Noname manuscript No.
(will be inserted by the editor)

A General Numerical Scheme for the Optimal Control of Fractional Birkhoffian Systems

Lin He · Chunqiu Wei · Jiang Sha · Delong Mao · Kangshuo Wang

Received: date / Accepted: date

Abstract This paper deals with the optimal control of fractional Birkhoffian systems based on the numerical method of variational integrators. Firstly, the fractional forced Birkhoff equations within Riemann–Liouville fractional derivatives are derived from the fractional Pfaff–Birkhoff–d'Alembert principle. Secondly, by directly discretizing the fractional Pfaff–Birkhoff–d'Alembert principle, we develop the equivalent discrete fractional forced Birkhoff equations, which are served as the equality constraints of the optimization problem. Together with the initial and final state constraints on the configuration space, the original optimal control problem is converted into a nonlinear optimization problem subjected to a system of algebraic constraints, which can be solved by the existing methods such as sequential quadratic programming. Finally, an example is given to show the efficiency and simplicity of the proposed method.

Keywords Fractional calculus · Fractional calculus of variations · Optimal control · Fractional Birkhoffian systems

Lin He, Corresponding author
CSSC Systems Engineering Research Institute, Beijing 100094, China
E-mail: hl617@126.com

Chunqiu Wei
School of Mathematics, Beijing University of Civil Engineering and Architecture, Beijing 102616, China

Jiang Sha
CSSC Systems Engineering Research Institute, Beijing 100094, China

Delong Mao
CSSC Systems Engineering Research Institute, Beijing 100094, China

Kangshuo Wang
CSSC Systems Engineering Research Institute, Beijing 100094, China

1 Introduction

In recent years, fractional calculus (FC, derivatives and integrals of arbitary order) has been a hot topic in different aspects of science and engineering, such as viscoelasticity, control theory, heat conduction, electricity, biomechanics, economics, polymers, liquid crystals, chaos and fractals, etc [1–6]. Among the diversified fields of FC, fractional optimal control problems (FOCPs, optimal control problems which contain fractional derivatives in the performance index or system dynamic constraints or both) are of great importance due to the fact that it has a more accurate description of dynamical systems when using fractional differential equations (FDEs) compared to classical differential equations.

Same to the role of calculus of variations (CVs) in the field of classical optimal control [7–12], the fractional calculus of variations (FCVs) play a nonnegligible role in the formulation of FOCPs. Since Riewe [13, 14] used FC and FCVs to develop Euler–Lagrange equations of motion for nonconservative mechanical systems, there have been significant works done in the area of fractional mechanics from both the Lagrangian [15–25] and Hamiltonian [26–34] viewpoints. Subsequently, various kinds of formulations and numerical schemes have been proposed and extended to FOCPs. Agrawal [35] firstly considered FOCPs with fractional dynamic constraints and obtained the corresponding Euler–Lagrange equations. Using an approach similar to a variational virtual work coupled with the Lagrange multiplier technique, the author solved the FOCPs numerically based on the shifted Legendre orthonormal polynomials. His formulation for FOCPs were extended to other FOCPs with several state and control variables [36, 37] and with dynamic constraints described in terms of Caputo derivatives [38–40].

Since "the Birkhoffian mechanics is the most general possible mechanics that can be constructed from the Hamiltonian mechanics via the transformation theory" [41] (see Ref. [42] and references therein for a concrete introduction of the Birkhoffian mechanics), the fractional Birkhoffian mechanics should have a wider application than the fractional Lagrangian or Hamiltonian mechanics when applied to fractional dynamics. There has been limited work in the area of fractional Birkhoffian mechanics. Ref. [43] constructed a general fractional dynamical model of fractional Birkhoffian mechanics. Ref. [44] gave a numerical scheme to solve the fractional Birkhoff equations using variational integrators. Other existing literatures mostly deal with the Noether symmetries and conserved quantities for fractional Birkhoffian systems [45–47]. Ref. [48] studied the optimal control of a Birkhoffian system based on variational discretization. For a wider range of theoretical and practical needs, this work will focus on the FOCPs in the Birkhoffian sense whose dynamic constraints are described by the fractional forced Birkhoff equations. The formulation is given in terms of Riemann–Liouville fractional derivative and can be easily generalized to Caputo and Riesz fractional derivatives or their combinations. By the aid of the fractional integration by parts formula, the fractional Pfaff–Birkhoff–d'Alembert principle is used to derive the fractional forced Birkhoff equations, which is same to the classical forced Birkhoff equations [48] if the fractional order α equals one. The cost functional depending on the state variables and their fractional derivatives along with the external control forces is minimized under the dynamic constraints governed by the fractional forced Birkhoff equations.

Numerical schemes such as finite difference methods [49], orthonormal polynomials basis [50–53], collocation methods [54, 55], Oustaloup's approximation [56] and so on have been applied to solve FOCPs. These indirect methods deal with the resulting fractional differential equations as necessary optimality conditions drived from the Pontryagin maximum principle [57]. For the numerical solution of FOCPs whose dynamic behaviours are described by the fractional variational principle, we make use of the discrete fractional calculus of variations, i.e., we start from the discrete fractional variational principle. The numerical treatment in the framework of variational integrators (VIs) can lead to structure-preserving numerical schemes for mechanical systems [58, 59]. The benefits of VIs have been handed down to the context of optimal control, which gave rise to the method of discrete mechanics and optimal control (DMOC) [60–62]. Here we apply DMOC to the optimal control of fractional Birkhoffian systems. Following the general strategy of VIs, we give a discretization of the fractional Pfaff–Birkhoff–d'Alembert principle and develop the discrete fractional forced Birkhoff equations. Together with the initial and final conditions on the configuration states, the obtained discrete algebraic equations are served as the constraints for the minimization of the given discretized cost functional. In this way, the original FOCP is transformed into a nonlinear optimization problem which can be solved by existing algorithms. The solution of the optimization problem produces a sequence of discrete configuration states and a sequence of discrete control forces, which drive the system from the initial state to the desired state in an optimal way.

An outline of the paper is as follows. Sect. 2 introduces the basic definitions and properties of fractional calculus needed in the following sections. In Sect. 3, we formulate the forced fractional Birkhoff equations based on the fractional Pfaff–Birkhoff–d'Alembert principle. The original OCP is transformed into a constrained nonlinear optimization problem. The main contribution is presented in Sect. 4. The discrete fractional forced Birkhoff equations are derived and served as constraints of the discrete cost functional, which result in a nonlinear constrained optimization problem. An example is given in Sect. 5 to illustrate our results and some conclusions are drawn in Sect. 6.

2 Preliminaries

To begin with we recall some definitions and basic properties of fractional calculus.

Definition 1 [1] Let $f(t) \in L_1(0,T)$, the left and right Riemann–Liouville fractional integrals of function $f(t)$ are defined by

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau $$

and

$$ {}_tI_T^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (\tau-t)^{\alpha-1} f(\tau) \mathrm{d}\tau $$

respectively, where $\alpha > 0$ is called the fractional order and $\Gamma(z)$ is the Euler gamma function defined by

$$ \Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} \mathrm{d}t. $$

There are various kinds of definitions for fractional derivatives. Here we focus on the Riemann–Liouville and Caputo ones.

Definition 2 [1] For a Lebesgue integrable function on the interval $[0,T]$, the left and right Riemann–Liouville fractional derivatives of order α of function $f \in L\left([0,T]\right)$ are defined by

$$ {}_0D_t^\alpha f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left[{}_0I_t^{n-\alpha} f(t)\right] = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) \mathrm{d}\tau $$

and

$$ {}_tD_T^\alpha f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left[{}_tI_T^{n-\alpha} f(t)\right] = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^T (\tau-t)^{n-\alpha-1} f(\tau) \mathrm{d}\tau $$

respectively, where $n = [\alpha] + 1$ is the smallest integer larger than α, i.e., $n - 1 \leqslant \alpha < n$.

Definition 3 [1] Let $AC([0,T])$ denote the space consisting of absolutely continuous functions on the interval $[0,T]$, we denote by $AC^n([0,T])$ the space of functions $f(t)$ which have continuous derivatives up to order $n-1$ on $[0,T]$ with $f^{(n-1)}(t) \in AC([0,T])$. For an function $f(t) \in AC^{(n)}([0,T])$, the left and right Caputo fractional derivatives are defined by

$$ {}_0^CD_t^\alpha f(t) = {}_0I_t^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) \mathrm{d}\tau $$

and

$$ {}_t^CD_T^\alpha f(t) = (-1)^n {}_tI_T^{n-\alpha} f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T (\tau-t)^{n-\alpha-1} f^{(n)}(\tau) \mathrm{d}\tau $$

respectively.

Remark 1 The fractional integral and fractional derivative are linear operators, i.e., we have

$$I^{\alpha}\left(c_1 f(t)+c_2 g(t)\right)=c_1 I^{\alpha} f(t)+c_2 I^{\alpha} g(t),$$
$$D^{\alpha}\left(c_1 f(t)+c_2 g(t)\right)=c_1 D^{\alpha} f(t)+c_2 D^{\alpha} g(t),$$

where $c_1, c_2 \in \mathbb{R}$, I^{α} denotes one of the Riemann–Liouville fractional integral operators ${}_0I_t^{\alpha}$, ${}_tI_T^{\alpha}$, and D^{α} denotes one of the fractional derivative operators ${}_0D_t^{\alpha}$, ${}_tD_T^{\alpha}$, ${}_0^CD_t^{\alpha}$, ${}_t^CD_T^{\alpha}$.

We will also need the following relationships between the Riemann–Liouville and Caputo fractional derivatives.

$${}_0D_t^{\alpha} f(t)={}_0^CD_t^{\alpha} f(t)+\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha},$$
$${}_tD_T^{\alpha} f(t)={}_t^CD_T^{\alpha} f(t)+\sum_{k=0}^{n-1}\frac{f^{(k)}(T)}{\Gamma(k-\alpha+1)}(T-t)^{k-\alpha}.$$

Obviously, if $f^{(k)}(0)=0$ (or respectively, $f^{(k)}(T)=0$) for any $k=0,1,\ldots,n-1$, then we have ${}_0D_t^{\alpha} f(t)={}_0^CD_t^{\alpha} f(t)$ (or respectively, ${}_tD_T^{\alpha} f(t)={}_t^CD_T^{\alpha} f(t)$).

In the following, we will always assume $\alpha \in (0,1)$ for simplicity unless emphasized. For the case of $\alpha \geq 1$, one can still obtain similar results.

Lemma 1 *[1] Let $f(t) \in L_p(0,T), g(t) \in L_q(0,T)$, where $p \geq 1, q \geq 1, 1/p+1/q \leq 1+\alpha$ and $p \neq 1, q \neq 1$ when $1/p+1/q=1+\alpha$, then we have the following formula:*

$$\int_0^T f(t)\cdot {}_0I_t^{\alpha} g(t)\mathrm{d}t=\int_0^T g(t)\cdot {}_tI_T^{\alpha} f(t)\mathrm{d}t. \qquad (1)$$

Proof Please see Ref. [1] for details. □

To develop the fractional forced Birkhoff equations in the next section, we need a formula for integration by parts involving fractional derivatives.

Lemma 2 (Fractional integration by parts) *[1] Suppose that ${}_0D_t^{\alpha} f(t), {}_tD_T^{\alpha} g(t)$ exist and are continuous on $[0,T]$, if $f(T)=0$ or $g(0)=0$, then we have*

$$\int_0^T f(t)\cdot {}_0D_t^{\alpha} g(t)\mathrm{d}t=\int_0^T g(t)\cdot {}_tD_T^{\alpha} f(t)\mathrm{d}t. \qquad (2)$$

Proof We start from the following calculation:

$$\begin{aligned}
&\int_0^T f(t)\cdot {}_0D_t^\alpha g(t)\mathrm{d}t = \int_0^T f(t)\cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{1-\alpha}g(t)\right]\mathrm{d}t\\
&= \left[f(t)\cdot {}_0I_t^{1-\alpha}g(t)\right]|_{t=0}^T - \int_0^T f'(t)\cdot {}_0I_t^{1-\alpha}g(t)\mathrm{d}t\\
&= \left[f(t)\cdot {}_0I_t^{1-\alpha}g(t)\right]|_{t=0}^T - \int_0^T g(t)\cdot {}_tI_T^{1-\alpha}f'(t)\mathrm{d}t\\
&= \left[f(t)\cdot {}_0I_t^{1-\alpha}g(t)\right]|_{t=T} + \int_0^T g(t)\cdot {}_t^CD_T^\alpha f(t)\mathrm{d}t,
\end{aligned} \tag{3}$$

where we apply Lemma 1 in the third equality.

If $f(T)=0$, then we have ${}_t^CD_T^\alpha f(t) = {}_tD_T^\alpha f(t)$, we can conclude immediately that Eq. (2) holds from Eq. (3).

Similarly, in the case of $g(0)=0$, we have ${}_0^CD_t^\alpha g(t) = {}_0D_t^\alpha g(t)$ and

$$\begin{aligned}
&\int_0^T g(t)\cdot {}_tD_T^\alpha f(t)\mathrm{d}t = \int_0^T g(t)\cdot -\frac{\mathrm{d}}{\mathrm{d}t}\left[{}_tI_T^{1-\alpha}f(t)\right]\mathrm{d}t\\
&= -\left[g(t)\cdot {}_tI_T^{1-\alpha}f(t)\right]|_{t=0}^T + \int_0^T g'(t)\cdot {}_tI_T^{1-\alpha}f(t)\mathrm{d}t\\
&= -\left[g(t)\cdot {}_tI_T^{1-\alpha}f(t)\right]|_{t=0}^T + \int_0^T f(t)\cdot {}_0I_t^{1-\alpha}g'(t)\mathrm{d}t\\
&= \int_0^T f(t)\cdot {}_0^CD_t^\alpha g(t)\mathrm{d}t = \int_0^T f(t)\cdot {}_0D_t^\alpha g(t)\mathrm{d}t,
\end{aligned} \tag{4}$$

which is equivalent to Eq. (2). □

3 Forced Fractional Birkhoff Equations

The aim of optimal control is to steer a dynamical system from the initial state to the desired final state such that a quantity given in advance, for example the control effort in our case, is minimal. Moreover, the given objective functional is optimized under the dynamical constraints governed by the dynamic equations. Since we study the fractional Birkhoff system in this work, the dynamical constraints are described by the following fractional forced Birkhoff equations (7).

Consider a dynamical system with a $2n$-dimensional configuration space M equiped with the local coordinates $\boldsymbol{a}=(a_1,a_2,\ldots,a_{2n})$. Given a time interval $[0,T]$, the system is driven from an initial state $\boldsymbol{a}(0)=\boldsymbol{a}_0$ to the final state $\boldsymbol{a}(T)=\boldsymbol{a}_T$ under the influence of a force $\boldsymbol{F}(t)=(F_1(t),F_2(t),\ldots,F_{2n}(t))$ such that the cost functional

$$J(\boldsymbol{a}(t),\boldsymbol{F}(t)) = \int_0^T C(\boldsymbol{a}(t), {}_0D_t^\alpha \boldsymbol{a}(t), \boldsymbol{F}(t))\mathrm{d}t \tag{5}$$

is minimal.

During the optimization process under control, the system is also subject to the following fractional Pfaff–Birkhoff–d'Alembert principle

$$\begin{aligned}\delta A_f = \delta \int_0^T \left[\sum_{i=0}^{2n} R_i(t, \boldsymbol{a}(t)) \cdot {}_0D_t^\alpha a_i(t) - B(t, \boldsymbol{a}(t)) \right] \mathrm{d}t \\ + \int_0^T \boldsymbol{F}(t) \cdot \delta \boldsymbol{a}(t) \mathrm{d}t = 0, \\ \delta a_i(0) = \delta a_i(T) = 0,\ {}_0D_t^\alpha \delta a_i(t) = \delta {}_0D_t^\alpha a_i(t), i = 1, 2, \ldots, 2n,\end{aligned} \tag{6}$$

where $R_i, i = 1, 2, \ldots, 2n$ are Birkhoff functions and B is the Birkhoffian of the system.

Proposition 1 *The fractional Pfaff–Birkhoff–d'Alembert principle* (6) *is equivalent to the following fractional forced Birkhoff equations:*

$$\sum_{i=1}^{2n} \frac{\partial R_i(t, \boldsymbol{a})}{\partial a_j} {}_0D_t^\alpha a_i + {}_tD_T^\alpha R_j(t, \boldsymbol{a}) - \frac{\partial B(t, \boldsymbol{a})}{\partial a_j} + F_j(t) = 0, j = 1, 2, ..., 2n. \tag{7}$$

Proof Firstly, we have

$$\begin{aligned}\delta A_f = \int_0^T \sum_{j=1}^{2n} \left[\sum_{i=1}^{2n} \frac{\partial R_i(t, \boldsymbol{a})}{\partial a_j} \cdot {}_0D_t^\alpha a_i(t) - \frac{\partial B(t, \boldsymbol{a})}{\partial a_j} \right] \delta a_j(t) \mathrm{d}t \\ + \int_0^T \sum_{i=1}^{2n} R_i(t, \boldsymbol{a}) \cdot {}_0D_t^\alpha \delta a_i(t) \mathrm{d}t + \int_0^T \sum_{j=1}^{2n} F_j(t) \cdot \delta a_j(t) \mathrm{d}t.\end{aligned} \tag{8}$$

Secondly, through the induction of fractional integration by parts formula (4), we can see that

$$\begin{aligned}\int_0^T \sum_{i=1}^{2n} R_i(t, \boldsymbol{a}) \cdot {}_0D_t^\alpha \delta a_i(t) \mathrm{d}t = \int_0^T \sum_{i=1}^{2n} \delta a_i(t) \cdot {}_tD_T^\alpha R_i(t, \boldsymbol{a}) \mathrm{d}t \\ - \sum_{i=1}^{2n} {}_tI_T^{1-\alpha} R_i(t, \boldsymbol{a})|_{t=0} \cdot \delta a_i(0).\end{aligned}$$

According to the fractional Pfaff–Birkhoff–d'Alembert principle, rearranging Eq. (8) reads

$$\begin{aligned}\delta A_f = \int_0^T \sum_{j=1}^{2n} \left[\sum_{i=1}^{2n} \frac{\partial R_i(t, \boldsymbol{a})}{\partial a_j} {}_0D_t^\alpha a_i(t) + {}_tD_T^\alpha R_j(t, \boldsymbol{a}) - \frac{\partial B(t, \boldsymbol{a})}{\partial a_j} + F_j(t) \right] \delta a_j \mathrm{d}t \\ - \sum_{j=1}^{2n} {}_tI_T^{1-\alpha} R_j(t, \boldsymbol{a})|_{t=0} \cdot \delta a_j(0)\end{aligned} \tag{9}$$

vanishes for any variations $\delta a_j, j = 1, 2, ..., 2n$.

Finally, we obtain

$$\sum_{i=1}^{2n} \frac{\partial R_i(t, \boldsymbol{a})}{\partial a_j} {}_0D_t^\alpha a_i + {}_tD_T^\alpha R_j(t, \boldsymbol{a}) - \frac{\partial B(t, \boldsymbol{a})}{\partial a_j} + F_j(t) = 0, j = 1, 2, ..., 2n. \quad (10)$$

following by the arbitrary variations $\delta a_j, j = 1, 2, ..., 2n$ vanishing at the end-points and the fundamental theorem in calculus of variations. □

Now we can formulate the optimal control problem as follows:

$$\begin{aligned} &\min \quad J(\boldsymbol{a}(t), \boldsymbol{F}(t)) = \int_0^T C(\boldsymbol{a}(t), {}_0D_t^\alpha \boldsymbol{a}(t), \boldsymbol{F}(t)) \mathrm{d}t \\ &\text{w.r.t.} \quad \boldsymbol{a}(t), \boldsymbol{F}(t), \\ &\text{s.t.} \quad \begin{cases} \boldsymbol{a}(0) = \boldsymbol{a}_0, \ \boldsymbol{a}(T) = \boldsymbol{a}_T, \\ \displaystyle\sum_{i=1}^{2n} \frac{\partial R_i(t, \boldsymbol{a})}{\partial a_j} {}_0D_t^\alpha a_i + {}_tD_T^\alpha R_j(t, \boldsymbol{a}) - \frac{\partial B(t, \boldsymbol{a})}{\partial a_j} \\ \qquad\qquad + F_j(t) = 0, j = 1, 2, \ldots, 2n. \end{cases} \end{aligned} \quad (11)$$

The problem (11) can be seen as a nonlinear constrained optimization problem. The necessary conditions for optimality can be derived from the Pontryagin maximum principle[57], which result in a system of FDEs and can be solved by numerical methods mentioned in the introduction. In the next section, we will not adopt this kind of treatment but start from a direct discretization of the fractional variational principle.

4 Discrete Optimal Control for Fractional Birkhoffian Systems

In this section, we will develop a general framework based on variational integrators to give a numerical solution for the optimal control problem of fractional Birkhoffian systems.

Given a time step h, we define an increasing sequence of discrete times $\mathbb{T} = \left\{t^k = kh | k = 0, 1, \ldots, N\right\}$ by splitting the time interval $[0, T]$ into N subintervals of equal length. The state variable $\boldsymbol{a}(t) : [0, T] \to M$ is replaced by its discrete counterpart $\boldsymbol{a}_d : \mathbb{T} \to M$, and $\boldsymbol{a}^k = \boldsymbol{a}_d(kh)$ is viewed as an approximation of $\boldsymbol{a}(kh)$. Similarly, the discrete force $\boldsymbol{F}^k = \boldsymbol{F}_d(kh)$ is regarded as the approximation of the control force $\boldsymbol{F}(kh)$.

We also denote by $R_i^k = (t^k, \boldsymbol{a}^k)$ and $B^k = (t^k, \boldsymbol{a}^k)$, for $i = 1, 2, \ldots, 2n, k = 0, 1, \ldots, N$ as discrete Birkhoff functions and the discrete Birkhoffian, respectively. Meanwhile, the left Riemann–Liouville fractional derivative ${}_0D_t^\alpha a_i(t)$ at $t = t^k$ is approximated by the shifted Grünwald–Letnikov fractional derivative, i.e., we have

$${}_0D_t^\alpha a_i^k \approx {}_0\Delta_{t^k} a_i^k = \frac{1}{h^\alpha} \sum_{l=0}^{k+1} w_\alpha^l a_i^{k-l+1}, \quad (12)$$

where $w_\alpha^0 = 1$ and $w_\alpha^k = \left(1 - \frac{\alpha+1}{k}\right) w_\alpha^{k-1}$ for all $k \in \mathbb{N}^+$.

In the discrete case, the discrete fractional Pfaff–Birkhoff–d'Alembert principle is given by

$$\delta A_{df} = \delta \sum_{k=0}^{N-1} \left[\sum_{i=1}^{2n} hR_i^k \cdot {}_0\Delta_{t^k}^\alpha a_i^k - hB^k \right] + \sum_{k=0}^{N-1} \left(\overleftarrow{\boldsymbol{F}}^k \cdot \delta \boldsymbol{a}^k + \overrightarrow{\boldsymbol{F}}^k \cdot \delta \boldsymbol{a}^{k+1} \right) = 0 \tag{13}$$

for all variations $\{\delta a_i^k\}_{k=0}^N$ satisfying

$$\delta a_i^0 = \delta a_i^N = 0, i = 1, 2, ..., 2n,$$

where $\overleftarrow{\boldsymbol{F}}^k = (\overleftarrow{F}_1^k, \overleftarrow{F}_2^k, \ldots, \overleftarrow{F}_{2n}^k)$ and $\overrightarrow{\boldsymbol{F}}^k = (\overrightarrow{F}_1^k, \overrightarrow{F}_2^k, \ldots, \overrightarrow{F}_{2n}^k)$ for all $k = 0, 1, \ldots, N-1$ are called the left and right forces, respectively.

Proposition 2 *The discrete fractional Pfaff–Birkhoff–d'Alembert principle is equivalent to the following system of equations*

$$\sum_{i=1}^{2n} \frac{\partial R_i^k}{\partial a_j^k} \sum_{l=0}^{k+1} w_\alpha^l a_i^{k-l+1} + \sum_{l=0}^{N-k} w_\alpha^l R_j^{k+l-1} - h^\alpha \frac{\partial B^k}{\partial a_j^k} + h^{\alpha-1} \left(\overleftarrow{F}_j^k + \overrightarrow{F}_j^{k-1} \right) = 0, \tag{14}$$

for all $k = 1, 2, ..., N-1, j = 1, 2, ..., 2n$, *which are called the discrete fractional forced Birkhoff equations.*

Proof Starting from the discrete fractional Pfaff–Birkhoff–d'Alembert principle (13) and using

$${}_0\Delta_{t^k}^\alpha a_i^k = \frac{1}{h^\alpha} \sum_{l=0}^{k+1} w_\alpha^l a_i^{k-l+1},$$

we have

$$\begin{aligned} &\sum_{k=0}^{N-1} \sum_{j=1}^{2n} \left[\sum_{i=1}^{2n} \frac{\partial R_i^k}{\partial a_j^k} \sum_{l=0}^{k+1} w_\alpha^l a_i^{k-l+1} - h^\alpha \frac{\partial B^k}{\partial a_j^k} \right] \delta a_j^k + \sum_{k=0}^{N-1} \sum_{i=1}^{2n} R_i^k \sum_{l=0}^{k+1} w_\alpha^l \delta a_i^{k-l+1} \\ &+ h^{\alpha-1} \sum_{k=0}^{N-1} \left(\overleftarrow{\boldsymbol{F}}^k \cdot \delta \boldsymbol{a}^k + \overrightarrow{\boldsymbol{F}}^k \cdot \delta \boldsymbol{a}^{k+1} \right) = 0. \end{aligned} \tag{15}$$

Taking into account that

$$\begin{aligned} \sum_{k=0}^{N-1} \sum_{i=1}^{2n} R_i^k \sum_{l=0}^{k+1} w_\alpha^l \delta a_i^{k-l+1} = &\sum_{i=1}^{2n} w_\alpha^0 R_i^{N-1} \delta a_i^N + \sum_{i=1}^{2n} \sum_{l=1}^{N} w_\alpha^l R_i^{l-1} \delta a_i^0 \\ &+ \sum_{k=1}^{N-1} \sum_{j=1}^{2n} \sum_{l=0}^{N-k} w_\alpha^l R_j^{k+l-1} \delta a_j^k \end{aligned} \tag{16}$$

and

$$
\begin{aligned}
&\sum_{k=0}^{N-1}\left(\overleftarrow{\boldsymbol{F}}^{k}\cdot\delta\boldsymbol{a}^{k}+\overrightarrow{\boldsymbol{F}}^{k}\cdot\delta\boldsymbol{a}^{k+1}\right)=\sum_{k=0}^{N-1}\sum_{j=1}^{2n}\left(\overleftarrow{F_j^k}\cdot\delta a_j^k+\overrightarrow{F_j^k}\cdot\delta a_j^{k+1}\right)\\
&=\sum_{k=1}^{N-2}\sum_{j=1}^{2n}\left(\overleftarrow{F}_j^k\cdot\delta a_j^k+\overrightarrow{F}_j^k\cdot\delta a_j^{k+1}\right)\\
&+\sum_{j=1}^{2n}\left(\overleftarrow{F}_j^0\cdot\delta a_j^0+\overrightarrow{F}_j^0\cdot\delta a_j^1\right)+\sum_{j=1}^{2n}\left(\overleftarrow{F}_j^{N-1}\cdot\delta a_j^{N-1}+\overrightarrow{F}_j^{N-1}\cdot\delta a_j^N\right)\\
&=\sum_{k=1}^{N-1}\sum_{j=1}^{2n}\left(\overleftarrow{F}_j^k\cdot\delta a_j^k+\overrightarrow{F}_j^{k-1}\cdot\delta a_j^k\right)+\sum_{j=1}^{2n}\overleftarrow{F}_j^0\cdot\delta a_j^0+\sum_{j=1}^{2n}\overrightarrow{F}_j^{N-1}\cdot\delta a_j^N,
\end{aligned}
\tag{17}
$$

we induce from Eqs. (15), (16) and (17) that

$$
\begin{aligned}
&\sum_{k=1}^{N-1}\sum_{j=1}^{2n}\Bigg[\sum_{i=1}^{2n}\frac{\partial R_i^k}{\partial a_j^k}\sum_{l=0}^{k+1}w_\alpha^l a_i^{k-l+1}+\sum_{l=0}^{N-k}w_\alpha^l R_j^{k+l-1}\\
&-h^\alpha\frac{\partial B^k}{\partial a_j^k}+h^{\alpha-1}\left(\overleftarrow{F}_j^k+\overrightarrow{F}_j^{k-1}\right)\Bigg]\delta a_j^k\\
&+\sum_{j=1}^{2n}\left[\sum_{i=1}^{2n}\frac{\partial R_i^0}{\partial a_j^0}\left(w_\alpha^0 a_i^1+w_\alpha^1 a_i^0\right)-h^\alpha\frac{\partial B^0}{\partial a_j^0}+\sum_{l=1}^{N}w_\alpha^l R_j^{l-1}+h^{\alpha-1}\overleftarrow{F}_j^0\right]\delta a_j^0\\
&+\sum_{j=1}^{2n}\left[w_\alpha^0 R_j^{N-1}+h^{\alpha-1}\overrightarrow{F}_j^{N-1}\right]\delta a_j^N\\
&=0
\end{aligned}
\tag{18}
$$

is valid for arbitary variations $\{\delta a_j^k\}_{k=0}^N$ vanish at the endpoints. Thus, the above expression implies that Eq. (14) holds for $k=1,2,...,N-1, j=1,2,...,2n$, provided that $\delta a_j^0=\delta a_j^N=0, j=1,2,\ldots,2n$, which ends the proof. □

Remark 2 Based on a similiar discussion in Ref. [44], it is easy to see that Eq. (14) is an α order approximation of Eq. (7).

In particular, the cost functional in Eq. (5) on the time slice $[kh,(k+1)h]$ is approximated by

$$C_d(\boldsymbol{F}^k,\boldsymbol{F}^{k+1})\approx\int_{kh}^{(k+1)h}C(\boldsymbol{a}(t),{}_0D_t^\alpha\boldsymbol{a}(t),\boldsymbol{F}(t))\mathrm{d}t,$$

which yields the discrete objective function

$$J_d(\boldsymbol{a}_d,\boldsymbol{F}_d)=\sum_{k=0}^{N-1}C_d(\boldsymbol{F}^k,\boldsymbol{F}^{k+1}).$$

In the next step, we also require that the given initial and final states $\boldsymbol{a}(0) = \boldsymbol{a}_0, \boldsymbol{a}(T) = \boldsymbol{a}_T$ take a part in the discrete formulation. Obviously, we can use $\boldsymbol{a}^0 = \boldsymbol{a}_0, \boldsymbol{a}^N = \boldsymbol{a}_T$ as constraints in a straightforward way. The discrete forces at the first time node can be incorporated into the formulation by comparing the coefficients of variations $\delta a_j(0)$ and δa_j^0. To be specific, we can set

$$-{}_tI_T^{1-\alpha} R_j(t, \boldsymbol{a})|_{t=0} = \sum_{i=1}^{2n} \frac{\partial R_i^0}{\partial a_j^0} \left(w_\alpha^0 a_i^1 + w_\alpha^1 a_i^0\right) - h^\alpha \frac{\partial B^0}{\partial a_j^0} + \sum_{l=1}^{N} w_\alpha^l R_j^{l-1} + \frac{h^\alpha}{h} \overleftarrow{F}_j^0 \tag{19}$$

through Eq. (18) by comparison with Eq. (9).

To summarize, the discrete optimal control problem is formulated as follows:

$$\begin{aligned}
&\min \quad J_d(\boldsymbol{a}_d, \boldsymbol{F}_d) = \sum_{k=0}^{N-1} C_d(\boldsymbol{F}^k, \boldsymbol{F}^{k+1}) \\
&\text{w.r.t} \quad F_j^k, j = 1, 2, \ldots, 2n, k = 0, 1, \ldots, N, \\
&\text{s.t.} \quad \begin{cases}
\boldsymbol{a}^0 = \boldsymbol{a}(0), \boldsymbol{a}^N = \boldsymbol{a}(T), \\
\displaystyle\sum_{i=1}^{2n} \frac{\partial R_i^k}{\partial a_j^k} \sum_{l=0}^{k+1} w_\alpha^l a_i^{k-l+1} + \sum_{l=0}^{N-k} w_\alpha^l R_j^{k+l-1} - h^\alpha \frac{\partial B^k}{\partial a_j^k} \\
\qquad\qquad\qquad\displaystyle + \frac{h^\alpha}{h}\left(\overleftarrow{F}_j^k + \overrightarrow{F}_j^{k-1}\right) = 0, \\
\displaystyle -{}_tI_T^{1-\alpha} R_j(t, \boldsymbol{a})|_{t=0} = \sum_{i=1}^{2n} \frac{\partial R_i^0}{\partial a_j^0} \left(w_\alpha^0 a_i^1 + w_\alpha^1 a_i^0\right) - h^\alpha \frac{\partial B^0}{\partial a_j^0} \\
\qquad\qquad\qquad\displaystyle + \sum_{l=1}^{N} w_\alpha^l R_j^{l-1} + \frac{h^\alpha}{h} \overleftarrow{F}_j^0,
\end{cases}
\end{aligned} \tag{20}$$

where $j = 1, 2, \ldots, 2n, k = 1, 2, \ldots, N-1$.

Now we have completed the general framework of optimal control for fractional Birkhoffian systems based on variational integrators. The direct discretization of the fractional Pfaff–Birkhoff–d'Alembert principle gives rise to the discrete fractional forced Birkhoff equations. After the approximation of the cost functional, the constraints on the configuration, the boundary conditions (19) including the discrete forces on the first time node, together with the discrete fractional forced Birkhoff equations are regarded as the equality constraints of the discrete objective functional. Thus, the original optimal control problem (11) is converted into a discrete optimal control problem, or a nonlinear optimization problem (20), which can be solved by the existing algorithms.

5 Example

Consider the following equation describing the dynamics of a linear damped oscillator

$$\ddot{x} + \gamma\dot{x} + x = 0, \tag{21}$$

where $\gamma > 0$ is a given constant.

We study its fractional version starting from the following fractional Pfaff action integral

$$A_f = \int_0^T (R_1 \cdot {}_0D_t^\alpha a_1 + R_2 \cdot {}_0D_t^\alpha a_2 - B)\mathrm{d}t, \tag{22}$$

with the boundary conditions

$$\delta a_i(t)|_{t=0} = \delta a_i(t)|_{t=T} = 0, i = 1, 2,$$

where $R_1 = \frac{1}{2}e^{\gamma t}a_2,\ R_2 = -\frac{1}{2}e^{\gamma t}a_1,\ B = \frac{1}{2}e^{\gamma t}\left[(a_1)^2 + (a_2)^2 + \gamma a_1 a_2\right].$

The resulting fractional Birkhoff equations from Eq. (22) are as follows:

$$\begin{aligned} &e^{\gamma t}{}_0D_t^\alpha a_2(t) - {}_tD_T^\alpha\left[e^{\gamma t})a_2(t)\right] + e^{\gamma t}(2a_1(t) + \gamma a_2(t)) = 0,\\ &e^{\gamma t}{}_0D_t^\alpha a_1(t) - {}_tD_T^\alpha\left[e^{\gamma t})a_1(t)\right] - e^{\gamma t}(2a_2(t) + \gamma a_1(t)) = 0, \end{aligned} \tag{23}$$

which are equivalent to the system (21) when $\alpha = 1$ if we set $a_1 = x, a_2 = \dot{a}_1$.

Suppose that the initial state of the system (23) is given by $a_1(0) = a_2(0) = 1$, the final state at $T = 10$ is expected to be $a_1(T) = a_2(T) = 0$ under the influence of the control force $u(t)$. Following the arguments in Ref. [48], we take $\boldsymbol{F} = (F_1, F_2) = (e^{\gamma t}u(t), 0)$, where u is the realistic control force to be optimized such that we can realize the forced Birkhoffian representation of a forced dynamical system.

It is required that the control effort is minimized during the control procedure. To include the control force in the formulation, we extend the fractional Pfaff action integral (22) and consider the following fractional Pfaff–Birkhoff–d'Alembert principle:

$$\bar{A}_f = \int_0^T (R_1 \cdot {}_0D_t^\alpha a_1 + R_2 \cdot {}_0D_t^\alpha a_2 - B)\mathrm{d}t + \int_0^T \left[F_1(t) \cdot \delta a_1(t) + F_2(t) \cdot \delta a_2(t)\right]\mathrm{d}t.$$

Then we obtain the following optimal control problem:

$$\begin{aligned} \min \quad & J(u(t)) = \int_0^T u^2(t)\mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} a_1(0) = a_2(0) = 1, \\ e^{\gamma t}{}_0D_t^\alpha a_2(t) - {}_tD_T^\alpha\left[e^{\gamma t}a_2(t)\right] \\ \qquad\qquad + e^{\gamma t}\left[2a_1(t) + \gamma a_2(t)\right] - e^{\gamma t}u(t) = 0, \\ e^{\gamma t}{}_0D_t^\alpha a_1(t) - {}_tD_T^\alpha\left[e^{\gamma t}a_1(t)\right] \\ \qquad\qquad - e^{\gamma t}\left[2a_2(t) + \gamma a_1(t)\right] = 0, \\ a_1(T) = a_2(T) = 0. \end{cases} \end{aligned} \tag{24}$$

On every sub-interval $[kh,(k+1)h]$, the virtual work is approximated by

$$\begin{aligned}\int_{kh}^{(k+1)h} \boldsymbol{F}(t)\cdot\delta\boldsymbol{a}(t)\mathrm{d}t &\approx \frac{\boldsymbol{F}^{\boldsymbol{k}}+\boldsymbol{F}^{k+1}}{2}\cdot\frac{\delta\boldsymbol{a}^k+\delta\boldsymbol{a}^{k+1}}{2}h\\ &=\frac{h}{4}(\boldsymbol{F}^k+\boldsymbol{F}^{k+1})\cdot\delta\boldsymbol{a}^k+\frac{h}{4}(\boldsymbol{F}^k+\boldsymbol{F}^{k+1})\cdot\delta\boldsymbol{a}^{k+1}.\end{aligned}\tag{25}$$

So we take $\overleftarrow{\boldsymbol{F}}^k=\overrightarrow{\boldsymbol{F}}^k=\frac{h}{4}(\boldsymbol{F}^k+\boldsymbol{F}^{k+1})$ to be the left and right discrete forces.

Finally, we obtain the discrete optimal control problem as follows:

$$\begin{aligned}&\min \quad \sum_{k=0}^{N-1} h\left(\frac{u^k+u^{k+1}}{2}\right)^2\\ &\text{w.r.t} \quad u^k, k=0,1,\dots,N,\\ &\text{s.t.} \quad \begin{cases} a_1^0=a_2^0=1,\\ a_1^N=a_2^N=0,\\ {}_tI_T^{1-\alpha}(e^{\gamma t}a_2)|_{t=0}=(w_\alpha^0a_2^1+w_\alpha^1a_2^0)+h^\alpha(2a_1^0+\gamma a_2^0)\\ \qquad -\displaystyle\sum_{l=1}^{N} w_\alpha^l e^{\gamma t^{l-1}}a_2^{l-1}-\frac{h^\alpha}{2}(u^0+e^{\gamma h}u^1),\\ {}_tI_T^{1-\alpha}(e^{\gamma t}a_1)|_{t=0}=(w_\alpha^0a_1^1+w_\alpha^1a_1^0)-h^\alpha(2a_2^0+\gamma a_1^0)\\ \qquad +\displaystyle\sum_{l=1}^{N} w_\alpha^l e^{\gamma t^{l-1}}a_1^{l-1},\\ \displaystyle\sum_{l=0}^{k+1} w_\alpha^l a_2^{k-l+1}-\sum_{l=0}^{N-k} e^{\gamma t^{l-1}}w_\alpha^l a_2^{k+l-1}\\ \quad +h^\alpha(2a_1^k+\gamma a_2^k)-\dfrac{h^\alpha}{2}\left(u^k+e^{\gamma h}u^{k+1}+e^{-\gamma h}u^{k-1}+u^k\right)=0,\\ \displaystyle\sum_{l=0}^{k+1} w_\alpha^l a_1^{k-l+1}-\sum_{l=0}^{N-k} e^{\gamma t^{l-1}}w_\alpha^l a_1^{k+l-1}-h^\alpha(2a_2^k+\gamma a_1^k)=0,\end{cases}\end{aligned}\tag{26}$$

where $k=1,2,\dots,N-1$ and we set $t^{-1}=-h$.

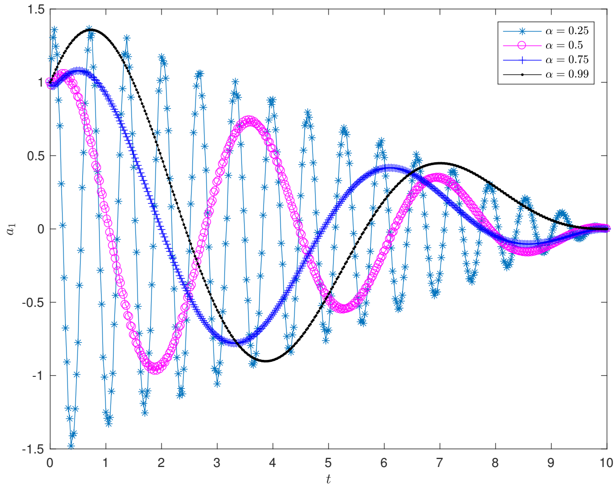

Fig. 1 The varying of a_1 over time with different fractional orders

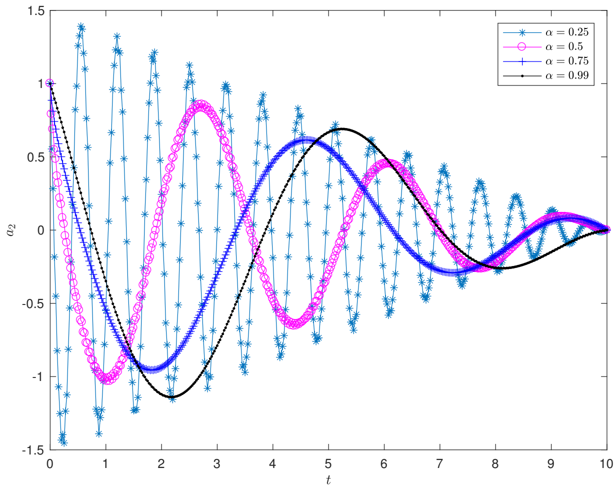

Fig. 2 The varying of a_2 over time with different fractional orders

We solve the above optimization problem by calling the fmincon function in Matlab, which finds the local minimum that satisfies the constraints in all the following numerical tests.

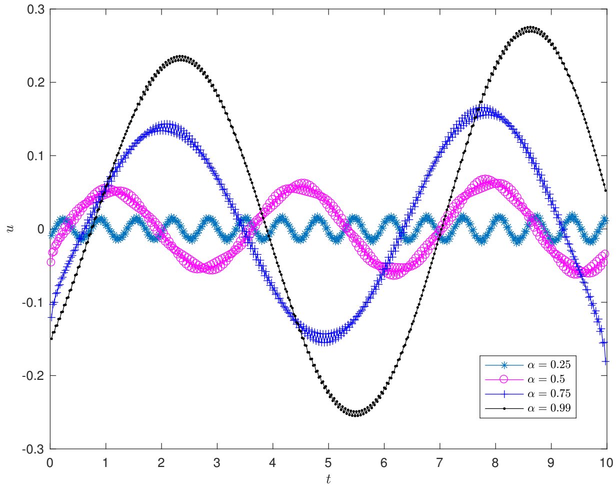

Fig. 3 The varying of discrete control u over time with different fractional orders

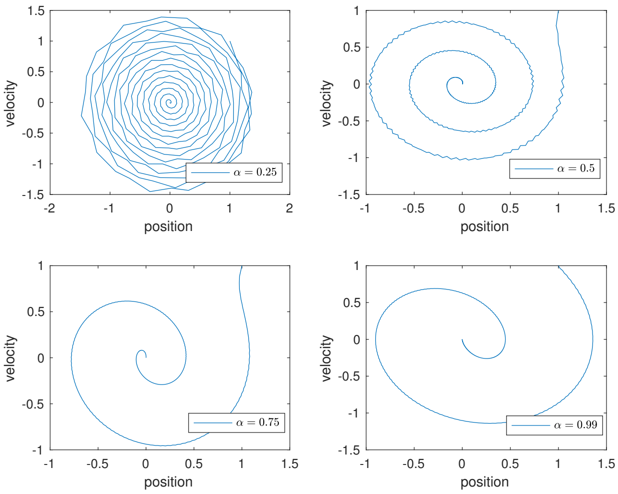

Fig. 4 The trajectory in phase space with different fractional orders

Figs. 1 ˜ 4 are plotted with $N = 400$ for $\alpha = 0.25, 0.5, 0.75, 0.99$. We can see from Figs. 1 and 2 that both the position and velocity of the fractional linear damped cscillator change regularly. The curves are similar to the case without imposing the control forces, which show obvious decay trends and periodic phenomena in a smooth way.

The discrete control $\frac{u^k+u^{k+1}}{2}$ in each time interval $[t^k, t^{k+1}]$ demonstrated in Fig. 3 almost varies continuously. This feature is favourable to the implementation of the control in practice. In Fig. 4, we depict the trajectories in phase space for $\alpha = 0.25, 0.5, 0.75, 0.99$. The trajectory becomes smoother when α increase from 0.25 to 0.99. And the period of oscillation goes up when α goes up.

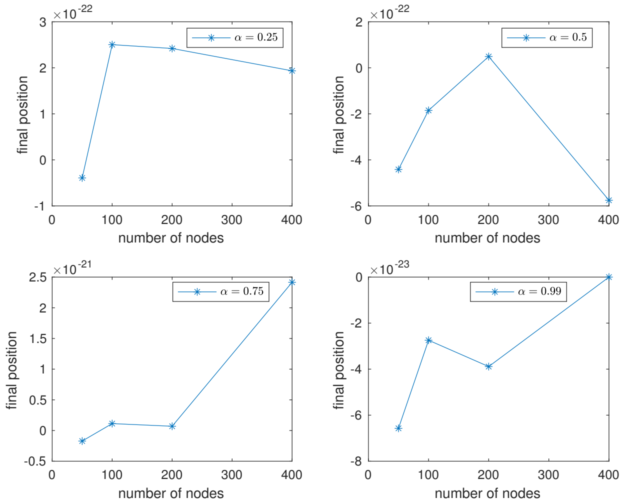

Fig. 5 The varying of final position over time with different fractional orders

We also present some numerical experiments to investigate the computed final position and velocity as indicated in Figs. 5 and 6 respectively. The obtained numerical results coinside closely with the given data at an extremely small scale of at most 10^{-21}, which verifies the validity and effectiveness of our method. Fig. 7 shows the control effort using different number of nodes. The more closely α approaches 1, the more closely the control effort stays at the same order of magnitude. In particular, the control effort, which has a relative difference at 10^{-2}, can be seen as a constant when $\alpha = 0.99$. This is consistent with the results in Ref. [48].

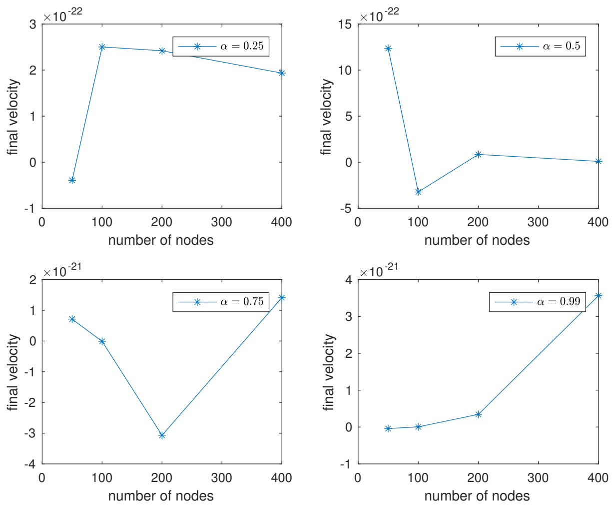

Fig. 6 The varying of final velocity over time with different fractional orders

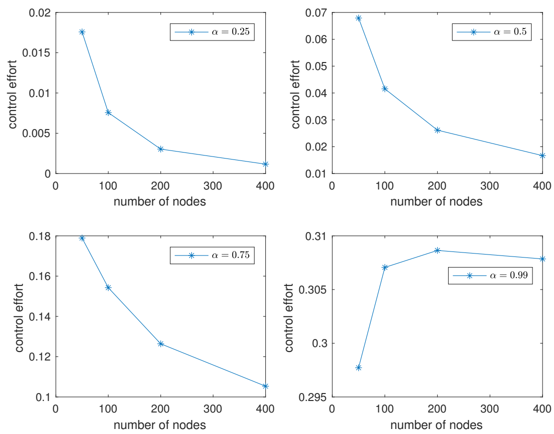

Fig. 7 The discrete constant of motion with different fractional orders

6 Conclusions

In this paper, we develop a general numerical scheme for the optimal control of fractional Birkhoffian systems. The fractional Birkhoffian system under consideration is described in terms of Riemann–Liouville fractional derivatives. The optimal control problem is to minimize the cost functional under the dynamic constraints governed by the fractional forced Birkhoff equations combined with the state constraints on the endpoints. In order to solve the optimal control problem numerically, we adopt the methodology of variational integrators. Specifically, we derive a system of algebraic equations called the discrete fractional forced Birkhoff equations from the discrete fractional Pfaff–Birkhoff–d'Alembert principle. The boundary conditions are also incorporated into the discrete discription through a comparison between the continuous and discrete variations on the endpoints when we develop the corresponding fractional forced Birkhoff equations. In this way, the discrete cost functional combined with the aforementioned algebraic equations served as constraints result in a nonlinear optimization problem, which can be solved by Matlab. We verify the validity and effectiveness of our method by the numerical example.

Acknowledgment

This project was supported by the National Natural Science Foundation of China (Grant No. 11672032).

Conflict of Interest

The authors declare that they have no conflict of interest.

References

1. Stefan G Samko, Anatoly A Kilbas, and Oleg I Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon, 1993.
2. Igor Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1998.
3. Anatoly A Kilbas, Hari M Srivastava, and Juan J Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.
4. Ricardo Enrique Gutiérrez, João Maurício Rosário, and José Tenreiro Machado. Fractional order calculus: basic concepts and engineering applications. *Math. Probl. Eng.*, 2010:19pp, 2010.
5. J Tenreiro Machado, Virginia Kiryakova, and Francesco Mainardi. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.*, 16(3):1140–1153, 2011.
6. Yong Zhou, Jin Rong Wang, and Lu Zhang. *Basic Theory of Fractional Differential Equations*. World Scientific, Singapore, 2016.
7. Laurence Chisholm Young. *Lecture on the Calculus of Variations and Optimal Control Theory*. Chelsea Publishing Company, New York, 1980.

8. A A Milyutin and N P Osmolovskii. *Calculus of Variations and Optimal Control*. American Mathematical Society, Providence, 1998.
9. Morton I Kamien and Nancy Lou Schwartz. *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*. Courier Corporation, North Chelmsford, 2012.
10. Daniel Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton University Press, Princeton, 2012.
11. George Leitmann. *The Calculus of Variations and Optimal Control: An Introduction*. Springer, London, 2013.
12. Francis Clarke. *Functional Analysis, Calculus of Variations and Optimal Control*. Springer, London, 2013.
13. Fred Riewe. Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E (3)*, 53(2):1890–1899, 1996.
14. Fred Riewe. Mechanics with fractional derivatives. *Phys. Rev. E (3)*, 55(3):3581–3592, 1997.
15. Om P Agrawal. Formulation of Euler–Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.*, 272(1):368–379, 2002.
16. D Baleanu and T Avkar. Lagrangians with linear velocities within Riemann–Liouville fractional derivatives. *Nuovo Cimento B Serie*, 119:73–79, 2004.
17. S Sh Rekhviashvili. The Lagrange formalism with fractional derivatives in problems of mechanics. *Technical Physics Letters*, 30(1):55–57, 2004.
18. Om P Agrawal. Generalized Euler–Lagrange equations and transversality conditions for FVPs in terms of the Caputo derivative. *J. Vib. Control*, 13(9-10):1217–1237, 2007.
19. Om P Agrawal. Fractional variational calculus in terms of Riesz fractional derivatives. *J. Phys. A*, 40(24):6287–6303, 2007.
20. Jacky Cresson. Fractional embedding of differential operators and Lagrangian systems. *J. Math. Phys.*, 48(3):033504(34pp), 2007.
21. TM Atanacković, Sanja Konjik, and S Pilipović. Variational problems with fractional derivatives: Euler–Lagrange equations. *J. Phys. A*, 41(9):095201(12pp), 2008.
22. Dumitru Baleanu and Juan I Trujillo. A new method of finding the fractional Euler–Lagrange and Hamilton equations within Caputo fractional derivatives. *Commun. Nonlinear Sci. Numer. Simul.*, 15(5):1111–1115, 2010.
23. Mohamed AE Herzallah and Dumitru Baleanu. Fractional Euler–Lagrange equations revisited. *Nonlinear Dynam.*, 69(3):977–982, 2012.
24. Agnieszka B Malinowska and D F M Torres. *Introduction to the Fractional Calculus of Variations*. Imperial College Press, London, 2012.
25. Agniezka B Malinowska, Tatiana Odzijewicc, and Delfim FM Torres. *Advanced Methods in the Fractional Calculus of Variations*. Springer, Cham, 2015.
26. Sami I Muslih and Dumitru Baleanu. Formulation of Hamiltonian equations for fractional variational problems. *Czechoslovak J. Phys.*, 55(6):633–642, 2005.

27. Sami I Muslih and Dumitru Baleanu. Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives. *J. Math. Anal. Appl.*, 304(2):599–606, 2005.
28. Vasily E Tarasov. Fractional generalization of gradient and Hamiltonian systems. *J. Phys. A*, 38(26):5929–5943, 2005.
29. Aleksander A Stanislavsky. Hamiltonian formalism of fractional systems. *The European Phys. J. B*, 49(1):93–101, 2006.
30. Eqab M Rabei, Khaled I Nawafleh, Raed S Hijjawi, Sami I Muslih, and Dumitru Baleanu. The Hamilton formalism with fractional derivatives. *J. Math. Anal. Appl.*, 327(2):891–897, 2007.
31. Eqab M Rabei, Ibtesam Almayteh, Sami I Muslih, and Dumitru Baleanu. Hamilton–Jacobi formulation of systems within Caputo's fractional derivative. *Physica Scripta*, 77(1):015101(6pp), 2008.
32. Ali Khalili Golmankhaneh, Alireza Khalili Golmankhaneh, Dumitru Baleanu, and Mihaela Cristina Baleanu. Hamiltonian structure of fractional first order Lagrangian. *Internat. J. Theoret. Phys.*, 49(2):365–375, 2010.
33. Shao Kai Luo and Lin Li. Fractional generalized Hamiltonian mechanics and Poisson conservation law in terms of combined Riesz derivatives. *Nonlinear Dynam.*, 73(1-2):639–647, 2013.
34. CÉSAR Torres. Existence of solution for a class of fractional Hamiltonian systems. *Electron. J. Differential Equations*, 2013(259):1–12, 2013.
35. Om Prakash Agrawal. A general formulation and solution scheme for fractional optimal control problems. *Nonlinear Dynam.*, 38(1-4):323–337, 2004.
36. Om P Agrawal, Ozlem Defterli, and Dumitru Baleanu. Fractional optimal control problems with several state and control variables. *J. Vib. Control*, 16(13):1967–1976, 2010.
37. AH Bhrawy, EH Doha, JA Tenreiro Machado, and SS Ezz-Eldien. An efficient numerical scheme for solving multi-dimensional fractional optimal control problems with a quadratic performance index. *Asian J. Control*, 17(6):2389–2402, 2015.
38. Om P Agrawal. A formulation and numerical scheme for fractional optimal control problems. *J. Vib. Control*, 14(9-10):1291–1299, 2008.
39. Om P Agrawal. A quadratic numerical scheme for fractional optimal control problems. *Journal of Dynamic Systems, Measurement, and Control*, 130(1):011010(6pp), 2008.
40. Tian Liang Guo. The necessary conditions of fractional optimal control in the sense of Caputo. *J. Optim. Theory Appl.*, 156(1):115–126, 2013.
41. Ruggero Maria Santilli. *Foundations of Theoretical Mechanics II*. Springer-Verlag, New York, 1982.
42. F X Mei, R C Shi, Y F Zhang, and H B Wu. *Dynamics of Birkhoffian System (in Chinese)*. Beijing Institute of Technology Press, Beijing, 1996.
43. Shao Kai Luo and Yan Li Xu. Fractional Birkhoffian mechanics. *Acta Mechanica*, 226(3):1–16, 2014.

44. Lin He, Hui Bin Wu, and Feng Xiang Mei. Variational integrators for fractional Birkhoffian systems. *Nonlinear Dynam.*, 87(4):2325–2334, 2016.
45. Xiang Hua Zhai and Yi Zhang. Noether symmetries and conserved quantities for Birkhoffian systems with time delay. *Nonlinear Dynam.*, 77(1-2):73–86, 2014.
46. Yi Zhang and Xiang Hua Zhai. Noether symmetries and conserved quantities for fractional Birkhoffian systems. *Nonlinear Dynam.*, 81(1-2):469–480, 2015.
47. Qiu Li Jia, Hui Bin Wu, and Feng Xiang Mei. Noether symmetries and conserved quantities for fractional forced Birkhoffian systems. *J. Math. Anal. Appl.*, 442(2):782–795, 2016.
48. Xin Lei Kong, Hui Bin Wu, and Feng Xiang Mei. Discrete optimal control for Birkhoffian systems. *Nonlinear Dynam.*, 74(3):711–719, 2013.
49. Dumitru Baleanu, Ozlem Defterli, and Om P Agrawal. A central difference numerical scheme for fractional optimal control problems. *J. Vib. Control*, 15(4):583–597, 2009.
50. Amir Lotfi, Mehdi Dehghan, and Sohrab Ali Yousefi. A numerical technique for solving fractional optimal control problems. *Comput. Math. Appl.*, 62(3):1055–1067, 2011.
51. Mohsen Alipour, Davood Rostamy, and Dumitru Baleanu. Solving multi-dimensional fractional optimal control problems with inequality constraint by Bernstein polynomials operational matrices. *J. Vib. Control*, 19(16):2523–2540, 2013.
52. AH Bhrawy and SS Ezz-Eldien. A new Legendre operational technique for delay fractional optimal control problems. *Calcolo*, 53(4):521–543, 2016.
53. E Keshavarz, Y Ordokhani, and M Razzaghi. A numerical solution for fractional optimal control problems via Bernoulli polynomials. *J. Vib. Control*, 22(18):3889–3903, 2016.
54. Sohrab Ali Yousefi, Amir Lotfi, and M Dehghan. The use of a Legendre multiwavelet collocation method for solving the fractional optimal control problems. *J. Vib. Control*, 17(13):2059–2065, 2011.
55. Emran Tohidi and Hassan Saberi Nik. A Bessel collocation method for solving fractional optimal control problems. *Appl. Math. Model.*, 39(2):455–465, 2015.
56. Christophe Tricaud and YangQuan Chen. An approximate method for numerically solving fractional order optimal control problems of general form. *Comput. Math. Appl.*, 59(5):1644–1655, 2010.
57. L. S. Pontryagin, V. G. Boltyanski, R. V. Gamkrelidze, and E. F. Miscenko. *The Mathematical Theory of Optimal Processes.* John Wiley & Sons, New York, 1962.
58. Jerrold E Marsden and Matthew West. Discrete mechanics and variational integrators. *Acta Numer.*, 10:357–514, 2001.
59. Adrian Lew, Jerrold E Marsden, Michael Ortiz, and Matthew West. An overview of variational integrators. *Finite Element Methods: 1970's and Beyond, CIMNE Barcelona*, pages 98–115, 2004.

60. Oliver Junge, Jerrold E Marsden, and Sina Ober-Blöbaum. Discrete mechanics and optimal control. *IFAC Proceedings Volumes*, 38(1):538–543, 2005.
61. Sigrid Leyendecker, Sina Ober-Blöbaum, Jerrold E Marsden, and Michael Ortiz. Discrete mechanics and optimal control for constrained systems. *Optimal Control Appl. Methods*, 31(6):505–528, 2010.
62. Sina Ober-Blöbaum, Oliver Junge, and Jerrold E Marsden. Discrete mechanics and optimal control: an analysis. *ESAIM Control Optim. Calc. Var.*, 17(2):322–352, 2011.