Transient Analysis of Power Management in Wireless Sensor Network With Start-Up Times and Threshold Policy

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Transient analysis of power management in wireless sensor network with start-up times and threshold policy

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Abstract

Queueing models play a significant role in analysing the performance of power management systems in various electronic devices and communication systems. This paper adopts a multiple vacation queueing model with a threshold policy to analyse the power-saving mechanisms of the wireless sensor network(WSN) using the Dynamic Power Management technique. The proposed system consists of a busy state(transmit state), wake-up state, shutdown state and inactive state. In this model, the server switches to a shutdown state for a random duration of time after serving all the events(data packets) in the busy state. Events that arrive during the shutdown period cannot be served until the system size reaches the predetermined threshold value of k and further it requires start-up time and a change of state to resume service. At the end of the shutdown period, if the system size is less than k, then the server begins the inactive period; otherwise, the server switches to the wake-up state. For this system, an explicit expression for the transient and steady-state solution is computed in a closed-form. Furthermore, performance indices such as mean, variance, probability that the server is in various stages of power management modes and mean power consumption are computed. Finally, graphical illustrations are made to understand the effect of the parameters on the performance of the system.

Key words: Single server, transient probabilities, steady-state probabilities, threshold policy, start-up times, mean power consumption .

1 Introduction

WSN is a smart technology used to collect information about the neighbouring environment by sensing and share the information with the user or with the base station. WSN uses smart sensors for sensing, computing and collecting information. These sensors are tiny, low-cost and battery-powered. The WSNs have a far-reaching application that has transformed human lives in many aspects and paved the way for intelligent living technology. It is applied in agriculture, defence, traffic, surveillance, natural disaster, etc. for tracking and monitoring. Although WSN is popular technology, it has got some limitations, particularly in battery life. The battery cannot be recharged during the surveillance because the nodes of WSNs may be deployed in an unfriendly environment. The lifetime of the WSN depends on the power consumption executed at each sensor node. Hence, the energy consumption in the WSNs is considered a serious issue that has attracted many researchers in recent times. To control the power consumption in sensor nodes, a variety of Dynamic Power Management(DPM) techniques have been proposed by researchers in recent times. It protocols the WSNs to switch power-saving modes and selectively shutting down the components during the idle time for saving power. It is also applied in many portable devices for power management. The DPM policies can be classified as predictive and stochastic [1]. In this model, a stochastic-based DPM approach is adopted. This paper analyses the functions of the DPM in WSNs using an M/M/1 queuing system with vacation and threshold policy.

Queueing systems with vacation play a vital role in analysing the power-saving mechanism(PSM) of computer and communication systems. Several authors have applied various vacation policies to study the performance of the PSM in various communication systems [See Dimitriou [4], Misra and Goswami [13], Sampath *et al.* [16] and Ren *et al.* [15] and references therein]. In these models, the server switches to busy and vacation states frequently. To avoid frequent switching, the system designers prefer an N-policy scheme which will be more effective than other vacation schemes. In the N-policy queueing systems, the server is turned OFF or it stays idle when there is no job(data packet) in the system and the server is turned ON when the system size reaches a threshold value N. The notion of an N-policy was first studied by Yadin and Naor [18]. Later, many researchers studied various queueing systems with N-policy in a different context [See Wang and Ke [17], Parthasarathy and Sudhesh [14] and references therein]. In the multiple vacations queuing system with N-policy, the server switches to a busy state from vacation only if the system accumulates N jobs.

Many researchers have analysed the PSM of WSNs based on the vacation queueing system with N-policy. Jiang *et al.* [7] analysed the PSM of WSNs using N-policy and discussed the steady-state results. In this paper, the authors considered two states namely busy and idle. Huang and Lee [5] studied the PSM of WSNs using an N-policy M/G/1/K queueing model and presented the steady-state results. Blondia [2] presented the steady-state analysis of a WSN using energy harvesting. In this paper the author considered two states namely transmit and vacation. Lee and Yang [10] analysed the PSM of WSNs using an N-policy Geo/G/1 queueing model. Chen *et al.* [3] proposed an improved stochastic model for the WSNs which consists of three power-saving states namely shutdown, wake-up and inactive. In this paper, the author proposed that power saving can be achieved by decreasing the number of shutdown and wake-up processes. Jayarajan *et al.* [6] applied M/D/1 priority queueing model with threshold policy to study the PSM of sensor network and obtained the steady-state results. Ma *et al.* [12] studied the PSM of WSNs using an M/M/2 queueing model with threshold policy and presented the steady-state results using the matrix-geometry method. From the literature survey, it is observed that most research on the mathematical modelling of WSNs has focussed mainly on the steady-state analysis of the system. Surprisingly the transient analysis of the system has not received as much attention. In many real-time applications, the system experiences a change and such changes can be measured by the transient analysis and not by steady-state analysis. The steady-state results cannot be used to determine the number of data packets waiting in the queue during the transmit state or in the vacation state at some time instant t. This motivates us to study the transient and steady-state analysis of the model.

The functions of the DPM are as follows. The DPM in WSNs consists of a service requestor (SR), a service provider (SP), a power manager (PM) and an event queue (EQ). The SR and the SP in the DPM are considered as incoming events and servers respectively. The incoming event joins the EQ and waits for service. After getting the service from the SP, the jobs are removed from the EQ. The PM in the DPM monitors the states of the SR, the SP and the EQ and controls the power management in the system. The power management modes of the DPM are shutdown state, wake-up state and inactive state. At the end of a busy period, the SP switches to the shutdown state during which the jobs can join the EQ. At this stage, the server cannot be activated until the EQ reaches a threshold value k. Once the EQ reaches a threshold value k, the SP enters into a wake-up state. From the wake-up state, the SP switches to the active mode. If no events arrive during the shutdown state, the SP will switch to an inactive state and wait for jobs to accumulate k jobs.

The significance and the advantages of the model are as follows. The key factors that affect the WSNs during their operation are power consumption in each state, switching cost and duration of time spent in each state. To optimize this system, mean power consumption is discussed in the paper and a threshold value k has been derived to acquire minimum power consumption for the WSNs while considering each different arrival rate. The transient analysis made in the paper enables the system analyst to understand the status of the system at any time t. The DPM with the threshold policy has many advantages over the other vacation schemes. It reduces the frequent switching of the wake-up and shutdown states, thereby increasing the life of sensor nodes and controlling the power consumption. It accumulates k events and then processes the job to make the system more energy efficient. The key feature of this work is to achieve minimum power consumption for each arrival rate using a threshold policy.

The remaining part of the paper is structured as follows. The description of the model is presented in Section 2. The transient probabilities of the wake-up, the shutdown, the inactive and the busy states are presented in Section 3. The mean, variance and the probability that the system in power-saving modes are presented in Section 4. The steady-state probabilities are derived explicitly in Section 5. The performance indices of the system in the steady-state are presented in section 6. The results obtained in Sections 3-6 are graphically illustrated in Section 7. The Conclusion and future work are presented in Section 8.

2 Description of the model

The model description of the WSN with start-up times and threshold policy is presented in this section.

1. The events join the queue according to a Poisson process with a rate of λ and the events receive service with a rate of μ_b which follows an exponential distribution.

2. After serving all the events in the busy state, the server switches to the shutdown mode of random duration V. The service is only provided in the busy state. During the shutdown period, events are allowed to join the queue, but the server will not resume service until the system accumulates k events.

3. At the end of period V, if the system reaches the threshold value k, then the system requires a start-up time which is exponentially distributed with the rate θ_1 to begin the service. To start-up, the system requires a change of state. The server switches from a shutdown state to an wake-up state which is exponentially distributed with a rate θ_2.

4. At the end of the shutdown period V, if the system size is less than k, then the server switches to an inactive mode with the rate θ_2 which follows an exponential distribution. Events can enter the system during the inactive period. At this epoch when the system size reaches the threshold value k, the system switches to the wake-up state

Let $\{M(t)), t \geq 0\}$ represent the status of the system at any time t and let $Q(t)$ denotes the number of events in the system at any time t.

$$M(t) = \begin{cases} 0, & \text{the server is in busy mode and operates with the rate } \mu_b \\ 1, & \text{the server is in wake-up mode} \\ 2, & \text{the server is in shutdown mode} \\ 3, & \text{the server is in inactive mode} \end{cases}$$

Then, $X(t) = \{M(t), Q(t), t \geq 0\}$ represents a continuous time Markov chain with state space

$$\begin{aligned} S = &\{(0,n) : n = 1,2,3,...\} \cup \{(1,n) : n = k, k+1, k+2, ...\} \\ &\times \cup \{(2,n) : n = 0,1,2,...\} \cup \big\{(3,n) : n = 0,1,2,...k-1.\big\} \end{aligned}$$

Let

$$\begin{aligned} P_{i,n}(t) &= P\big\{M(t) = i, Q(t) = n\big\}, i = 0\,; n = 1,2,3..., \\ P_{i,n}(t) &= P\big\{M(t) = i, Q(t) = n\big\}, i = 1\,; n = k, k+1, k+2..., \\ P_{i,n}(t) &= P\big\{M(t) = i, Q(t) = n\big\}, i = 2\,; n = 0,1,2,..., \\ P_{i,n}(t) &= P\big\{M(t) = i, Q(t) = n\big\}, i = 3\,; n = 0,1,2,...k-1. \end{aligned}$$

Then $P_{i,n}(t)$ satisfies the following forward Kolmogorov equation. Figure 1 presents the pictorial representation of the model.

$$P'_{0,1}(t) = -(\lambda + \mu_b) P_{0,1}(t) + \mu_b P_{0,2}(t). \tag{2.1}$$

$$P'_{0,n}(t) = -(\lambda + \mu_b) P_{0,n}(t) + \lambda P_{0,n-1}(t) + \mu_b P_{0,n+1}(t), n = 2,3,4,...,k-1. \tag{2.2}$$

$$P'_{0,n}(t) = -(\lambda + \mu_b) P_{0,n}(t) + \lambda P_{0,n-1}(t) + \mu_b P_{0,n+1}(t) + \theta_1 P_{1,n}(t), n = k, k+1, k+2, ... \tag{2.3}$$

$$P'_{1,k}(t) = -(\lambda + \theta_1) P_{1,k}(t) + \theta_2 P_{2,k}(t) + \lambda P_{3,k-1}(t). \tag{2.4}$$

$$P'_{1,n}(t) = -(\lambda + \theta_1) P_{1,n}(t) + \theta_2 P_{2,n}(t) + \lambda P_{1,n-1}(t), n = k+1, k+2, k+3..... \tag{2.5}$$

$$P'_{2,0}(t) = -(\lambda + \theta_2) P_{2,0}(t) + \mu_b P_{0,1}(t). \tag{2.6}$$

$$P'_{2,n}(t) = -(\lambda + \theta_2) P_{2,n}(t) + \lambda P_{2,n-1}(t), n = 1,2,3, ... \tag{2.7}$$

$$P'_{3,0}(t) = -\lambda P_{3,0}(t) + \theta_2 P_{2,0}(t). \tag{2.8}$$

$$P'_{3,n}(t) = -\lambda P_{3,n}(t) + \lambda P_{3,n-1}(t) + \theta_2 P_{2,n}(t), n = 1,2,...,k-1. \tag{2.9}$$

To setup initial conditions, we assume that the server is on inactive state initially. Therefore, $P_{3,0}(0) = 1$, $P_{0,n}(0) = 0$ for n=1,2,3...; $P_{1,n}(0) = 0$ for $n = k, k+1, k+2, ...$; $P_{2,n}(0) = 0$ for $n = 0,1,2,...$; $P_{3,n}(0) = 0$ for $n = 1,2,...,k-1$.

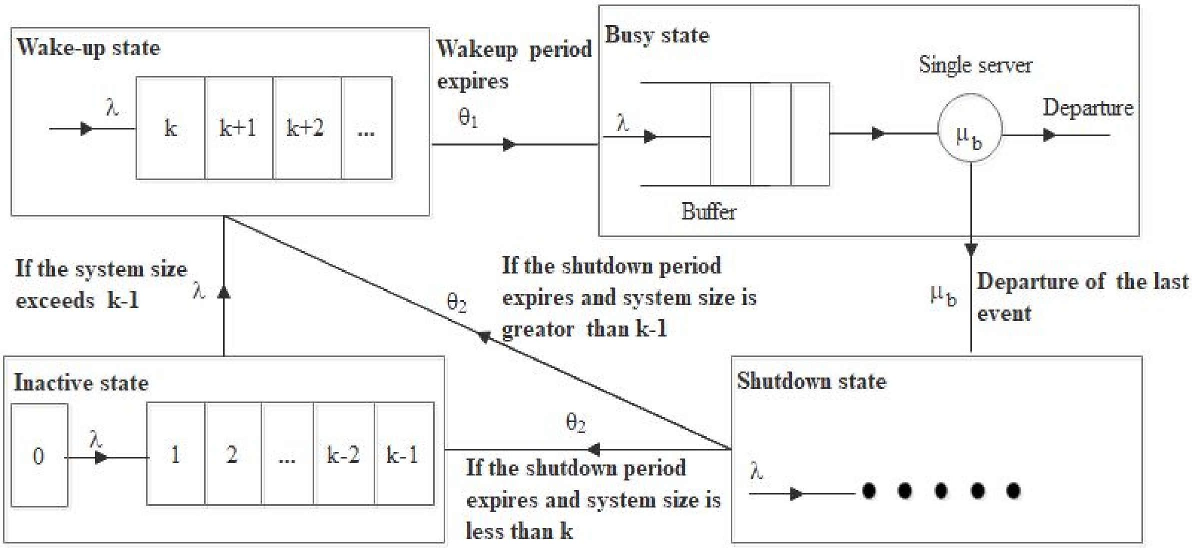

Figure 1: Pictorial representation of the model.

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3 Transient analysis

This section presents the time-dependent probabilities of the wake-up state $P_{1,n}(t)$, the shutdown state $P_{2,n}(t)$, the inactive state $P_{3,n}(t)$ and the busy state $P_{0,n}(t)$.

3.1 Evaluation of $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{3,n}(t)$

Let $\hat{P}_{i,n}(s)$ denote the Laplace transform of $P_{i,n}(t)$ for $i = 1, 2, 3\,; n = 0, 1, 2, ...$ Taking Laplace transform on Equations (2.4)-(2.9), we get

$$s\hat{P}_{1,k}\left(s\right) = -\left(\lambda+\theta_1\right)\hat{P}_{1,k}\left(s\right) + \theta_2\hat{P}_{2,k}\left(s\right) + \lambda\hat{P}_{3,k-1}\left(s\right), \tag{3.1}$$

$$s\hat{P}_{1,n}\left(s\right) = -\left(\lambda+\theta_1\right)\hat{P}_{1,n}\left(s\right) + \theta_2\hat{P}_{2,n}\left(s\right) + \lambda\hat{P}_{1,n-1}\left(s\right), n = k+1, k+2, k+3, ..., \tag{3.2}$$

$$s\hat{P}_{2,0}\left(s\right) = -\left(\lambda+\theta_2\right)\hat{P}_{2,0}\left(s\right) + \mu_b\hat{P}_{0,1}\left(s\right), \tag{3.3}$$

$$s\hat{P}_{2,n}\left(s\right) = -\left(\lambda+\theta_2\right)\hat{P}_{2,n}\left(s\right) + \lambda\hat{P}_{2,n-1}\left(s\right), n = 1, 2, ..., \tag{3.4}$$

$$s\hat{P}_{3,0}\left(s\right) - 1 = -\lambda\hat{P}_{3,0}\left(s\right) + \theta_2\hat{P}_{2,0}\left(s\right), \tag{3.5}$$

$$s\hat{P}_{3,n}\left(s\right) = -\lambda\hat{P}_{3,n}\left(s\right) + \lambda\hat{P}_{3,n-1}\left(s\right) + \theta_2\hat{P}_{2,n}\left(s\right), n = 1, 2, ..., k-1. \tag{3.6}$$

Let $\beta_1 = \lambda + \theta_1$ and $\beta_2 = \lambda + \theta_2$. Using Equations (3.3) and (3.4), we get

$$\hat{P}_{2,n}\left(s\right) = \frac{\lambda^n\mu_b}{\left(s+\beta_2\right)^{n+1}}\hat{P}_{0,1}\left(s\right), n = 0, 1, 2, ... \tag{3.7}$$

Substituting Equation (3.3) in (3.5) and further using it in Equation (3.6) after some algebraic manipulation, we obtain

$$\hat{P}_{3,n}(s) = \frac{\mu_b \theta_2 \lambda^n}{(s+\lambda)(s+\beta_2)} \left\{ \left(\frac{1}{s+\lambda}\right)^n + \sum_{i=1}^{n} \left(\frac{1}{s+\lambda}\right)^{n-i} \left(\frac{1}{s+\beta_2}\right)^i \right\} \hat{P}_{0,1}(s) + \frac{\lambda^n}{(s+\lambda)^{n+1}}, n = 0, 1, 2, ..., k-1. \tag{3.8}$$

Using Equations (3.7) and (3.8) in Equation (3.1) after some manipulation, we get

$$\hat{P}_{1,k}(s) = \frac{\theta_2 \mu_b \lambda^k}{(s+\beta_1)(s+\beta_2)} \left\{ \frac{1}{(s+\beta_2)^k} + \left(\frac{1}{s+\lambda}\right)^k + \sum_{i=1}^{k-1} \left(\frac{1}{s+\lambda}\right)^{k-i} \left(\frac{1}{s+\beta_2}\right)^i \right\} \times \hat{P}_{0,1}(s) + \left(\frac{\lambda}{s+\lambda}\right)^k \frac{1}{(s+\beta_1)}. \tag{3.9}$$

Using Equations (3.7) and (3.9) in Equation (3.2), we get

$$\begin{aligned}\hat{P}_{1,n}(s) = {} & \frac{\mu_b \lambda^n \theta_2}{(s+\beta_1)(s+\beta_2)} \left[\sum_{j=k+1}^{n} \left(\frac{1}{s+\beta_1}\right)^{n-j} \left(\frac{1}{s+\beta_2}\right)^j + \left(\frac{1}{s+\beta_1}\right)^{n-k} \right. \\ & \left. \times \left\{ \frac{1}{(s+\beta_2)^k} + \left(\frac{1}{s+\lambda}\right)^k + \sum_{i=1}^{k-1} \left(\frac{1}{s+\lambda}\right)^{k-i} \times \left(\frac{1}{s+\beta_2}\right)^i \right\} \right] \hat{P}_{0,1}(s) \\ & + \lambda^n \left(\frac{1}{s+\beta_1}\right)^{n-k+1} \left(\frac{1}{s+\lambda}\right)^k, n = k, k+1, k+2, ... \end{aligned} \tag{3.10}$$

Inversion on Equations (3.10), (3.7) and (3.8) respectively yields

$$\begin{aligned} P_{1,n}(t) = {} & \lambda^n \mu_b \theta_2 \exp\{-(\beta_1)t\} * \exp\{-(\beta_2)t\} * \left[\sum_{j=k+1}^{n} \frac{\exp\{-(\beta_1)t\} t^{n-j-1}}{(n-j-1)!} \right. \\ & * \frac{\exp\{-(\beta_2)t\} t^{j-1}}{(j-1)!} + \frac{\exp\{-(\beta_1)t\} t^{n-k-1}}{(n-k-1)!} * \left\{ \frac{\exp\{-(\beta_2)t\} t^{k-1}}{(k-1)!} \right. \\ & \left. \left. + \frac{\exp(-\lambda t)^{k-1}}{(k-1)!} + \sum_{i=1}^{k-1} \frac{\exp(-\lambda t)^{k-i-1}}{(k-i-1)!} * \frac{\exp\{-(\beta_2)t\} t^{i-1}}{(i-1)!} \right\} \right] * P_{0,1}(t) \\ & + \frac{\lambda^n \exp\{-(\beta_1)t\} t^{n-k}}{(n-k)!} * \frac{\exp(-\lambda t)^{k-1}}{(k-1)!}, n = k, k+1, k+2, ..., \end{aligned} \tag{3.11}$$

$$P_{2,n}(t) = \frac{\lambda^n \mu_b \exp\{-(\beta_2)t\} t^n}{n!} * P_{0,1}(t), n = 0, 1, 2, ..., \tag{3.12}$$

$$\begin{aligned} P_{3,n}(t) = {} & \left\{ \frac{\exp(-\lambda t) t^{n-1}}{(n-1)!} + \sum_{i=1}^{n} \frac{\exp(-\lambda t) t^{n-i-1}}{(n-i-1)!} * \frac{\exp\{-(\beta_2)t\} t^{i-1}}{(i-1)!} \right\} * P_{0,1}(t) \\ & * \theta_2 \mu_b \lambda^n \exp(-\lambda t) * \exp\{-(\beta_2)t\} + \frac{\lambda^n \exp(-\lambda t) t^n}{n!}, n = 0, 1, 2, ...k-1. \end{aligned} \tag{3.13}$$

The time-dependent probabilities $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{3,n}(t)$ are expressed in-terms of $P_{0,1}(t)$ and an explicit expression for $P_{0,1}(t)$ is given in Equation (3.24).

3.2 Evaluation of $P_{0,n}(t)$

The busy-state probability $P_{0,n}(t)$; n=1,2,3,... is obtained using Equations (2.1)-(2.3) by applying generating function.
Let

$$H(z,t)=\sum_{n=1}^{\infty}P_{0,n}(t)z^n, H(z,0)=0.$$

Using Equations (2.1)-(2.3), we get

$$\frac{\partial}{\partial t}H(z,t)=H(z,t)\left[-(\lambda+\mu_b)+\mu_b z^{-1}+\lambda z\right]+\theta_1\sum_{n=k}^{\infty}P_{1,n}(t)z^n-\mu_b P_{0,1}(t).$$

On solving,

$$\begin{aligned}H(z,t)&=\theta_1\int_0^t\sum_{m=k}^{\infty}P_{1,m}(w)\,z^m\exp\left\{-\beta+\left(\lambda z+\mu_b z^{-1}\right)\right\}(t-w)\,dw\\&-\mu_b\int_0^t P_{0,1}(w)\exp\left\{-\beta+\left(\lambda z+\mu_b z^{-1}\right)\right\}(t-w)\,dw,\end{aligned}\tag{3.14}$$

where $\beta=\lambda+\mu_b$. Let $\kappa=2\sqrt{\lambda\mu_b}$ and $\nu=\sqrt{\lambda\mu_b^{-1}}$, then

$$\exp\left[\left(\lambda z+\frac{\mu_b}{z}\right)(t-w)\right]=\sum_{n=-\infty}^{\infty}(\nu z)^n I_n(\kappa(t-w)).\tag{3.15}$$

where $I_m(t)$ represents the modified Bessel function of the first kind of order m. Applying Equation (3.15) in Equation (3.14) and equating the coefficient of z^n on both sides for $n=k+1,k+2,k+3...$

$$\begin{aligned}P_{0,n}(t)&=\theta_1\int_0^t\sum_{m=k}^{\infty}P_{1,m}(w)\exp\{-\beta(t-w)\}\,\nu^{n-m}I_{n-m}(.)\,dw\\&-\mu_b\int_0^t P_{0,1}(w)\exp\{-\beta(t-w)\}\,\nu^n I_n(.)\,dw.\end{aligned}\tag{3.16}$$

Equating the coefficients of z^{-n} on both sides of Equation (3.14) for $n=k+1,k+2,k+3...$ and using $I_{-n}(.)=I_n(.)$, we get

$$\begin{aligned}0&=\theta_1\int_0^t\sum_{m=k}^{\infty}P_{1,m}(w)\exp\{-\beta(t-w)\}\,\nu^{-n-m}I_{n+m}(.)\,dw\\&-\mu_b\int_0^t P_{0,1}(w)\exp\{-\beta(t-w)\}\,\nu^{-n}I_n(.)\,dw..\end{aligned}\tag{3.17}$$

Multiplying ν^{2n} on both sides of Equation (3.17) and subtracting it from Equation (3.16) for n=1,2,3..., we arrive

$$P_{0,n}(t) = \theta_1 \int_0^t \sum_{m=k}^{\infty} P_{1,m}(w) \exp\{-\beta(t-w)\} \nu^{n-m} [I_{n-m}(\cdot) - I_{n+m}(\cdot)]\, dw. \tag{3.18}$$

Taking Laplace transform on Equation (3.18), we obtain

$$\hat{P}_{0,n}(s) = \frac{\theta_1}{\sqrt{d^2-\kappa^2}} \sum_{m=1}^{\infty} \hat{P}_{1,m}(s) \nu^{n-m} \left(\hat{\varsigma}(s)^{n-m} - \hat{\varsigma}(s)^{n+m}\right) \tag{3.19}$$

where

$$\hat{\varsigma}(s) = \frac{d-\sqrt{d^2-\kappa^2}}{\kappa} \text{ and } d = s+\lambda+\mu_b.$$

Using Equation (3.10) in Equation (3.19), we obtain

$$\hat{P}_{0,n}(s) = \frac{\theta_1}{\sqrt{d^2-\kappa^2}} \sum_{m=k}^{\infty} \nu^{n-m} \Bigg[\frac{\mu_b\theta_2\lambda^m}{(s+\beta_1)(s+\beta_2)} \Bigg[\sum_{j=k+1}^{m} \left(\frac{1}{s+\beta_1}\right)^{m-j} \left(\frac{1}{s+\beta_2}\right)^{j}$$
$$+ \left(\frac{1}{s+\beta_1}\right)^{m-k} \left\{ \frac{1}{(s+\beta_2)^k} + \left(\frac{1}{s+\lambda}\right)^k + \sum_{i=1}^{k-1} \left(\frac{1}{s+\lambda}\right)^{k-i} \left(\frac{1}{s+\beta_2}\right)^{i} \right\}$$
$$P_{0,1}(t) \Bigg] + \lambda^m \left(\frac{1}{s+\beta_1}\right)^{m-k+1} \left(\frac{1}{s+\lambda}\right)^{k} \hat{\varsigma}(s)^{n-m} - \hat{\varsigma}(s)^{n+m} \Bigg].$$

On inversion,

$$P_{0,n}(t) = \theta_1 \sum_{m=k}^{\infty} \nu^{n-m} \Bigg[\mu_b\theta_2\lambda^m \exp\{-(\beta_1)t\} * \exp\{-(\beta_2)t\}$$
$$* \Bigg[\sum_{j=k+1}^{m} \left\{ \frac{\exp\{-(\beta_1)t\} t^{m-j-1}}{(m-j-1)!} * \frac{\exp\{-(\beta_2)t\} t^{j-1}}{(j-1)!} \right\}$$
$$+ \frac{\exp\{-(\beta_1)t\} t^{m-k-1}}{(m-k-1)!} * \Bigg[\frac{\exp\{-(\beta_2)t\} t^{k-1}}{(k-1)!} + \frac{\exp(-\lambda t)^{k-1}}{(k-1)!}$$
$$+ \sum_{i=1}^{k-1} \left\{ \frac{\exp(-\lambda t)^{k-i-1}}{(k-i-1)!} * \frac{\exp\{-(\beta_2)t\} t^{i-1}}{(i-1)!} \right\} \Bigg] \Bigg] * P_{0,1}(t) + \Bigg\{ \frac{\exp(-\lambda t)^{k-1}}{(k-1)!}$$
$$* \frac{\exp\{-(\beta_1)t\} t^{m-k}}{(m-k)!} \Bigg\} * [I_{n-m}(\kappa t) - I_{n+m}(\kappa t)] \exp\{-(\lambda+\mu_b)t\} \Bigg]. \tag{3.20}$$

3.3 Evaluation of $P_{0,1}(t)$

Setting $n=1$ in Equation (3.5), we obtain

$$P_{0,1}(t) = \theta_1 \int_0^t \sum_{m=k}^{\infty} P_{1,m}(w) \exp\{-\beta(t-w)\} \nu^{1-m} [I_{1-m}(\cdot) - I_{1+m}(\cdot)]\, dw.$$

On inversion

$$\hat{P}_{0,1}(s) = 2\theta_1 \sum_{m=k}^{\infty} \nu^{1-m} \hat{P}_{1,m}(s) \frac{\kappa^{m-1}}{\left(d+\sqrt{d^2-\kappa^2}\right)^m}. \tag{3.21}$$

where $d = s + \lambda + \mu_b$. Substituting Equation (3.15) in Equation (3.19) after some manipulation, we get

$$\hat{P}_{0,1}(s) = \theta_1 \left(\frac{1}{s+\lambda}\right)^k \sum_{m=k}^{\infty} \nu^{1-m} \lambda^m \left(\frac{1}{s+\beta_1}\right)^{m-k+1} \frac{2\kappa^{m-1}}{\left(d+\sqrt{d^2-\kappa^2}\right)^m} \sum_{h=0}^{\infty} \left(\hat{G}_1(s)\right)^h, \tag{3.22}$$

where

$$\begin{aligned}\hat{G}_1(s) = {} & \frac{\mu_b \theta_1 \theta_2}{(s+\beta_1)(s+\beta_2)} \sum_{m=k}^{\infty} \nu^{1-m} \lambda^m \left[\sum_{j=k+1}^{m} \left(\frac{1}{s+\beta_1}\right)^{m-j} \left(\frac{1}{s+\beta_2}\right)^j\right. \\ & + \left(\frac{1}{s+\beta_1}\right)^{m-k} \left\{\frac{1}{(s+\beta_2)^k} + \left(\frac{1}{s+\lambda}\right)^k + \sum_{i=1}^{k-1} \left(\frac{1}{s+\lambda}\right)^{k-i}\right. \\ & \times \left.\left.\left(\frac{1}{s+\beta_2}\right)^i\right\}\right] \frac{2\kappa^{m-1}}{\left(d+\sqrt{d^2-\kappa^2}\right)^m}. \end{aligned} \tag{3.23}$$

Inversion on (3.20) gives

$$\begin{aligned}P_{0,1}(t) = {} & \frac{\theta_1 \exp(-\lambda t)^{k-1}}{(k-1)!} * \sum_{m=k}^{\infty} \frac{\lambda^m \exp\{-(\beta_1)t\} t^{m-k}}{(m-k)!} * \nu^{1-m} \left[I_{1-m}(\kappa t) - I_{1+m}(\kappa t)\right] \\ & * \sum_{h=0}^{\infty} (G_1(t))^{*h}, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned}G_1(t) = {} & \theta_1 \theta_2 \mu_b \exp\{-(\beta_1)t\} * \exp\{-(\beta_2)t\} * \sum_{m=k}^{\infty} \lambda^m \left[\sum_{j=k+1}^{m} \exp\{-(\beta_1)t\}\right. \\ & \times \frac{t^{m-j-1}}{(m-j-1)!} * \frac{\exp\{-(\beta_2)t\} t^{j-1}}{(j-1)!} + \frac{\exp\{-(\beta_1)t\} t^{m-k-1}}{(m-k-1)!} \\ & * \left\{\frac{t^{k-1}}{(k-1)!} \exp\{-(\beta_2)t\} + \frac{\exp(-\lambda t)^{k-1}}{(k-1)!} + \sum_{i=1}^{k-1} \frac{\exp(-\lambda t)^{k-i-1}}{(k-i-1)!}\right. \\ & \left.\left. * \frac{\exp\{-(\beta_2)t\} t^{i-1}}{(i-1)!}\right\}\right] * \nu^{1-m} \left[I_{1-m}(\kappa t) - I_{1+m}(\kappa t)\right] \end{aligned} \tag{3.25}$$

and '$*h$' denotes h-fold convolution. Thus we have obtained an explicit expression for $P_{0,1}(t)$.

4 Performance measures

The mean and variance of the system size are presented in this section.

4.1 Mean

Let $\Omega(t)$ denote the expected system size at time t. For $t > 0$, we have

$$\Omega(t) = E\left[X\left(t\right)\right] = \sum_{n=1}^{\infty} nP_{0,n}\left(t\right) + \sum_{n=k}^{\infty} nP_{1,n}\left(t\right) + \sum_{n=1}^{\infty} nP_{2,n}\left(t\right) + \sum_{n=0}^{k-1} nP_{3,n}(t).$$

Then using Equations (2.1)-(2.9), we get

$$\frac{d}{dt}\Omega\left(t\right) = \lambda - \mu_b \sum_{n=1}^{\infty} P_{0,n}\left(t\right) + \lambda \sum_{n=k}^{\infty} P_{1,n}\left(t\right) + \lambda \sum_{n=0}^{k-1} P_{3,n}\left(t\right).$$

The above equation gives

$$\Omega\left(t\right) = \lambda t - \mu_b \sum_{n=1}^{\infty} \int_0^t P_{0,n}\left(y\right) dy + \lambda \sum_{n=k}^{\infty} \int_0^t P_{1,n}\left(y\right) dy + \lambda \sum_{n=0}^{k-1} \int_0^t P_{3,n}\left(y\right) dy.$$

4.2 Variance

Let $V(t)$ denote the variance system size at time t. For $t > 0$,

$$V(t) = E[X^2(t)] - (E[X(t)])^2,$$

where

$$E\left[X^2\left(t\right)\right] = \sum_{n=1}^{\infty} n^2 P_{0,n}\left(t\right) + \sum_{n=k}^{\infty} n^2 P_{1,n}\left(t\right) + \sum_{n=1}^{\infty} n^2 P_{2,n}\left(t\right) + \sum_{n=1}^{k-1} n^2 P_{3,n}\left(t\right).$$

Using Equations (2.1)-(2.9), we obtain

$$\frac{d}{dt} E\left[X^2\left(t\right)\right] = \lambda - \mu_b \sum_{n=0}^{\infty} \left(2n+1\right) P_{0,n+1}\left(t\right) + 2\lambda\Omega(t).$$

Then

$$E\left[X^2\left(t\right)\right] = \lambda t - \mu_b \sum_{n=0}^{\infty} \left(2n+1\right) \int_0^t P_{0,n+1}\left(y\right) dy + 2\lambda \int_0^t \Omega(y) dy.$$

4.3 Probability that the server is in power-saving modes

Let $P_{u_i,\bullet}\left(t\right); i = 1,2,3$ denote the probability that the server is on wake-up state, shutdown state and inactive state respectively, then

$$\hat{P}_{1,\bullet}\left(s\right) = \sum_{n=k}^{\infty} \hat{P}_{1,n}\left(s\right),$$

$$\hat{P}_{2,\bullet}\left(s\right) = \sum_{n=0}^{\infty} \hat{P}_{2,n}\left(s\right),$$

$$\hat{P}_{3,\bullet}(s) = \sum_{n=0}^{k-1} \hat{P}_{3,n}(s).$$

Using Equations (3.10), (3.7) and (3.8) in the above expression and taking inversion, respectively yield

$$\begin{aligned} P_{1,\bullet}(t) &= \mu_b \theta_2 \exp\{-(\beta_1)t\} * \exp\{-(\beta_2)t\} * \Bigg[\lambda^{k+1}\left\{\frac{\delta'(t) + (\beta_1)\delta(t)}{\theta_2 - \theta_1}\right\} \\ &* \frac{\exp\{-(\beta_2)t\}t^{k-1}}{(k-1)!} * \{\exp(-\theta_1 t) - \exp(-\theta_2 t)\} + \{\lambda^{k+1}\exp(-\theta_1 t) \\ &+ \lambda^k \delta(t)\} * \left\{\frac{\exp\{-(\beta_2)t\}t^{k-1}}{(k-1)!} + \frac{\exp(-\lambda t)^{k-1}}{(k-1)!} + \sum_{i=1}^{k-1} \frac{\exp(-\lambda t)^{k-i-1}}{(k-i-1)!}\right. \\ &\left. * \frac{\exp\{-(\beta_2)t\}t^{i-1}}{(i-1)!}\right\}\Bigg] * P_{0,1}(t) + \frac{\lambda^k \exp(-\lambda t)t^{k-1} * \exp(-\theta_1 t)}{(k-1)!}, \end{aligned}$$

$$P_{2,\bullet}(t) = \mu_b \exp(-\theta_2 t) * P_{0,1}(t),$$

$$\begin{aligned} P_{3,\bullet}(t) &= \delta(t) - \frac{\lambda^k \exp(-\lambda t)t^{k-1}}{(k-1)!} + \mu_b[\exp(-\lambda t) - \exp\{-(\beta_2)t\}] \\ &* \Bigg[\{\delta(t) + \lambda\} * \left\{\delta(t) - \frac{\lambda^k \exp(-\lambda t)t^{k-1}}{(k-1)!}\right\} + \left\{\frac{\delta'(t) + \lambda\delta(t)}{\theta_2}\right\} \\ &* \Bigg[\lambda\left\{\delta(t) - \frac{\lambda^{k-1}\exp(-\lambda t)t^{k-2}}{(k-2)!}\right\} - \lambda\exp(-\theta_2 t) \\ &* \left\{\delta(t) - \frac{\lambda^{k-1}\exp\{-(\beta_2)t\}t^{k-2}}{(k-2)!}\right\}\Bigg]\Bigg] * P_{0,1}(t). \end{aligned}$$

where $\delta(t)$ represents Dirac delta function.

4.4 Probability that the server is in busy state

Let $P_{b,\bullet}(t)$ denote the probability that the server is on busy state, then

$$\hat{P}_{b,\bullet}(s) = \sum_{n=1}^{\infty} \hat{P}_{b,n}(s).$$

Using Equation (3.19) and taking inversion, we get

$$P_{b,\bullet}(t) = \theta_1 \sum_{n=1}^{\infty}\sum_{m=k}^{\infty} P_{1,m}(t)\nu^{n-m} * [I_{n-m}(\kappa t) - I_{n+m}(\kappa t)]\exp(-\beta t).$$

4.5 Probability that the server is either in busy state or wake-up state or shutdown state or inactive state

Let $P(t)$ denote the probability that the server is either in busy state or in wake-up state or in shutdown state or in inactive state, then

$$P(t) = \sum_{n=1}^{\infty} P_{0,n}(t) + \sum_{n=k}^{\infty} P_{1,n}(t) + \sum_{n=0}^{\infty} P_{2,n}(t) + \sum_{n=0}^{k-1} P_{3,k-1}(t).$$

5 Steady-state probabilities

The system size probabilities of the wake-up state, the shutdown state and the inactive state are presented in this section.
Let $\{\pi_{k,n}; k = v_1, v_2, c, b, n \geq 0\}$ represent the steady-state probability distributions for the model considered. Applying the steady-state condition $\lim_{s\to 0} s\hat{P}_{i,n} = \pi_{i,n}$ on Equations (2.1)-(2.9), we get

$$
\begin{aligned}
(\lambda + \mu_b)\,\pi_{0,1} &= \mu_b \pi_{0,2}, && (5.1)\\
(\lambda + \mu_b)\,\pi_{0,n} &= \lambda \pi_{0,n-1} + \mu_b \pi_{0,n+1}, n = 2, 3, 4, ..., k-1, && (5.2)\\
(\lambda + \mu_b)\,\pi_{0,n} &= \lambda \pi_{0,n-1} + \mu_b \pi_{0,n+1} + \theta_1 \pi_{1,n}, n = k, k+1, k+2, ..., && (5.3)\\
(\lambda + \theta_1)\,\pi_{1,k} &= \theta_2 \pi_{2,k} + \lambda \pi_{3,k-1}, && (5.4)\\
(\lambda + \theta_1)\,\pi_{1,n} &= \theta_2 \pi_{2,n} + \lambda \pi_{1,n-1}, n = k+1, k+2, k+3, ..., && (5.5)\\
(\lambda + \theta_2)\,\pi_{2,0} &= \mu_b \pi_{0,1}, && (5.6)\\
(\lambda + \theta_2)\,\pi_{2,n} &= \lambda \pi_{2,n-1}, n = 1, 2, 3, ..., && (5.7)\\
\lambda \pi_{3,0} &= \theta_2 \pi_{2,0}, && (5.8)\\
\lambda \pi_{3,n} &= \lambda \pi_{3,n-1} + \theta_2 \pi_{2,n}, n = 1, 2, ..., k-1. && (5.9)
\end{aligned}
$$

Using Equations (5.6) and (5.7), we get

$$\pi_{2,n} = \left(\frac{\lambda}{\beta_2}\right)^n \left(\frac{1}{\beta_2}\right) \mu_b \pi_{0,1}, n = 0, 1, 2, ..., \tag{5.10}$$

where $\beta_2 = \lambda + \theta_2$.
Substituting Equation (5.10) in Equation (5.9), we get

$$\pi_{3,n} = \pi_{3,n-1} + \frac{\theta_2}{\lambda\beta_2}\left(\frac{\lambda}{\beta_2}\right)^n \mu_b \pi_{0,1}, n = 1, 2, ..., k-1. \tag{5.11}$$

Equation (5.11) recursively yields

$$\pi_{3,n} = \pi_{3,0} + \frac{\theta_2}{\lambda\beta_2}\sum_{i=1}^{n}\left(\frac{\lambda}{\beta_2}\right)^i \mu_b \pi_{0,1}. \tag{5.12}$$

Using Equation (5.8) in (5.12), we get

$$\pi_{3,n} = \frac{\theta_2}{\lambda\beta_2}\left[1 + \sum_{i=1}^{n}\left(\frac{\lambda}{\beta_2}\right)^i\right] \mu_b \pi_{0,1}, n = 0, 1, 2, ..., k-1. \tag{5.13}$$

Using Equations (5.10) and (5.13) in Equation (5.4), we get

$$\pi_{1,k} = \frac{\theta_2}{\beta_1\beta_2}\sum_{i=0}^{k}\left(\frac{\lambda}{\beta_2}\right)^i \mu_b \pi_{0,1}, \tag{5.14}$$

where $\beta_1 = \lambda + \theta_1$.
Equation (5.5) recursively yields

$$\pi_{1,n} = \frac{\theta_2 \lambda^n}{\beta_1\beta_2}\sum_{i=k+1}^{n}\left(\frac{1}{\beta_1}\right)^{n-i}\left(\frac{1}{\beta_2}\right)^i \mu_b \pi_{0,1} + \left(\frac{\lambda}{\beta_1}\right)^{n-k} \pi_{1,k}. \tag{5.15}$$

Substituting Equation (5.14) in Equation (5.15), we get

$$\pi_{1,n} = \frac{\theta_2}{\beta_1\beta_2}\left[\lambda^n \sum_{i=k+1}^{n}\left(\frac{1}{\beta_1}\right)^{n-i}\left(\frac{1}{\beta_2}\right)^{i} + \left(\frac{\lambda}{\beta_1}\right)^{n-k}\left(\frac{\beta_2}{\theta_2}\right)\left\{1-\left(\frac{\lambda}{\beta_2}\right)^{k+1}\right\}\right]\mu_b\pi_{0,1}. \tag{5.16}$$

The steady-state probabilities of the wake-up mode, the shutdown mode and the inactive mode are expressed in-terms of $\pi_{0,1}$. To get an explicit expression for $\pi_{0,1}$, we define a generating function as follows:

$$G_0(z) = \sum_{n=1}^{\infty}\pi_{0,n}z^n, \tag{5.17}$$

$$G_1(z) = \sum_{n=k}^{\infty}\pi_{1,n}z^n, \tag{5.18}$$

$$G_1'(z) = \sum_{n=k}^{\infty}n\pi_{1,n}z^{n-1}. \tag{5.19}$$

Using Equations (5.1)-(5.3), we obtain

$$G_0(z)\left(\lambda+\mu_b-\lambda z-\frac{\mu_b}{z}\right) = -\mu_b\pi_{0,1}+\theta_1 G_1(z).$$

After some algebra, we get

$$G_0(z) = \frac{z\lambda\theta_1\{G_1(z)-G_1(1)\}}{(z\lambda-\mu_b)(1-z)}. \tag{5.20}$$

Setting $z=1$, we get

$$G_0(1) = \frac{\lambda\theta_1 G'_1(1)}{\mu_b-\lambda}.$$

Using Equation (5.19) for $z=1$ and the result (5.16) in the above equation, we get

$$\begin{aligned} G_0(1) = &\sum_{n=k}^{\infty} n\left[\lambda^n \sum_{i=k+1}^{n}\left(\frac{1}{\beta_1}\right)^{n-i}\left(\frac{1}{\beta_2}\right)^{i} + \left(\frac{\lambda}{\beta_1}\right)^{n-k}\left(\frac{\beta_2}{\theta_2}\right)\left\{1-\left(\frac{\lambda}{\beta_2}\right)^{k+1}\right\}\right]\mu_b\pi_{0,1} \\ &\times \frac{\rho}{1-\rho}\frac{\theta_1\theta_2}{\beta_1\beta_2}, \end{aligned} \tag{5.21}$$

where $\rho = \frac{\lambda}{\mu_b} < 1$. The normalization condition is given by

$$\sum_{n=1}^{\infty}\pi_{0,n} + \sum_{n=k}^{\infty}\pi_{1,n} + \sum_{n=0}^{\infty}\pi_{2,n} + \sum_{n=0}^{k-1}\pi_{3,n} = 1.$$

Substituting the results (5.10), (5.13), (5.16) and (5.21) in the above condition, we obtain

$$\begin{aligned} \pi_{0,1} = &\left[\frac{\theta_1\theta_2\mu_b}{\beta_1\beta_2}\sum_{n=k}^{\infty}\left[\left(\frac{n\rho}{1-\rho}+\frac{1}{\theta_1}\right)\left\{\lambda^n\sum_{i=k+1}^{n}\left(\frac{1}{\beta_1}\right)^{n-i}\left(\frac{1}{\beta_2}\right)^{i} + \left(\frac{\lambda}{\beta_1}\right)^{n-k}\left(\frac{\beta_2}{\theta_2}\right)\right.\right.\right. \\ &\left.\left.\left.\left(1-\left(\frac{\lambda}{\beta_2}\right)^{k+1}\right)\right\}\right] + \frac{1}{\theta_2} + \frac{\theta_2}{\lambda\beta_2} + \frac{\theta_2}{\lambda\beta_2}\sum_{n=0}^{k-1}\left\{1+\left(\frac{\lambda}{\beta_2}\right)\left\{1-\left(\frac{\lambda}{\beta_2}\right)^{n}\right\}\frac{\lambda}{\theta_2}\right\}\right]^{-1}. \end{aligned} \tag{5.22}$$

5.1 Probability that the server is on wake-up state, shutdown state and inactive state

Let $\pi_{u_i\bullet}; i = 1, 2, 3$ denote the probability that the server is on wake-up state, shutdown state and inactive state respectively, then

$$\pi_{1,\bullet} = \sum_{n=k}^{\infty} \pi_{1,n}.$$

$$\pi_{2,\bullet} = \sum_{n=0}^{\infty} \pi_{2,n}.$$

$$\pi_{3,\bullet} = \sum_{n=0}^{k-1} \pi_{3,n}.$$

Using the results (5.16), (5.10) and (5.13) in the above equation, we obtain

$$\pi_{1,\bullet} = \frac{\theta_2}{\beta_1\beta_2} \sum_{n=k}^{\infty} \left[\lambda^n \sum_{i=k+1}^{n} \left(\frac{1}{\beta_1}\right)^{n-i} \left(\frac{1}{\beta_2}\right)^{i} + \left(\frac{\lambda}{\beta_1}\right)^{n-k} \left(\frac{\beta_2}{\theta_2}\right) \left\{1 - \left(\frac{\lambda}{\beta_2}\right)^{k+1}\right\} \right] \mu_b \pi_{0,1}, \tag{5.23}$$

$$\pi_{2,\bullet} = \frac{\mu_b}{\theta_2} \pi_{0,1}, \tag{5.24}$$

$$\pi_{3,\bullet} = \frac{1}{\lambda} \left[k - \frac{1}{\theta_2} \left\{1 - \left(\frac{\lambda}{\beta_2}\right)^{k}\right\} \right] \mu_b \pi_{0,1}. \tag{5.25}$$

6 Performance measures

This section presents expected system size in the steady-state. Let $E[N_0]$, $E[N_1]$, $E[N_2]$ and $E[N_3]$ be the mean number of events in the busy, wake-up, shutdown and inactive states respectively and let $E[N_s]$ denote the expected system size. Then,

$$E\left[N_s\right] = E\left[N_0\right] + E\left[N_1\right] + E\left[N_2\right] + E[N_3].$$

6.1 Expected system size in the busy state

The mean number of events in the the busy state is given by

$$E(N_0) = \lim_{z\to 1} G_0'\left(z\right).$$

Differentiating Equation (5.20) and setting $z = 1$, after some algebraic manipulation, we get

$$E\left(N_0\right) = \frac{\rho}{1-\rho} \frac{\theta_1\theta_2}{\beta_1\beta_2} \sum_{n=k}^{\infty} \left[\lambda^n \sum_{i=k+1}^{n} \left(\frac{1}{\beta_1}\right)^{n-i} \left(\frac{1}{\beta_2}\right)^{i} + \left(\frac{\lambda}{\beta_1}\right)^{n-k} \frac{\beta_2}{\theta_2} \left\{1 - \left(\frac{\lambda}{\beta_2}\right)^{k+1}\right\} \right]$$
$$\times \left[\frac{n}{1-\rho} + \frac{n(n-1)}{2} \right] \mu_b \pi_{0,1}. \tag{6.1}$$

where $\rho < 1$.

6.2 Expected system size in the wake-up state

Let $E[N_1]$ be the mean number of events in the system during wake-up mode. Then

$$E\left[N_1\right] = \sum_{n=k}^{\infty} n\pi_{1,n}.$$

Using the result (5.16), we obtain

$$E[N_1] = \frac{\theta_2}{\beta_1\beta_2}\sum_{n=k}^{\infty} n\left[\lambda^n \sum_{i=k+1}^{n}\left(\frac{1}{\beta_1}\right)^{n-i}\left(\frac{1}{\beta_2}\right)^{i} + \left(\frac{\lambda}{\beta_1}\right)^{n-k}\frac{\beta_2}{\theta_2}\left\{1-\left(\frac{\lambda}{\beta_2}\right)^{k+1}\right\}\mu_b\pi_{0,1}\right]. \tag{6.2}$$

6.3 Expected system size in the shutdown state

Let $E[N_2]$ be the mean number of events in the system during shutdown mode. Then,

$$E\left[N_2\right] = \sum_{n=0}^{\infty} n\pi_{2,n}.$$

Using the result (5.10), we obtain

$$E\left[N_2\right] = \frac{\lambda}{{\theta_2}^2}\mu_b\pi_{0,1}. \tag{6.3}$$

6.4 Expected system size in the inactive state

Let $E[N_2]$ be the mean number of events in the system during inactive mode. Then,

$$E\left[N_3\right] = \sum_{n=0}^{k-1} n\pi_{3,n}.$$

Using the result (5.13), we obtain

$$E\left[N_3\right] = \frac{1}{\lambda}\sum_{n=0}^{k-1} n\left\{1-\left(\frac{\lambda}{\beta_2}\right)^{n+1}\right\}\mu_b\pi_{0,1}. \tag{6.4}$$

6.5 Expected number of events waiting in the system

Let $E[W_s]$ denote the mean number of events waiting in the system. Then,

$$E\left[W_s\right] = \frac{E\left[N_s\right]}{\lambda}.$$

The mean number of events waiting in the queue is given by

$$E\left[W_q\right] = \sum_{n=k}^{\infty} n\pi_{1,n} + \sum_{n=0}^{\infty} n\pi_{2,n} + \sum_{n=0}^{k-1} n\pi_{3,n} + \sum_{n=1}^{\infty}(n-1)\pi_{0,n}.$$

6.6 Mean power consumption

This section presents the mean power consumed by the sensor node in each cycle. To minimise the mean power consumption, we define a function

$$F(k) = C_h E(N_s) + C_0\pi_{0,\bullet} + C_1\pi_{1,\bullet} + C_2\pi_{2,\bullet} + C_3\pi_{3,\bullet} + C_{\mu_b}\mu_b + C_{\theta_1}\theta_1 + C_{\theta_2}\theta_2.$$

Using the results (6.1)-(6.4) in the above expression, we get

$$\begin{aligned} F(k) = \Bigg[& \frac{\theta_2}{\beta_1\beta_2}\sum_{n=k}^{\infty}\left\{\lambda^n \sum_{i=k+1}^{n}\left(\frac{1}{\beta_1}\right)^{n-i}\left(\frac{1}{\beta_2}\right)^{i} + \left(\frac{\lambda}{\beta_1}\right)^{n-k}\frac{\beta_2}{\theta_2}\left(1-\left(\frac{\lambda}{\beta_2}\right)^{k+1}\right)\right\} \\ & \times\left\{\theta_1 C_h\frac{\rho}{1-\rho}\left(\frac{n}{1-\rho}+\frac{n^2-n}{2}\right) + n\theta_1 C_0\frac{\rho}{1-\rho} + C_1\right\} + \frac{C_h\lambda}{{\theta_2}^2} + \frac{C_2\mu_b}{\theta_2} \\ & + \frac{C_h}{\lambda}\sum_{n=0}^{k-1} n\left(1-\left(\frac{\lambda}{\beta_2}\right)^{n+1}\right) + \frac{C_3}{\lambda}\left\{k - \frac{1}{\theta_2}\left(1-\left(\frac{\lambda}{\beta_2}\right)^{k}\right)\right\} + C_{\mu_b}\mu_b \\ & + C_{\theta_1}\theta_1 + C_{\theta_2}\theta_2\Bigg]\mu_b\pi_{0,1}, \end{aligned}$$

The mean power consumption for all cycles can be formulated as $\frac{\sum_{i=1}^{R} iF(k)}{R}$ where
C_h: Power consumed to hold an event,
C_0: Power consumed during the busy state,
C_1: Power consumed during the wake-up state,
C_2: Power consumed during the shutdown state,
C_3: Power consumed during the inactive state,
C_{μ_b}: Power consumed during the event transmission,
C_{θ_2}: Power consumed during the change of state,
C_{θ_1}: Power consumed during the start-up time, R: Number of cycles.

7 Numerical illustrations

In this section, the results obtained in the sections 3-6 are numerically computed through MATLAB software and graphically illustrated.

7.1 Numerical illustrations of transient solutions

The system size probabilities and the expected system probabilities in the transient state are computed and their behaviour in the system is analysed in this section. The parameter values are chosen as follows: $\lambda = 1$, $\mu_b = 1.25$, $\theta_1 = 0.3$, $\theta_2 = 0.5$ and $k = 5$.

Figure 2 illustrates the behaviour of the busy state $P_{0,n}(t)$. From this graph, it is observed that all the probability curves start at 0 and increases to a certain extent as 't' increases and attains a steady-state. Figure 3 portrays the behaviour of the wake-up state $P_{1,n}(t)$. The curves of $P_{1,n}(t)$ increase to a certain extent as t increases and then the curves decrease and further, it attains the steady-state. We also notice that the probability values of $P_{1,5}(t)$ are greater than $P_{1,n}(t), n = 6, 7, 8$. This is because during the inactive state if the system reaches $k = 5$, the system switches to a wake-up state. As a result, the curves of $P_{1,n}(t)$ start decreasing. Figure 4 depicts the behaviour of the shutdown state $P_{2,n}(t)$. All the probability curves of $P_{2,n}(t)$ start at zero, increase as t increases and attain the steady-state. Figure 5 demonstrates the behaviour of the inactive state $P_{3,n}(t)$. The probability curves of $P_{3,0}(t)$ start at 1 and decrease as t increases and attains the steady-state. The renaming curves of $P_{3,n}(t)$ increase to certain extent as t increases and attains the steady-state. From Figures 6 and 7, it is evident that as the arrival rate λ increases, the mean system size decreases.

7.2 Numerical illustrations of steady-state solutions

The system size probabilities and the expected system probabilities in the steady-state are computed using MATLAB, and their behaviour in the system is analysed in this section. The parameter values are chosen as follows: $\lambda = 1$, $\mu_b = 2$, $k = 10$, $\theta_1 = 0.3$, $\theta_2 = 0.4$, $C_h = 3$, $C_b = 35$, $C_1 = 5$, $C_2 = 4$, $C_3 = 1$,$C_{\mu_b} = 30$, $C_{\theta_1} = 6$ and $C_{\theta_2} = 5$.

Figures 8, 9 and 10 demonstrate the probability of n events in the wake-up state $\pi_{1,n}$, the shutdown state $\pi_{2,n}$ and the inactive state $\pi_{3,n}$ respectively. These graphs are plotted against n for varying arrival rate λ. In Figure 8, we notice that for fixed λ as n increases, the curves of $\pi_{1,n}$ decrease. This is because the server switches to a busy state from an wake-up state if the system size reaches the threshold of $k = 10$. In Figure 9, the curves of $\pi_{2,n}$ decrease as n increases for fixed λ. This is because the server immediately switches to an inactive state after the expiry of the shutdown period. In Figure 10 we notice that for fixed λ as n increases, the curves of $\pi_{3,n}$ also increase. As the server turns to this state from the shutdown state, the system size probability of this state increases as n increases.

Figure 11 delineates the mean number of events in the functional state $E(N_b)$. From this figure, we notice that for a fixed value of θ_1, the system size increases as the arrival rate λ increases. It is also observed that for fixed λ, the value of $E(N_b)$ decreases as θ_1 increases. This is because when θ_1 increases the server will quickly turn to the busy state from the wake-up state and hence the system size decreases. Figure 12 explains the mean number of events in the wake-up state $E(N_1)$. This graph is plotted against λ for fixed $\theta_2 = 0.4$ and varying values of θ_1. As the server switches to the busy state from the wake-up state, the curves of $E(N_1)$ decrease as the arrival rate λ increases.
Figure 13 and Figure 14 present the mean number of events in the shutdown state $E(N_2)$ and the inactive state $E(N_3)$ respectively. The graphs are plotted against λ for varying values of θ_2. As the server turns to an inactive state from the shutdown state, the curves of $E(N_2)$ decrease as the arrival rate λ increases.

7.3 Numerical illustrations of the mean power consumption

Table 1-4 presents the minimum power consumed by the system for a fixed arrival rate λ and varying threshold value k. In Table 1, we observe that the mean power consumption $F(k)$ starts decreasing initially as we increase the threshold value k and it starts increasing at $k = 20$. Further, we notice that the minimum power consumed at $k = 19$. Table 2, 3 and 4 present minimum power is consumed by the system for $\lambda = 1.2$, 1.3 and 1.4 respectively. In table 2, the minimum power consumed at $k = 50$, in Table 3, the minimum power consumed at $k = 74$ and in Table 4, the minimum power consumed at $k = 101$. We also note that as the arrival rate increases, the threshold value k for which the power consumption is minimum also increases. Hence, we conclude that there exists a pair $(k, F(k))$ for each arrival rate λ for which the power consumption is minimum.

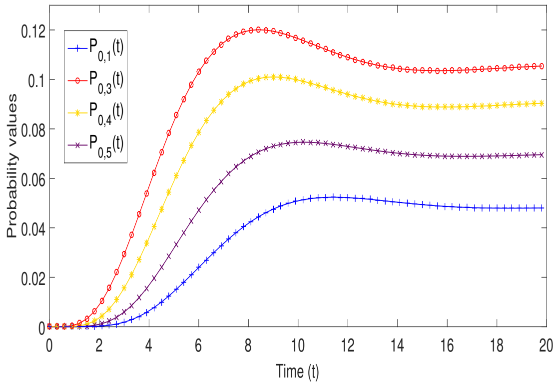

Figure 2: Probabilities of the busy-state $P_{0,n}(t)$

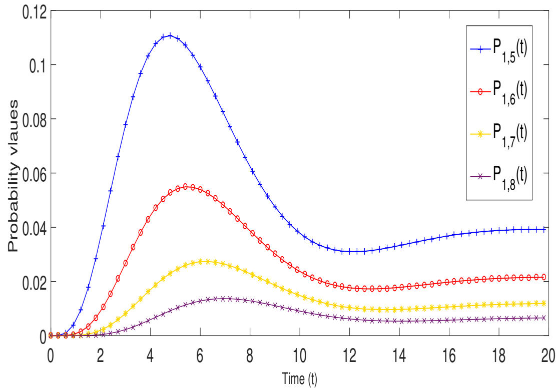

Figure 3: Probabilities of the wake-up state $P_{1,n}(t)$

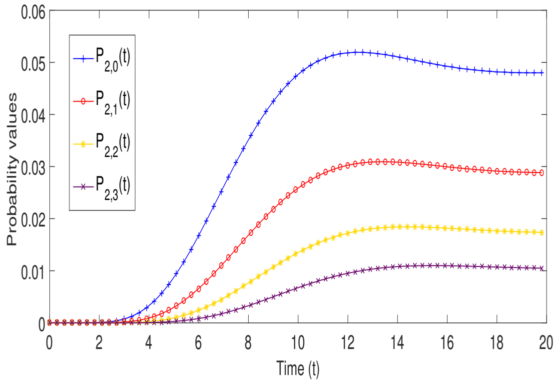

Figure 4: Probabilities of the shutdown state $P_{2,n}(t)$

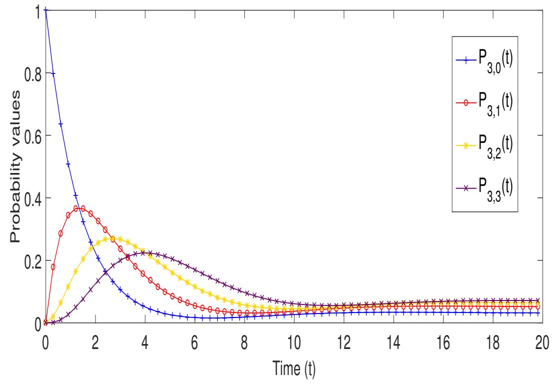

Figure 5: Probabilities of the inactive state $P_{3,n}(t)$

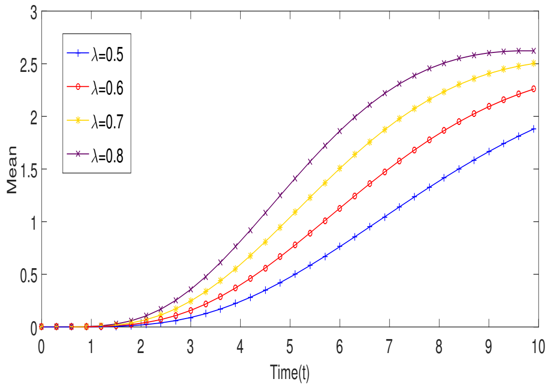

Figure 6: Mean system size for varying λ rates

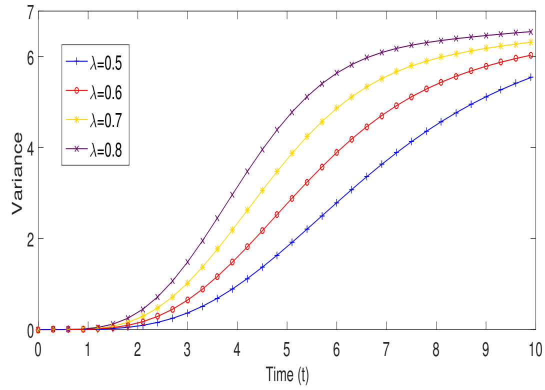

Figure 7: Variance of the system for varying λ rates

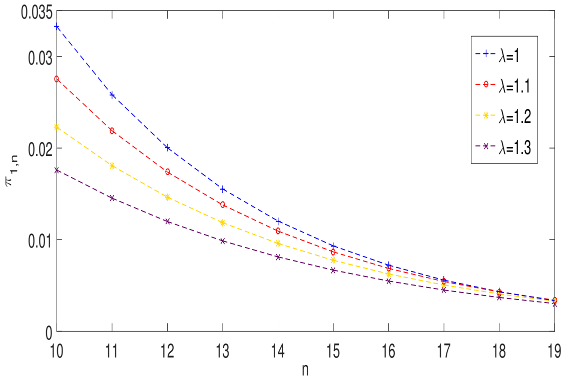

Figure 8: Probability of n events in the wake-up state

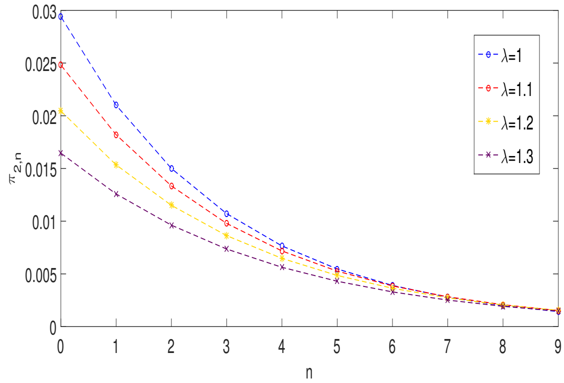

Figure 9: Probability of n events in the shutdown state

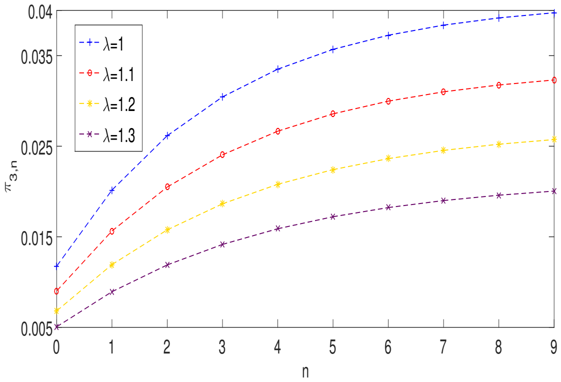

Figure 10: Probability of n events in the inactive state

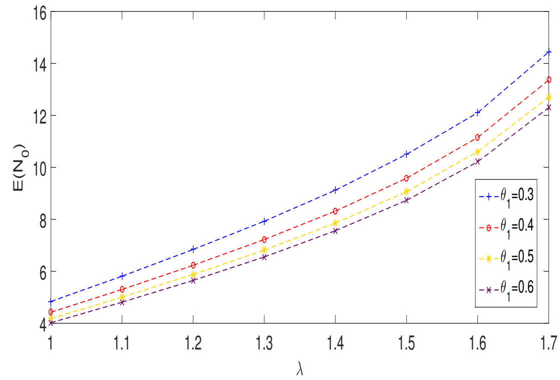

Figure 11: Mean number of events in the busy state $E(N_0)$ versus arrival rate λ

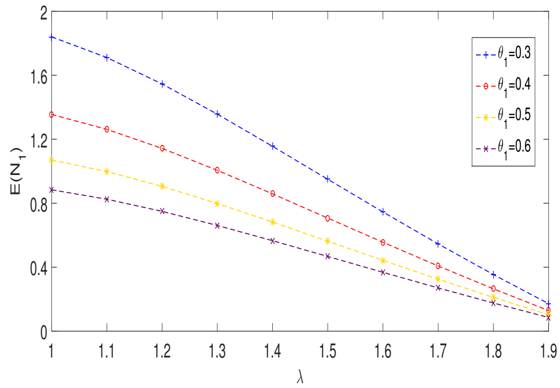

Figure 12: Mean number of events in the wake-up state $E(N_1)$ versus arrival rate λ

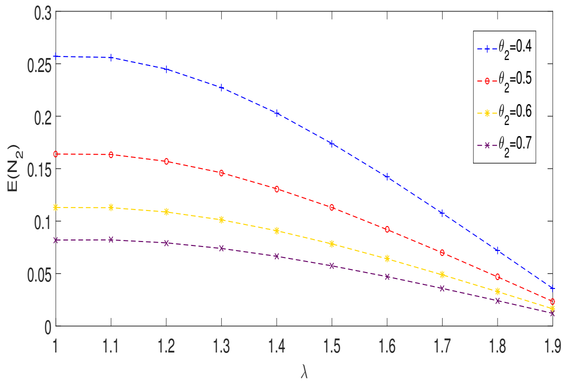

Figure 13: Mean number of events in the shutdown state $E(N_2)$ versus arrival rate λ

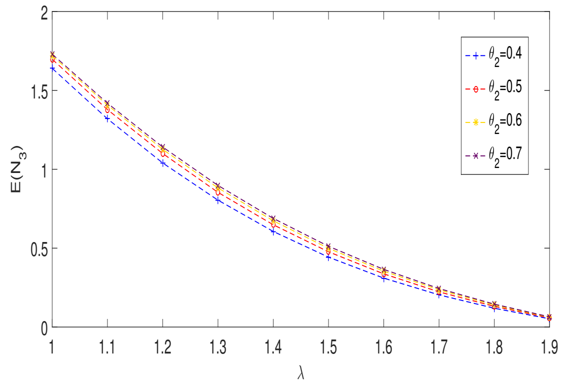

Figure 14: Mean number of events in the inactive state $E(N_3)$ versus arrival rate λ

k	7	9	11	13	15	17	19	20	21
$F(k)$	217.27	191.64	174.79	163.97	157.39	153.98	153.13	153.59	154.65

Table 1: Mean power consumption for λ=1.1

k	17	19	21	30	40	50	51	52
$F(k)$	21458.7	17162.1	13841.3	5711.08	2583.8	2561.94	2788.57	3163.81

Table 2: Mean power consumption for λ=1.2

k	60	70	72	74	75	76	80
$F(k)$	254067.42	126767.18	117310.20	113292.47	113487.16	115273.21	140920.50

Table 3: Mean power consumption for λ=1.3

k	98	100	101	102	103	105	110
$F(k)$	492892.34	66675.07	97.92	55170.46	258280.81	1263740.03	140920.50

Table 4: Mean power consumption for λ=1.4

8 Conclusion and future work

The performance of the DPM in the WSNs with threshold policy is discussed in the paper. The transient and steady-state system size probabilities of the system are obtained in a closed form. The performance indices such as mean, variance, probability that the system is in power-saving modes and mean power consumption are obtained. It is observed that there exists a threshold value for each arrival rate to minimise the power consumption. This work may be extended to an M/M/C queueing model with working vacation and close-down times.

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