)$. Therefore, according to the hypothesis, we obtain

$$\begin{aligned}
&\alpha \|y - x\| \preceq \mathbf{F}_{\mathcal{D}}(x)(y - x) \\
\implies &\alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathcal{D}}(x)(y - x) \text{ by Definition 18} \\
\implies &\alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ by Lemma 5} \\
\implies &\mathbf{F}(x) \oplus \alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \text{ by (i) of Lemma 1,}
\end{aligned}$$

which shows that $\bar{S}$ is a set of WSM of $\mathbf{F}$ over S .

(a) $\iff$ (f). Let the statement (a) holds. Let $y \in S$ and $p = P(y \mid \bar{S})$. Thus, the statement (a) gives

$$\mathbf{F}(p) \oplus \alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y), \text{ i.e., } \mathbf{F}(p) \oplus \alpha \|y - p\| \preceq \mathbf{F}(y). \quad (39)$$

Define $z_\lambda = \lambda y + (1 - \lambda)p$ for $\lambda \in [0, 1]$. Then, $p = P(z_\lambda \mid \bar{S})$ for all $\lambda \in [0, 1]$. From (39), we have

$$\begin{aligned}
&\mathbf{F}(p) \oplus \alpha \|z_\lambda - p\| \preceq \mathbf{F}(z_\lambda) \\
\implies &\mathbf{F}(p) \oplus \alpha \lambda \|y - p\| \preceq \mathbf{F}(z_\lambda) \\
\implies &\alpha \|x - p\| \preceq \frac{1}{\lambda} \odot \left(\mathbf{F}(p + \lambda(y - p)) \ominus_{gH} \mathbf{F}(p) \right). \quad (40)
\end{aligned}$$

By taking limit as $\lambda \downarrow 0$ in (40), we get

$$\alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathcal{D}}(p)(y - p), \text{ where } p \in P(y \mid \bar{S}).$$

Conversely, let $y \in S$ and set $x = P(y \mid \bar{S})$. Then, from the statement (f), we get

$$\begin{aligned}
&\alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}_{\mathcal{D}}(x)(y - x) \\
\implies &\alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(x) \text{ by Lemma 5} \\
\implies &\mathbf{F}(x) \oplus \alpha \text{dist}(y, \bar{S}) \preceq \mathbf{F}(y) \text{ by (i) of Lemma 1,}
\end{aligned}$$

which is the required result.

6 Conclusion and future scopes

In this article, the conventional concepts of support function and subdifferentiability have been extended for IVFs (Definitions 22 and Definition 23). Also, some important characteristics of the gH -subdifferential set like nonemptiness (Lemma 13), boundedness (Theorem 7), convexity and closedness (Theorem 4) have been presented. Subsequently, we have provided few necessary results (Lemma 8, Theorem 3 and Lemma 9) based on the support function of a subset of $I(\mathbb{R})^n$. It has been reported that the gH -subdifferential set of a gH -differentiable convex IVF is a singleton set containing the gH -gradient (Theorem 5). The relationship between gH -directional derivative and the support function of gH -subdifferential set of convex IVF has been also established (Theorem 6). Further, we have introduced the notion of WSM for convex IVFs (Definition 24). With the help of the proposed concepts of gH -subdifferentiability and support function, a primal characterization (Theorem 9) and a few dual characterizations (Theorem 10) of WSM have been presented.

In future, we shall apply proposed theory on WSM to derive necessary and sufficient conditions under which a global error bound may exist for a convex inequality system as follows:

$$\mathbf{H}_\lambda(x) \preceq \mathbf{0}, \lambda \in \Lambda \text{ and } x \in C, \quad (41)$$

where Λ is an index set, and for each $\lambda \in \Lambda$, $\mathbf{H}_\lambda : X \subseteq \mathbb{R}^n \rightarrow I(\mathbb{R})$ is gH -lsc, convex, proper, and the set C is closed convex subset of X . By a global error bound for the inequality system (41), we mean the existence of a constant $\beta > 0$ such that

$$\beta \text{dist}(x, \Omega) \preceq \text{dist}(x, C) \oplus \mathbf{H}_{\lambda+}(x) \text{ for each } \lambda \in \Lambda \text{ and } x \in X, \quad (42)$$

where $\Omega = \{x : x \in C \text{ and } \mathbf{H}_\lambda(x) \preceq \mathbf{0}\}$ and $\mathbf{H}_{\lambda+}(x) = \max\{\mathbf{0}, \mathbf{H}_\lambda(x)\}$. For the sake of convenience, we define $\mathbf{H}(x) = \sup\{\mathbf{H}_\lambda(x) : \lambda \in \Lambda\}$ for each $x \in X$. Note that if a constant $\bar{\beta} > 0$ exists such that

$$\bar{\beta} \text{dist}(x, \Omega) \preceq \text{dist}(x, C) \oplus \mathbf{H}_+(x) \text{ for all } x \in X, \quad (43)$$

where $\bar{\Omega} = \{x : x \in C \text{ and } \mathbf{H}(x) \preceq \mathbf{0}\}$ and $\mathbf{H}_+(x) = \max\{\mathbf{0}, \mathbf{H}(x)\}$, then (42) holds for $\beta > 0$. The following observation can be useful to solve the problem. If we define an IVF $\mathbf{F} : X \rightarrow I(\mathbb{R})$ such that

$$\mathbf{F}(x) = \text{dist}(x, C) \oplus \mathbf{H}_+(x),$$

then $\mathbf{F}$ has Ω as a set of WSM with modulus $\bar{\beta} > 0$, which is equivalent to the condition in (43).

A Proof of Lemma 1

(i). Let $\mathbf{A} = [\underline{a}, \bar{a}]$, $\mathbf{B} = [\underline{b}, \bar{b}]$, and $\mathbf{C} = [\underline{c}, \bar{c}]$. We have

$$r \leq \underline{a} \text{ and } r \leq \bar{a}. \quad (44)$$

Similarly, by $\mathbf{A} \preceq \mathbf{B} \ominus_{gH} \mathbf{C}$, we have

$$\underline{a} \leq \min\{\underline{b} - \underline{c}, \bar{b} - \bar{c}\} \text{ and } \bar{a} \leq \max\{\underline{b} - \underline{c}, \bar{b} - \bar{c}\}. \quad (45)$$

- Case 1. Let $\min\{\underline{b} - \underline{c}, \bar{b} - \bar{c}\} = \bar{b} - \bar{c}$ and $\max\{\underline{b} - \underline{c}, \bar{b} - \bar{c}\} = \underline{b} - \underline{c}$. Then, from (44) and (45), we get

$$\bar{c} + r \leq \bar{b} \text{ and } \underline{c} + r \leq \underline{b}.$$

Hence, $\mathbf{C} \oplus [r, r] \preceq \mathbf{B}$.

- Case 2. When $\min\{\underline{b} - \underline{c}, \bar{b} - \bar{c}\} = \underline{b} - \underline{c}$ and $\max\{\underline{b} - \underline{c}, \bar{b} - \bar{c}\} = \bar{b} - \bar{c}$. Proof contains similar steps as in Case 1.

(ii). Let $\mathbf{A} = [\underline{a}, \bar{a}]$ and $\mathbf{B} = [\underline{b}, \bar{b}]$. Then,

$$\begin{aligned} \left\{ (1-\lambda) \odot \mathbf{A} \oplus \lambda \mathbf{B} \right\} \ominus_{gH} \mathbf{A} &= \left\{ (1-\lambda) \odot [\underline{a}, \bar{a}] \oplus \lambda \odot [\underline{b}, \bar{b}] \right\} \ominus_{gH} [\underline{a}, \bar{a}] \\ &= \left[(1-\lambda)\underline{a} + \lambda\underline{b}, (1-\lambda)\bar{a} + \lambda\bar{b} \right] \ominus_{gH} [\underline{a}, \bar{a}] \text{ because } \lambda \in [0, 1] \\ &= \left[\min \left\{ \lambda\underline{b} - \lambda\underline{a}, \lambda\bar{b} - \lambda\bar{a} \right\}, \max \left\{ \lambda\underline{b} - \lambda\underline{a}, \lambda\bar{b} - \lambda\bar{a} \right\} \right] \\ &= \lambda \odot \left\{ \mathbf{A} \ominus_{gH} \mathbf{B} \right\}. \end{aligned}$$

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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