

An Exceptional Dark Matter from Cayley–Dickson Algebras

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An Exceptional Dark Matter from Cayley–Dickson Algebras

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Abstract

In this article I propose a new criterion to individuate the origin and the properties of the dark matter particle sector. The emerging candidates come from a straightforward algebraic conjecture: the symmetries of physical microscopic forces originate from the automorphism groups of main Cayley–Dickson algebras, from complex numbers to octonions and sedenions. This correspondence leads to a natural enlargement of the Standard Model color sector, from a $SU(3)$ gauge group to an exceptional Higgs-broken $G(2)$ group, following the octonionic automorphism relation guideline. In this picture, dark matter is a relic heavy $G(2)$ -gluons ensemble, separated from the particle dynamics of the Standard Model due to the high mass scale of its constituents.

1 Introduction

Dark matter (DM) is a very long-standing problem of modern physics with no evident nor univocal solution: all the efforts made, from particle theory [1; 2] to modified gravities [3; 4; 5], have not been successful in clarifying its nature. The most convincing particle candidates, the weakly interacting massive particles (WIMPs), have not been discovered yet: direct, indirect and collider searches show no evidence of new particles approximately up to the 1 TeV scale [6; 2; 7; 8; 9]. This is a strong hint that the *Naturalness* criterion [10] for the Higgs sector and the so-called *WIMP Miracle* [11] in particular, which postulate the existence of a thermal particle relic of the Big Bang at the electroweak scale $O(100GeV)$ which interacts via weak force, could not be a prerogative of *Nature* or, at any rate, not sufficient to individuate the origin of dark matter. Even the possibility that the weak interaction between DM and Standard Model (SM) particles is disfavored must be considered: so a DM candidate could hide at a higher energy scale and it could be not capable of interacting with the visible world, at least at the experimentally explored energies. Therefore, to proceed in the dark sector investigation, one has to fill up the lack of a theoretical guideline and integrate some new simplicity criteria to select reliable candidates and explain the complex astrophysical and cosmological observations [12; 13; 14]. Today physics seems to need some extra inputs to go beyond current paradigms and reach a deepest understanding of the dark matter conundrum: in this complex situation mathematics could provide fresh insights and conjectures to overcome physical prejudices.

Here we propose an approach based on a division algebras conjecture capable of selecting a unique branch of heavy dark matter particles from a simple and minimally high symmetry. The criterion

is to identify fundamental interactions with the automorphism groups¹ of Cayley-Dickson algebras. Then, from the automorphism of octonions (and sedenions) algebra, the promising exceptional symmetry group $G(2)$ can be pinpointed to solve the DM problem. We will demonstrate that, once broken through a Higgs-like mechanism, $G(2)$ represents the optimal gauge group to describe strong interaction and dark matter at the same time, shedding light on DM origin and present behavior. To the best of our knowledge, no existing work in literature is devoted to the possibility that dark matter is formed by the heavy gluons from a broken- $G(2)$ gauge group, which naturally incorporates the standard $SU(3)$ color Quantum Chromodynamics (QCD): even if $G(2)$ lattice models have been largely applied to simplify standard QCD computations [15; 16], the implications for dark matter theory have not been explored. Hence, the present dissertation is not intended as a mere review of the current status of Cayley–Dickson algebras applied to particle physics, but as a phenomenological proposal to incorporate DM in the Standard Model framework.

2 Fundamental forces from division algebras automorphisms: a conjecture for dark matter

In the last decades many attempts to connect the Standard Model of elementary particles with division algebras have been made, showing it is worthwhile establishing relations between algebraic structures and symmetry groups [17; 18; 19; 20; 21; 22; 23; 24; 25].

It is well-known that following the Cayley–Dickson construction process [19; 17], one can build up a sequence of larger and larger algebras, adding new imaginary units. In detail, from Hurwitz and Zorn theorem [18], one can identify the so-called division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, *i.e.* the only four alternative algebraic fields with no non-trivial zero divisors [26; 19], which are real numbers, complex numbers, quaternions and octonions, respectively. During the construction process, the algebras lose some peculiar properties, one at a time. For example, complex numbers are not ordered but commutative, quaternions are not commutative but associative, whereas octonions lose all the familiar commutative and associative properties, but they are still an alternative algebra [26]. The process does not terminate with octonions: applying the Cayley–Dickson construction, greater 2^n -dimensional algebras can be constructed, for any positive integer n . For $n > 3$, however, as anticipated, they all include non-trivial zero divisors², *i.e.* they have problems in a general definition of norm. This was considered an obstacle for the use of these extended algebras, such as $n = 4$ sedenions, in science. But, as shown in [28; 29], sedenions should not be ruled out as playing a role in particle physics on the basis that they do not constitute a division algebra. We will return to this topic later.

The link between unitary groups and division algebras \mathbb{A}_n has been diffusely studied [30; 31; 32]. Unitary groups are the fundamental bricks to build the particle Standard Model, because each fundamental force can be described by a unitary or special unitary group [33; 34; 35; 36; 37], being $G = SU(3) \times SU(2) \times U(1)$ the SM group of strong $SU(3)$, weak $SU(2)$ and electromagnetic $U(1)$ interactions [33]. Besides its symmetry, the SM includes three fermions families: between these three generations, particles differ by their flavour quantum number and

¹An automorphism is a bijective way of mapping a mathematical object to itself preserving its structure: the set of all automorphisms form the automorphism group, *i.e.* the symmetry group of the object.

²In abstract algebra, a non-zero element a of a ring R is called a zero divisor if there exists a non-zero x such that $ax = 0$. For general properties of zero divisors see [27].

mass, but their interactions are identical.

In the following, we want to briefly highlight the relations between the automorphisms of Cayley-Dickson algebras and these important physical gauge groups, including some considerations about the tripartite structure of the Standard Model.

Starting from the most simple complex algebra and SM symmetry group, it is easy to find a direct connection between the electromagnetism (or Quantum Electrodynamics) $U(1)$ formalism and the complex number field \mathbb{C} : in fact the group $U(1)$, the smallest compact real Lie group, corresponds to the circle group S^1 , consisting of all complex numbers with absolute value 1 under multiplication, which is isomorphic to the $SO(2)$ group of rotation [38]. All the unitary groups contain copies of this fundamental group. For $n \geq 1$, one can also consider for the comparison the n -torus T_n , that is defined to be $\mathbb{R}^n/\mathbb{Z}^n \cong U(n) \cong SO(2)^n \cong (S^1)^n$, where $/$ denotes the quotient group between reals and integers, which shows off the deep connection between $U(1)$ gauge symmetry and other representations strictly connected to complex numbers [38; 39]. It is also true that the $n \times n$ complex matrices which leave the scalar product \langle, \rangle invariant form the group $U(n) = \text{Aut}(\mathbb{C}^n, \langle, \rangle)$, *i.e.* the group of automorphisms of \mathbb{C}^n as a Hilbert space [40]. These links are not surprising because, from a mathematical point of view, the existence of infinite distinct *wild* automorphisms of the complex numbers, beyond identity and complex conjugation, is well-known [20; 41]. We find another noteworthy examination in [39], where the unitary group $U(1)$ is showed as defining binary complex relations $\mathbb{C} \times \mathbb{C}$, *i.e.* the $U(1)$ numbers effectively operate as automorphisms of \mathbb{C} via multiplication of a phase factor. As we know, the complex numbers can be expressed in polar coordinates and this implies that the general linear multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus 0 = e^{\mathbb{C}} \cong GL(1, \mathbb{C})$ is uniquely decomposable ($e^z = e^x \cdot e^{iy}$) into the totally ordered group with real exponential $|\mathbb{C}^*| = e^{\mathbb{R}}$ and into the phase group with imaginary exponentials $\mathbb{C}^*/|\mathbb{C}^*| = e^{i\mathbb{R}} \cong U(1)$, which is approximately $U(1)$ (see [39; 42] for details). This is another way to underline the intimate connection between the unitary group and the complex numbers.³

Even $SU(2)$ weak isospin can be clearly represented with the algebraic quaternionic basis, *i.e.* *Pauli matrices* [22]: $SU(2)$ naturally embeds into \mathbb{H} as the group of quaternion elements of norm 1, with a perfect analogy with respect to $U(1)$ and complex numbers. More precisely, the group $SU(2)$ is isomorphic to the group of quaternions of norm 1, and it is thus diffeomorphic⁴ to the 3-sphere S^3 . Indeed, since unit quaternions can be used to represent rotations in 3-dimensional space (up to a sign), there is a surjective homomorphism⁵ from $SU(2)$ to the rotation group $SO(3)$ [22]: one can show that the local $SU(2)$ spinors⁶ are exactly the same two-component spinors derived from the local quaternion matrix representation, *i.e.* the four Pauli matrices. In other words, the correspondence between the automorphism of quaternion algebra and the Standard Model symmetry group of weak force can be clearly shown: for quaternions $\text{Aut}(\mathbb{H}) = SO(3)$, where $SO(3)$ is homomorphic to $SU(2)$ in turn, and the universal cover of $SO(3)$ is the spin group $\text{Spin}(3)$, which is isomorphic to $SU(2)$. So $SU(2)$ and $SO(3)$ algebraic structures are

³Furthermore, from a physical point of view, one can also think at the Riemann-Silberstein field reformulation of the electromagnetism [43] in terms of a complex vector that combines the electric field E , as the real part, and the magnetic field B , as the imaginary part, in order to put in evidence this essential relation.

⁴A diffeomorphism is an isomorphism of smooth manifolds, *i.e.* a map between manifolds which is differentiable and has a differentiable inverse.

⁵An homomorphism is a structure-preserving map between two algebraic structures of the same type.

⁶Spinors are defined as vectors of a representation of the group of automorphisms of a Clifford algebra defined on space-time.

equivalent. An interesting demonstration of the correspondence between the two groups using *Möbius* transformation is described in [44]. The quaternionic representation of (electro-) weak isospin has been used by many authors [45; 46].

Hence, both in the $U(1)$ electromagnetic case and in the $SU(2)$ weak interaction, the solutions can be expressed in terms of division algebras, respectively the complex and the quaternion algebras: the division property is important to define the mathematical structure and in the determination of solutions. This could be a coincidence, but the possibility that fundamental gauge interactions can be described by the apparatus of division algebras should be explored.

It seems logical to revise the next division algebra, the octonion algebra \mathbb{O} (which is not a Clifford algebra, unlike \mathbb{R} , \mathbb{C} and \mathbb{H} , because non associative) [47; 26] for a possible description of the $SU(3)$ gauge field [48; 45], but the result is less clear than in quaternion case for the $SU(2)$ gauge field. The interesting fact to be considered is that the group of automorphisms of the octonion algebra, the largest of the normed division algebras, corresponds to the exceptional Lie algebra $G(2)$, the smallest among the known exceptional Lie algebras: $Aut(\mathbb{O}) = G(2)$ [49]. So it is noteworthy to point out that the Standard Model gauge group $SU(3)$ is not isomorphic to the group of automorphisms of the octonions, which is $G(2)$. Nonetheless, it is possible to fix one of the octonion basis elements to obtain seven possible subalgebras, each of which has a subgroup of automorphisms isomorphic to $SU(3)$. For example, $SU(3)$ itself may be defined as the subgroup of $G(2)$ which leaves the octonionic unit e_7 invariant [22]. Of course, alternative $SU(3)$ subgroups of $G(2)$ may be found, corresponding to other imaginary units. In addition, recent works in the framework of particle physics show the possibility to rewrite *Gell-Mann matrices* of $SU(3)$ strong force (the group generators) with octonions [45]. Also split-octonions representations have been proposed as alternative formalism for $SU(3)$ color gauge symmetry [48].

But here a crucial difference appears: it must be noted that for \mathbb{C} and \mathbb{H} the direct automorphism groups contain an equal, or comparable, amount of “mathematical information” than $U(1)$ and $SU(2)$ themselves (through the approximate algebraic correspondences, via homomorphism in $SU(2)$ case), whereas the exceptional $G(2)$ group is certainly bigger than SM $SU(3)$, as it includes $SU(3)$ and is equipped with six additional generators [50]. In other words, if we want to study the application of the octonion automorphism in physics, it is mandatory to invoke a gauge group which is not the strong color symmetry $SU(3)$.

Summarizing, for non real division algebras it turns out that:

$$Aut(\mathbb{C}) \cong U(1), Aut(\mathbb{H}) \cong SU(2), Aut(\mathbb{O}) \equiv G(2). \quad (1)$$

These relations show an ordered correspondence between (approximate) automorphisms of algebras and gauge groups useful for Standard Model description, where $G(2)$ contains $SU(3)$ color force. We will see in the next section that, besides $SU(2)$ Pauli matrices, also $G(2)$ generators can be written in terms of $SU(3)$ Gell-Mann matrices as 14 unitary matrices.

Fundamental correlations between division algebras and symmetry groups, as anticipated, have been already stressed in the last decades. Using division algebras, Dixon proposed an elegant representation of particle physics in [19]. Furey has recently suggested the appealing possibility to reformulate the SM group $G = SU(3) \times SU(2) \times U(1)$ in terms of a $\mathbb{A} = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ tensor product algebra, restarting from Dixon’s work, using the concept of *Ideals*, i.e. using subspaces of proper Clifford Algebras as “particles” (see [23; 51; 52; 53; 54] for details). Also string theory and supersymmetrical theories invoked division algebra to study particle interactions [55; 56; 57; 31; 58; 59]. Moreover, $G(2)$ as automorphism of \mathbb{O} has important applications in terms of the so-called G_2 structure or G_2 manifolds [60], in the context of M-theory [61]. Indeed,

one solid reason for studying division algebras in relation to particle symmetries is that, unlike Lie algebras and Clifford algebras [20], there is a finite number of division algebras and corresponding automorphisms (see again the extensive works of Dixon [19]). If we start with a division algebra, the physical symmetries are dictated by the mathematical structure and the choice of a proper symmetry group is constrained.

To proceed with the reasoning, we are going to see why 16-dimensional sedenions can be easily added to this picture and how dark matter description can benefit from this algebraic facts, summarizing the main features of sedenions algebra.

The sedenion algebra is the fifth Cayley-Dickson algebra $\mathbb{A}_4 = \mathbb{S}$, where $\mathbb{A}_{0,1,2,3}$ correspond to reals, complex numbers, quaternions and octonions. This is not a division algebra, it is non-commutative, non-associative, and non-alternative⁷, hence it cannot be a composition algebra⁸ [63; 28]. However sedenion algebra is power-associative and flexible⁹ (and satisfies the weak inversive properties for non-zero elements¹⁰ - see [62; 29] for details). In principle, the Cayley-Dickson construction can be indefinitely carried on and, at each step, a new power-associative and flexible algebra is produced, doubling in size. So, in first approximation, no new fundamental properties and information are added nor lost enlarging the algebra beyond sedenions. One can choose a canonical basis for \mathbb{S} to be $E_{16} = \{e_i \in \mathbb{S} | i = 0, 1, \dots, 15\}$ where e_0 is the real unit and e_1, \dots, e_{15} are anticommuting imaginary units. In this basis, a general element $A \in \mathbb{S}$ is written as

$$A = \sum_{i=0}^{15} a_i e_i = a_0 + \sum_{i=1}^{15} a_i e_i, \quad a_i \in \mathbb{R}. \quad (2)$$

The basis elements satisfy the multiplication rules

$$\begin{aligned} e_0 &= 1, & e_0 e_i &= e_i e_0 = e_i, \\ e_i^2 &= -e_0, & i &\neq 0, \\ e_i e_j &= \gamma_{ij}^k e_k & i &\neq 0, \quad i \neq j, \end{aligned} \quad (3)$$

with γ_{ij}^k the real structure constants, which are completely antisymmetric. For two sedenions A, B , one has

$$AB = \left(\sum_{i=0}^{15} a_i e_i \right) \left(\sum_{i=0}^{15} b_i e_i \right) = \sum_{i,j=0}^{15} a_i b_j (e_i e_j) = \sum_{i,j,k=0}^{15} f_{ij} \gamma_{ij}^k e_k, \quad (4)$$

where $f_{ij} \equiv a_i b_j$.

Because the sedenion algebra is not a division algebra, it contains zero divisors: for \mathbb{S} these are elements of the form

$$(e_a + e_b) \circ (e_c + e_b) = 0, \quad e_a, e_b, e_c, e_d \in \mathbb{S}. \quad (5)$$

⁷An algebra A is alternative if the subalgebra generated by any two elements is associative, *i.e.* iff for all $a, b \in A$ we have $(aa)b = a(ab)$, $(ba)a = b(aa)$ [62].

⁸It is an algebra A over a field K with a non-degenerate quadratic form N , called *norm*, that satisfies $N(ab) = N(a)N(b)$ for all a, b in A [18].

⁹An algebra is power-associative if the subalgebra generated by any one element is associative: it is a sort of lowest level of associativity [26]. The flexible property, for any $a, b \in A$, can be defined as $a(ba) = (ab)a$.

¹⁰Each Cayley-Dickson algebra satisfies the weak inversive property: $a^{-1}(ab) = a(a^{-1}b)$, $(ba^{-1})a = (ba)a^{-1}$, $a^{-1}(ab) = (ba)a^{-1}$.

There are 84 such zero divisors in sedenion space and the subspace of zero divisors with unit norm is homeomorphic to $G(2)$ [64; 27]. To understand the role and emergency of zero divisors, one has to consider not only single algebras but also compositions of them. For example, whereas \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} are by themselves division algebras, their tensor products, such as $\mathbb{C} \otimes \mathbb{H}$, $\mathbb{C} \otimes \mathbb{O}$ and $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, largely applied in SM algebraic extensions, are not, and in fact the zero divisors of these algebras play a crucial role in the construction of Furey’s Ideals [52; 23; 51; 53]. Moreover, the two by two compositions of division algebras, which are not division algebras and contain zero divisors, are the subjects of the well-known *Freudenthal–Tits magic square* [65; 66; 67]:

\otimes	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}
\mathbb{R}	$SO(3)$	$SU(3)$	$Sp(3)$	F_4
\mathbb{C}	$SU(3)$	$SU(3)^2$	$SU(6)$	E_6
\mathbb{H}	$Sp(3)$	$SU(6)$	$SO(12)$	E_7
\mathbb{O}	F_4	E_6	E_7	E_8

a symmetric square¹¹ which exhibits the “unexpected” relation between octonions products and exceptional groups (F_4 , E_6 , E_7 , E_8) [49], except for the exceptional $G(2)$ which represents octonions automorphism itself. The exceptional groups on the last line/row are not exactly automorphisms of the octonions products, because of mathematical problems in the definition of projective planes, due to the appearance of zero divisors: they are called bioctonions ($\mathbb{C} \otimes \mathbb{O}$), quateroctonions ($\mathbb{H} \otimes \mathbb{O}$) and octooctonions ($\mathbb{O} \otimes \mathbb{O}$) and find correspondence into Jordan’s algebras [26]. Exceptional E_i are also largely used in supergravity and string theory [57; 35]. Therefore it seems reasonable to continue the Cayley-Dickson algebraic construction into the non-division algebras, such as \mathbb{S} .

Interestingly, in [68; 28] the authors put in evidence an important relation between sedenions and the exceptional group $G(2)$, demonstrated by Brown in [69]:

$$\text{Aut}(\mathbb{S}) = \text{Aut}(\mathbb{O}) \times S_3. \quad (6)$$

where we know that $\text{Aut}(\mathbb{O}) = G(2)$ and S_3 is the permutation group of degree three. So the inner symmetries of this non-division algebra can be again extracted from the automorphism group of octonions and, in particular, from a proper product of the exceptional $G(2)$ group with a symmetric group. The only difference between octonions and sedenions automorphism groups is a factor of the permutation group S_3 (this permutation group can be constructed from the *triality*¹² automorphism of the spin group $Spin(8)$, see [20; 26] for details). Eq.(6) suggests that the fundamental symmetries of \mathbb{S} are the same as those of \mathbb{O} , even if the factor S_3 introduces a three copies scenario, that is exactly what we need in order to describe the observed three generations of fermions in the Standard Model of particles.

The previous formula can be generalized, for an arbitrary algebra constructed via Cayley-Dickson process (for $n > 3$), into [28; 70]

$$\text{Aut}(\mathbb{A}_n) \cong \text{Aut}(\mathbb{O}) \times (n - 3)S_3. \quad (7)$$

¹¹ $SO(N)$ and $SU(N)$ are the usual special orthogonal and unitary groups of order N , $Sp(3)$ is the symplectic group of order three.

¹²Triality is a trilinear map among three vector spaces, most commonly described as a special symmetry between vectors and spinors in 8-dimensional euclidean space.

This tells us that the underlying symmetry is always $G(2)$, the automorphism group of the octonions. The higher Cayley-Dickson algebras only add additional *trialities*, *i.e.* copies of $G(2)$, and reasonably no new physics beyond sedenions. Furthermore, sedenion algebra might represent the archetype of all non-associative and non-division flexible algebras, if $n > 3$ Cayley-Dickson algebras do not differ from sedenions for what concerns the multilinear identities (or algebraic properties) content, as suggested in [71].

In this picture, sedenion algebra could constitute the searched simplicity criterion to select the full symmetry of a three generations Standard Model strong force and include a new particle physics content, which might represent the unknown dark matter sector. This could be also read as a sort of a *naive* indirect proof that fundamental forces should be a small number (only three), because all algebras beyond octonions point towards the very same exceptional group, adding only copies (particle generations). Finally, as it will be discussed in the next section, to recover the usual $SU(3)$ strong force the octonions-sedenions automorphism group must be broken at our energy scales and new physics extracted: this enlarged algebraic content is going to be associated to dark matter.

So, without the presumption of a rigorous and definitive mathematical definition of the problem, we can reformulate and summarize *the algebraic phenomenological conjecture for the dark matter sector* in a general way as follows.

The fundamental symmetry of the Standard Model of particle physics with three fermion families might be the realization of some tensor products between the associative division algebras and the most comprehensive non-division algebra obtained through the Cayley–Dickson construction, *i.e.* the sedenion algebra. The sedenionic description, like the octonionic one, corresponds, via automorphism, to the simplest exceptional group $G(2)$, but tripled. It could provide an explanation to the $N = 3$ fermion families of the Standard Model, which lie in the sedenions S_3 automorphism factor, as suggested by [28]. This is consistent with the proposal of a S_3 -invariant extension of the Standard Model, as discussed in [72; 73; 74; 75].

The gauge groups $U(1)$, $SU(2)$, $SU(3)$, describing the three fundamental forces, find mathematical correspondence into the division algebras \mathbb{C} , \mathbb{H} , \mathbb{O} respectively: Table 1 summarizes this correspondence. However, whereas $U(1)$ and $SU(2)$ are approximate isomorphisms of complex and quaternion algebras automorphisms (see Eq.(1)), the octonion and sedenion automorphism relations point towards a different group, which is manifestly larger than the usual 8-dimensional $SU(3)$ color group of the Standard Model, *i.e.* the 14-dimensional $G(2)$ group; $SU(3)$ and $G(2)$ differ for 6 dimensions-generators. Therefore

$$\text{Aut}(\mathbb{C}) \times \text{Aut}(\mathbb{H}) \times \text{Aut}(\mathbb{S}) = \text{Aut}(\mathbb{C}) \times \text{Aut}(\mathbb{H}) \times \text{Aut}(\mathbb{O}) \times S_3 = U(1) \times SU(2) \times G(2) \times S_3 \quad (8)$$

could give the overall unbroken Standard Model symmetry. This is the first main statement of the present dissertation. Here the automorphism selection is invoked to predict something beyond current SM, and $SU(3)$ in particular, and it works as a guideline to replace $SU(3)$ color itself with the smallest exceptional group: *fundamental forces must be isomorphic to the automorphisms groups of the division algebras built up through the Cayley–Dickson construction*. Tensor products between the corresponding algebras (see Freudenthal–Tits magic square) could be effective symmetries but not fundamental forces.

Dark matter constituents come from the aforementioned difference between $G(2)$ and $SU(3)$ groups and lie in the spectrum gap between them. Following the Cayley–Dickson algebraic automorphism criterion, no more physics is needed nor predicted, except for the six additional degrees

Charge (n_g)	Group	Force	Algebra	Dim	Commutative	Associative	Alternative	Normed	Flexible
$Q(1)$	$U(1)$	EM	\mathbb{C}	2	Yes	Yes	Yes	Yes	Yes
$T(3)$	$SU(2)$	Weak	\mathbb{H}	4	No	Yes	Yes	Yes	Yes
$C(8)$	$SU(3)$	Strong	\mathbb{O} or \mathbb{S}	8/16	No	No	Yes	Yes	Yes
$DC(6)$	broken- $G(2)$	Dark Strong	\mathbb{O} or \mathbb{S}	8/16	No	No	No	No	Yes

Table 1 Schematic correspondence between forces, groups and algebras. In the first column the charge of the physical interaction is displayed along with the number n_g of associated generators (bosons). Q, T, C are usual SM electric charge, weak isospin and color charge, respectively; here DC stands for “dark-colored”, to indicate the six broken generators which originate the massive exceptional $G(2)$ bosons which have quark and anti-quark color quantum numbers (see next section). The second and third columns associate gauge groups and forces, highlighting the link between $G(2)$ and the 6 new dark-colored particles, separated from visible strong phenomena. $G(2)$ algebraic automorphism representation is valid for both octonions and sedenions (the only difference is the S_3 factor). In principle, strong force and dark sector represent the same interaction but they are disconnected, coming from the broken exceptional symmetry. For this reason their algebras are displayed as \mathbb{O} or \mathbb{S} . Algebraic dimensions are showed in the fifth column. As shown in the subsequent columns, each division algebra loses inner properties hierarchically, from commutativity to alternativity, as the dimensions increase. All algebras are flexible (and power-associative). See [63; 26] for proper descriptions of the algebraic properties and insights.

of freedom, i.e. boson fields, which represent the discrepancy between $G(2)$ and $SU(3)$ generators. Hence, the automorphism selection rule extends the color sector and provides a rich exceptional phenomenology.

The novelty is the definition of a new algebraic criterion to predict dark matter physics, which substitutes Higgs Naturalness and the Wimp Miracle. In this scenario, the strong force acquires a more complex structure, which includes the usual color sector and an enlarged strong dark dynamics, due to six residual generators of exceptional $G(2)$, which might gain mass via a symmetry breaking: to recover standard $SU(3)$ color strong force description, the new $G(2)$ color sector should be broken by a Higgs-like mechanism and separated into two parts, one visible and the other excluded from the dynamics due to its high mass.

In the next section the emergency of these massive exceptional dark bosons is discussed, starting from a deep analysis of the exceptional $G(2)$ group.

3 A $G(2)$ gauge theory for dark matter

$G(2)$ can be described as the automorphism group of the octonion algebra or, equivalently, as the subgroup of the special orthogonal group $SO(7)$ that preserves any chosen particular vector in its 8-dimensional real spinor representation [47; 76; 20]. The group $G(2)$ is the simplest among the exceptional Lie groups [30]; it is well known that the compact simple Lie groups are completely described by the following classes: $A_N (= SU(N + 1))$, $B_N (= SO(N + 1))$, $C_N (= Sp(N))$, $D_N (= SO(2N))$ and exceptional groups G_2, F_4, E_6, E_7, E_8 , with $N = 1, 2, 3, \dots$ (for D_N , $N > 2$) [77]. Among them, only $SU(2), SU(3), SO(4)$ and symplectic $Sp(1)$ have 3-dimensional irreducible representations and only one, $SU(3)$, has a complex triplet representation (this was one of the historical criteria to associate $SU(3)$ to the three color strong force, with quark states different from antiquarks states [78]). There is only one non-Abelian simple compact Lie algebra of rank 1, i.e. the one of $SO(3) \simeq SU(2) = Sp(1)$, which describes the weak force, whereas there are four of rank 2, which generate the groups $G(2), SO(5) \simeq Sp(2), SU(3)$ and $SO(4) \simeq SU(2) \otimes SU(2)$, with 14, 10, 8 and 6 generators, respectively [50].

If we want to enlarge the QCD sector to include dark matter, it is straightforward we have to choose $G(2)$ or $SO(5)$. The group $G(2)$, beside its clear relation with division algebras described in the previous section, is of particular interest because it has a trivial center, the identity, and it is its own universal covering group, meanwhile $SO(5)$ has \mathbb{Z}_2 as a center (and $SU(3)$ has \mathbb{Z}_3); $SO(N)$ in general are not simply connected and their universal covering groups for $n > 2$ are spin $Spin(N)$ [79]. It is also well-know in literature that $G(2)$, thanks to its aforementioned peculiarities, can be used to mimic QCD in lattice simulations, avoiding the so-called sign problem [80] which afflicts $SU(3)$. Proposing to enlarge QCD above the TeV scale and have the SM as a low energy theory is surely not an unprecedented nor odd idea: for example, modern composite Higgs theories [81; 82; 83] try to introduce (cosets) gauge groups beyond $SU(3)$, such as $SU(6)/SO(6)$, $SO(7)/SO(6)$ or $SO(5)/SO(4)$, dealing with multiple Higgs, strong composite states and dark matter candidates.

Focusing on the present proposal, $G(2)$ can be constructed as a subgroup of $SO(7)$, which has rank 3 and 21 generators [50; 79]. The 7×7 real matrices U of the group $SO(7)$ have determinant 1, orthogonal relation $UU^\dagger = 1$ and fulfill the constraint $U_{ab}U_{ac} = \delta_{bc}$. The $G(2)$ subgroup is described by the matrices that also satisfy the cubic constraint

$$T_{abc} = T_{def}U_{da}U_{eb}U_{fc} \quad (9)$$

where T is an anti-symmetric tensor defining the octonions multiplication rules, whose non-zero elements are

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1. \quad (10)$$

To explicitly construct the matrices in the fundamental representation, one can choose the first eight generators of $G(2)$ as [50; 79]:

$$\Lambda_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_a & 0 & 0 \\ 0 & -\lambda_a^* & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

where λ_a (with $a \in \{1, 2, \dots, 8\}$) are the Gell-Mann generators of $SU(3)$, which indeed is a subgroup of $G(2)$, with standard normalization $\text{Tr}\lambda_a\lambda_b = \text{Tr}\Lambda_a\Lambda_b = 2\delta_{ab}$. Λ_3 and Λ_8 are diagonal and represent the Cartan generators w.r.t. $SU(3)$. The $G(2)$ coset space by its subgroup $SU(3)$ is a 6-sphere $G(2)/SU(3) \cong S^6 \cong SO(7)/SO(6)$ [84], in analogy with the composite Higgs proposal [82].

The remaining six generators can be found studying the root and weight diagrams of the group [85; 86; 87], and can be written as:

$$\Lambda_9 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_2 & \sqrt{2}e_3 \\ i\lambda_2 & 0 & \sqrt{2}e_3 \\ \sqrt{2}e_3^T & \sqrt{2}e_3^T & 0 \end{pmatrix}, \Lambda_{10} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -\lambda_2 & i\sqrt{2}e_3 \\ -\lambda_2 & 0 & -i\sqrt{2}e_3 \\ -i\sqrt{2}e_3^T & i\sqrt{2}e_3^T & 0 \end{pmatrix}, \quad (12)$$

$$\Lambda_{11} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & i\lambda_5 & \sqrt{2}e_2 \\ -i\lambda_5 & 0 & \sqrt{2}e_2 \\ \sqrt{2}e_2^T & \sqrt{2}e_2^T & 0 \end{pmatrix}, \Lambda_{12} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & \lambda_5 & i\sqrt{2}e_2 \\ \lambda_5 & 0 & -i\sqrt{2}e_2 \\ -i\sqrt{2}e_2^T & i\sqrt{2}e_2^T & 0 \end{pmatrix}, \quad (13)$$

$$\Lambda_{13} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_7 & \sqrt{2}e_1 \\ i\lambda_7 & 0 & \sqrt{2}e_1 \\ \sqrt{2}e_1^T & \sqrt{2}e_1^T & 0 \end{pmatrix}, \Lambda_{14} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -\lambda_7 & i\sqrt{2}e_1 \\ -\lambda_7 & 0 & -i\sqrt{2}e_1 \\ -i\sqrt{2}e_1^T & i\sqrt{2}e_1^T & 0 \end{pmatrix}, \quad (14)$$

where e_i are the unit vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (15)$$

In the chosen basis of the generators it is manifest that, under $SU(3)$ subgroup transformations, the 7-dimensional representation decomposes into [50; 79]

$$\{7\} = \{3\} \oplus \{\bar{3}\} \oplus \{1\}. \quad (16)$$

Since all $G(2)$ representations are real, the $\{7\}$ representation is identical to its complex conjugate, so that $G(2)$ “quarks” and “anti-quarks” are conceptually indistinguishable. This representation describes a $SU(3)$ quark $\{3\}$, a $SU(3)$ anti-quark $\{\bar{3}\}$ and a $SU(3)$ singlet $\{1\}$. The generators transform under the 14-dimensional adjoint representation of $G(2)$ [50; 79], which decomposes into [50; 79; 88]

$$\{14\} = \{8\} \oplus \{3\} \oplus \{\bar{3}\}. \quad (17)$$

So the $G(2)$ “gluons” ensemble is made of $SU(3)$ gluons $\{8\}$ plus six additional “gluons” which have $SU(3)$ quark and anti-quark color quantum numbers. As mentioned before, the center of $G(2)$ is trivial, containing only the identity, and the universal covering group of $G(2)$ is $G(2)$ itself. This has important consequences for confinement [79; 88; 89; 90]: we will see that the color string between $G(2)$ “quarks” is capable of breaking via the creation of dynamical gluons. As discussed in [50], the product of two fundamental representations

$$\{7\} \otimes \{7\} = \{1\} \oplus \{7\} \oplus \{14\} \oplus \{27\}, \quad (18)$$

shows a singlet $\{1\}$: as a noteworthy implication, two $G(2)$ “quarks” can form a color-singlet, or a “diquark”. Moreover, just as for $SU(3)$ color, three $G(2)$ “quarks” can form a color-singlet “baryon”:

$$\{7\} \otimes \{7\} \otimes \{7\} = \{1\} \oplus 4 \{7\} \oplus 2 \{14\} \oplus 3 \{27\} \oplus 2 \{64\} \oplus \{77\}. \quad (19)$$

Due to the fact that “quarks” and “antiquarks” are indistinguishable, it is straightforward to show for the one flavor $N_f = 1$ case that the $U(1)_{L=R} = U(1)_B$ baryon number symmetry of $SU(3)$ QCD is reduced to a $\mathbf{Z}(2)_B$ symmetry [50; 91]: one can only distinguish between states with an even and odd number of “quark” constituents.

Another useful example is

$$\{7\} \otimes \{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus \dots \quad (20)$$

From this composition it is clear that three $G(2)$ “gluons” are sufficient to screen a $G(2)$ “quark”, producing a color-singlet hybrid $qGGG$. It is also true that:

$$\{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} = 4\{1\} \oplus \dots \quad (21)$$

so that the product contains four singlets.

Summarizing: a $G(2)$ gauge theory has colors, anticolors and color-singlet, and 14 generators. So it is characterized by 14 gluons, 8 of them transforming as ordinary gluons (as an octuplet of $SU(3)$), while the other 6 $G(2)$ gauge bosons separates into $\{3\}$ and $\{\bar{3}\}$, keeping the color

quarks/antiquarks quantum numbers, but they are still vector bosons. A general Lagrangian for $G(2)$ Yang-Mills theory can be written as [33; 50; 79]:

$$\mathcal{L}_{YM}[A] = -\frac{1}{2}\text{Tr}[F_{\mu\nu}^2], \quad (22)$$

with the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_G[A_\mu, A_\nu], \quad (23)$$

obtained from the vector potential

$$A_\mu(x) = A_\mu^a(x) \frac{\Lambda_a}{2}. \quad (24)$$

with g_G a proper coupling constant for all the gauge bosons and Λ_a the $G(2)$ generators. The Lagrangian is invariant under non-Abelian gauge transformations $A'_\mu = U(A_\mu + \partial_\mu)U^\dagger$, with $U(x) \in G(2)$. $G(2)$ Yang-Mills theory is asymptotically free like all non-Abelian $SU(N)$ gauge theories and, on the other hand, we expect confinement at low energies [79]. The $G(2)$ confinement is surely peculiar with a different realization with respect to $SU(3)$, where gluons cannot screen quarks (and screening arises due to dynamical quark-antiquark pair creation). In particular, as we have already seen in Eq.(20), $G(2)$ admits a new form of exceptional confinement. It has been showed that $G(2)$ lattice Yang-Mills theory is indeed in the confined phase in the strong coupling limit [50].

But we know that $G(2)$ is not a proper gauge theory for a real Quantum Chromodynamics theory. Therefore we must add a Higgs-like field in the fundamental $\{7\}$ representation in order to break $G(2)$ down to $SU(3)$. The consequence is simple and fundamental: 6 of the 14 $G(2)$ “gluons” gain a mass proportional to the vacuum expectation value (vev) w of the Higgs-like field, the other 8 $SU(3)$ gluons remaining untouched and massless. The Lagrangian of such a $G(2)$ -Higgs model can be written as [16; 50; 79; 88]:

$$\mathcal{L}_{G_2H}[A, \Phi] = \mathcal{L}_{YM}[A] + (D_\mu \Phi)^2 - V(\Phi) \quad (25)$$

where $\Phi(x) = (\Phi^1(x), \Phi^2(x), \dots, \Phi^7(x))$ is the real-valued Higgs-like field, $D_\mu \Phi = (\partial_\mu - ig_G A_\mu) \Phi$ is the covariant derivative and

$$V(\Phi) = \lambda(\Phi^2 - w^2)^2 \quad (26)$$

the quadratic scalar potential, with $\lambda > 0$. Because of the $\{7\} \otimes \{7\} \otimes \{7\} = \{1\} \oplus \dots$ singlet state seen before, in the fundamental representation a Higgs cubic term should be considered but, according to the antisymmetric property of T_{abc} , such a term disappears. Following the product in Eq.(21), the four singlets corresponds to $w^2 \Phi^2$, Φ^4 and two vanish due to antisymmetry, making the aforementioned potential general and consistent with $G(2)$ symmetry breaking and renormalizability. We can choose a simple vev like $\Phi_0 = \frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, 0, w)$ to break $G(2)$ and re-obtain the familiar unbroken $SU(3)$ symmetry: it is easy to notice from the diagonal and non-diagonal structure of Eq. (11-14) that

$$\Lambda_{1-8} \Phi_0 = 0 \quad (\text{unbroken generators}) \quad (27)$$

$$\Lambda_{9-14} \Phi_0 \neq 0 \quad (\text{broken generators}) \quad (28)$$

Plugging this scalar field vev into the square of the Higgs covariant derivative, we get the usual quadratic term in the gauge fields

$$g_G^2 \Phi_0^\dagger \frac{\Lambda_a}{2} \frac{\Lambda_b}{2} \Phi_0 A_\mu^a(x) A^{\mu,b}(x) = \frac{1}{2} M_{ab} A_\mu^a(x) A^{\mu,b}(x) \quad (29)$$

that gives the diagonal mass matrix M_{ab} for the gauge bosons, of which we can use the aforementioned trace normalization relation, which Gell-Mann matrices and $G(2)$ generators share, to put the squared masses terms $g_G^2 w^2$ in evidence. This new scalar Φ , which acquires a typical mass of

$$M_H = \sqrt{2\lambda}w \quad (30)$$

from the expansion of the potential about its minimum [33], should be a different Higgs field w.r.t. the SM one, with a much higher vev, in order to disjoin massive gluons dynamics from SM one, and a strong phenomenology. Such a strongly coupled massive field could be ruled out by future LHC and Future Circular Collider searches [92] (it is enough to think of heavy scalars models searches, such as the two-Higgs doublet model [93] or the composite Higgs models [94]). In this picture, as anticipated, following the standard Higgs mechanism to build up the dark candidates, 6 massless Goldstone bosons are eaten and become the longitudinal components of $G(2)$ vector gluons corresponding to the broken generators, which acquire the eigenvalue mass

$$M_G = g_G w \quad (31)$$

through the Higgs mechanism [33], according to Eq. (29), and exhibit the color quarks/antiquarks quantum numbers. No additional Yukawa-like terms are needed for the purpose of the present proposal, so that quarks remain massless at the scale of $G(2)$ symmetry breaking, since the SM Higgs has not yet acquired its vev. Then, if the sedenions description via automorphisms group is invoked, the symmetry breaking process could in principle act on three different copies of $G(2)$, expressed by the permutation factor S_3 which keeps track of the three fermion families. In other words, a Higgs sector (the SM one or an additional strong-coupled one for $G(2)$) of a S_3 -invariant extension of the SM could also break the flavour symmetry in order to produce the correct patterns of different masses and mixing angles for fermions families (see [72; 95] for insights). Additionally, it has been shown that, in a phenomenologically viable electroweak S_3 extension of the SM, S_3 symmetry should be broken to prevent flavor changing neutral currents [72] and the Higgs potential becomes more complicated due to the presence of three Higgs fields [74]. For simplicity, we could assume that this hypothetical process, involving S_3 breaking and Yukawa fermion masses generation, triggers at the electroweak scale, without interfering with the $G(2)$ Higgs potential.

Using the Higgs mechanism to smoothly interpolate between $SU(3)$ and $G(2)$ Yang-Mills theory, we can study the deconfinement phase transition. In the $SU(3)$ case this transition is weakly first order. In fact, in $(3 + 1)$ dimensions only $SU(2)$ Yang-Mills theory manifest a second order phase transition, whereas, in general, $SU(N)$ Yang-Mills theories with any higher N seem to have first order deconfinement phase transitions [96; 97; 98; 99; 100], which are more markedly first order for increasing N . The peculiarities of the phase transition from lattice $G(2)$ -Higgs to $SU(3)$ have been extensively studied in [15; 90; 101; 102; 103], confirming that $G(2)$ gauge theory has a finite-temperature deconfining phase transition mainly of first order and a similar but discernable behavior with respect to $SU(N)$ [103].

Moving back to the $G(2)$ color string, the breaking of this string between two static $G(2)$ “quarks” happens due to the production of two triplets of $G(2)$ “gluons” which screen the quarks. Hence, the string breaking scale is related to the mass of the six $G(2)$ “gluons” popping out of the vacuum. The resulting quark-gluons bound states (colorless $qGGG$ states) coming from the string breaking, must be both $G(2)$ -singlets and $SU(3)$ -singlets. When we switch on the interaction with the Higgs field, six $G(2)$ gluons acquire a mass thanks to the Higgs mechanism. The larger is M_G ,

the greater is the distance where string breaking occurs. When the expectation value of the Higgs-like field is sent to infinity, so that the 6 massive $G(2)$ “gluons” are completely removed from the dynamics, also the string breaking scale is infinite. Thus the scenario of the usual $SU(3)$ string potential reappears. For small w (on the order of Λ_{QCD}), on the other hand, the additional $G(2)$ “gluons” could be light and participate in the dynamics. As long as w remains finite, as we know it should be in the SM and in its extensions, the heavy $G(2)$ “gluons” can mediate weak baryon number violating processes [50] (only in the $w \rightarrow \infty$ limit baryon number is an exact discrete symmetry of the Lagrangian). Finally, for $w = 0$ the Higgs mechanism disappears and we come back to $G(2)$. As stressed before, hereafter only high w values (with w much greater than the SM Higgs vev) are considered in order to realize a consistent dark matter scenario.

For what concerns the hadronic spectrum of a hypothetical $G(2)$ -QCD, the physics appears to be qualitatively similar to $SU(3)$ QCD [104], but richer. This can be easily demonstrated from the decomposition of representations products, like Eq.(18), (19), (20), (21). In the (massless) spectrum of the unbroken $G(2)$ phase there are many more states beyond standard mesons and baryons: one-quark-three- $G(2)$ gluons states (and, in general, the quark confinement for one-quark- N - $G(2)$ gluons, with $N \geq 3$), diquarks, ($qqqq$) tetraquarks and ($qqqqq$) pentaquarks. $G(2)$ and $SU(3)$ also share glueballs states, for any numbers of $G(2)$ gluons (2 and 3 in the ground states) and hexaquarks¹³. Even collective manifestations of $G(2)$ exceptional matter could be different: for example, a $G(2)$ -QCD neutron star could display a distinct behavior with respect to a $SU(3)$ neutron gas star [105].

In addition, and even more fundamental for the present dissertation, we gain an *exceptional* particle sector: if we move away the six $G(2)$ gluons from the dynamics according to a TeV-ish (or beyond TeV) mass scale, these bosons must be secluded and separated from the visible SM sector in first approximation, without experimentally accessible electroweak interactions, unlike WIMPs, and extreme energies (and distances) should be mandatory to access the $G(2)$ string breaking. This could be due to the very high energy scale of the $G(2) - SU(3)$ phase transition, occurring at much greater energies than electroweak breaking scale. This is the realization of a *beyond Naturalness and WIMP Miracle* criterion for dark matter search. Indeed, $G(2)$ gluons, as $SU(3)$ ones, are electrically neutral and immune to interactions with light and weak W, Z bosons at tree level. Another advantage of a $G(2)$ broken theory is that no additional families are added to the Standard Model, unlike $SU(N)$ theories.

Overall, this seems to be a good scenario for a cold¹⁴ dark matter (CDM) theory if we find a stable or long-lived candidate. The six *dark* gluons can form heavy dark glueballs constituted by two or three (or multiples) $G(2)$ gluons, according to $\{14\} \otimes \{14\} = \{1\} \oplus \dots$ and $\{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus \dots$ representations [104], with integer total angular momentum $J = 0, 2$ and $J = 1, 3$ for 2-gluons and 3-gluons balls respectively. In principle, dark-colored broken- $G(2)$ glueballs should not be stable, if these heavy composite states themselves have no extra symmetries to prevent their decay; since the proposed $G(2)$ theory includes QCD, unlike

¹³The further explicit decompositions of the products are:

$$\begin{aligned} \{7\} \otimes \{14\} \otimes \{14\} \otimes \{14\} &= \{1\} \oplus 10\{7\} \oplus 6\{14\} \oplus 15\{27\} \oplus 20\{64\} \oplus 13\{77\} \oplus 13\{77'\} \oplus 10\{182\} \oplus 15\{189\} \oplus \\ &9\{286\} \oplus 3\{378\} \oplus 6\{448\} \oplus 3\{729\} \oplus \{896\} \oplus 2\{924\} \\ \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} &= 4\{1\} \oplus 10\{7\} \oplus 9\{14\} \oplus 12\{27\} \oplus 8\{64\} \oplus 6\{77\} \oplus 2\{77'\} \oplus \{182\} \oplus 3\{189\} \\ \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} &= 10\{1\} \oplus 35\{7\} \oplus 30\{14\} \oplus 45\{27\} \oplus 40\{64\} \oplus 30\{77\} \oplus 11\{77'\} \oplus 10\{182\} \oplus 20\{189\} \oplus \\ &5\{286\} \oplus \{378\} \oplus 4\{448\} \\ \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} &= 35\{1\} \oplus 120\{7\} \oplus 120\{14\} \oplus 180\{27\} \oplus 176\{64\} \oplus 145\{77\} \oplus 65\{77'\} \oplus \\ &65\{182\} \oplus 120\{189\} \oplus 5\{273\} \oplus 40\{286\} \oplus 15\{378\} \oplus 40\{448\} \oplus \{714\} \oplus 9\{729\} \oplus 5\{924\} \\ \{14\} \otimes \{14\} &= \{1\} \oplus \{14\} \oplus \{27\} \oplus \{77\} \oplus \{77'\} \\ \{14\} \otimes \{14\} \otimes \{14\} &= \{1\} \oplus \{7\} \oplus 5\{14\} \oplus 3\{27\} \oplus 2\{64\} \oplus 4\{77\} \oplus 3\{77'\} \oplus \{182\} \oplus 3\{189\} \oplus \{273\} \oplus 2\{448\} \end{aligned}$$

¹⁴Cold means non-relativistic and refers to the standard Lambda-CDM cosmological model.

hidden Yang-Mills theories with no direct connections with the SM [106; 107], there exist states that couple to both the dark-colored glueballs and $SU(3)$ particles (for example the $G(2)$ -breaking Higgs fields): hence, whether at tree-level or via loops, these heavy glueballs would not be stable. To avoid this, first of all the new $G(2)$ Higgs should be at least more massive than the lightest 2-gluons glueball, so that $M_H > M_{GG}$, which implies the qualitative constraint $\sqrt{2\lambda} > 2g_G$ from Eq.(30), (31). Secondly, the decays into meson states should be also forbidden. In principle, a lightest $J^{PC} = 0^{++}$ state, in analogy with standard QCD, could dominate the glueball spectrum [108], but this could be unstable, like the lightest meson π^0 and the other known scalar particle, *i.e.* the SM Higgs h^0 . The possibility of a conserved charge or a peculiar phenomenon which guarantees stability to the lightest glueball states should not be ruled out *a priori*, considering that analytical and topological properties of Yang-Mills theory solutions are still not completely understood: even the fundamental problem of color confinement has not a definitive answer nor an analytical proof.

For example, in analogy with the baryon number conservation and the forbidden proton decay into π^0 , one can introduce a conserved additive *gluon number* Γ for the glueball states, which counts the number of massive $G(2)$ gluons (and “antigluons”), preventing the glueball from decaying into SM mesons, which are not made of $G(2)$ gluons (one has to keep in mind that these peculiar bosons do maintain the color quarks/antiquarks quantum number). Indeed also the $U(1)_B$ global symmetry of the Standard Model which prevents the proton decay is an *accidental symmetry* and not a fundamental law, that can be broken by quantum effects.

We have the same lack of knowledge for the glueballs interactions with their own environments: as for residual nuclear force between hadrons in nuclei, the possibility of a residual binding interaction between glueballs, which prevents the decay (like neutron decay), must be investigated.

Possibly, one can also postulate a suppression scale $1/f_G(\Delta M)$ for the couplings with SM which depends on the relative mass difference between the interacting particles, *i.e.* the dark glueball and the quarks: if the masses of the glueballs are too high w.r.t. the QCD scale, their decay might be highly suppressed.

Another interesting chance is to invoke the $J^{PC} = 0^{++}, 2^{++}$ dark $G(2)$ glueball states as graviton counterparts in a AdS/CFT correspondence framework, as discussed in [109] for QCD glueballs (even if this does not exhaust the quest for stability). Moreover, the scalar glueball could be part of a scalar-tensor gravity approach [110], whereas the tensor glueball could play the role of a massive graviton-like particle, for example in the context of bimetric gravity theory [111; 112], where the massive gravitational dark matter can non-trivially interact with gravity itself. The couplings to SM quarks of such a tensor DM can be by far too weak [112], making it undetectable in collider searches; furthermore, the requirement of a correct DM abundance and stability constrains a non-thermal spin-2 mass to be $\gg 1 \text{ TeV}$ [113].

Furthermore, if we assume there exists a global symmetry, a peculiar feature or a collective phenomenon capable of stabilizing the lightest $G(2)$ exceptional glueballs states, it must be considered that bosonic ensembles could eventually clump together to form a Bose-Einstein condensate (BEC): once the temperature of a cosmological boson gas is less than the critical temperature, a Bose-Einstein condensation process can always take place during the cosmic history of the Universe, even if the high mass of these candidates should disfavor this scenario (for example, the occurrence of glueballs condensates and glueball stars have been recently discussed in [106; 114; 115; 116]).

Concluding, it is reasonable to assert that is no more pretentious to hypothesize a not yet clarified stabilization mechanism for heavy $G(2)$ glueballs than to postulate *ad hoc* a Standard Model–disconnected gauge symmetry to avoid dark matter candidates decay.

Such a dark sector can naturally accommodate the fact that there is only gravitational evidence for dark matter so far, certainly disfavoring direct and indirect searches, and it can also qualitatively account for the observation that dark matter and ordinary matter are in commensurable quantities (approximately 5:1 from recent Planck experiment measurements [117]), as they come from the same broken gauge group. Given the forbidden or extremely weak interactions between the heavy glueball states and ordinary matter, the usual WIMP-like scenario in which the DM relic abundance is built via the freeze-out mechanism cannot be achieved, since these bosons are not in thermal equilibrium with the baryon-photon fluid in the early Universe: their production should be abruptly triggered by a first-order cosmological phase transition, fixing their abundance one and for all. It is well-known that several non freeze-out models has been proposed in literature, such as the *FIMP* (Feebly Interacting Massive Particle) cosmology, via a freeze-in mechanism [118]. In analogy, a two-gluons $G(2)$ glueball, with an extremely small coupling ($O(10^{-7})$ or less) with the visible sector, could be *ab initio* produced out of thermal equilibrium for instance by the heavy visible mediator decay, such as the new Higgs ($H \rightarrow DM DM$), if $M_H \geq M_{GG}$ and the coupling between DM and the heavy Higgs is very weak, realizing a sort of exotic-Higgs portal DM. The most general renormalizable scalar $G(2)$ -glueball potential of this type reads

$$V(\Phi, S) = V(\Phi) + \frac{M_S^2}{2} S^2 + \frac{\lambda_S}{4} S^4 + \frac{\lambda_{HS}}{2} \Phi^2 S^2 \quad (32)$$

where S is the scalar $G(2)$ glueball field and M_S its mass, λ_S its quartic self-interaction strength and λ_{HS} the heavy Higgs-scalar glueball coupling. If the portal coupling is sufficiently small [118], one can recover a correct dark matter relic abundance $\Omega h^2 \cong 0.12$ [117]. The situation can change if λ_S is large, *i.e.* scalar self-interactions are active: this could be the case of a *SIMP* (Strongly Interacting Massive Particle) scenario, with a possible dark freeze-out mechanism [118; 119], which prescribes DM thermalisation within the dark sector itself (through generic $n_{DM} \rightarrow n'_{DM}$ processes), which was initially populated by previous freeze-in. A proper balance, to be determined, between λ_{HS} and λ_S could produce the right abundance for the complex $G(2)$ phenomenology. This issue will certainly deserve a dedicated study.

Finally we are going to briefly discuss possible manifestations of $G(2)$ dark gluons in the present Universe. Many vector bosons composite states have been proposed as DM candidates in the last two decades: light hidden glueballs [120; 107; 121; 122], gluon condensates [123], exceptional dark matter referring to a composite Higgs model with $SO(7)$ symmetry broken to the exceptional $G(2)$ [124], $SU(N)$ vector gauge bosons [125], vector BECs [126] and, in general, non-Abelian dark forces [127]. These studies demonstrate the growing interest in beyond SM non-Abelian frameworks where to develop consistent DM theories, without invoking string theory and keeping the theoretical apparatus sufficiently minimal.

In our case of six dark heavy gluons from the broken exceptional $G(2)$ group, mainly as stable $J^{PC} = 0^{++}$ and/or $J^{PC} = 2^{++}$ glueballs, it must be considered they could clump and organize into dark matter halos, in form of a heavy bosons gas or in some fluid systems [128; 129], for sufficiently low temperatures and/or enough high densities. Indeed, the idea of a superfluid dark matter has recently attracted attention in literature, from Khoury's promising proposal of a unified superfluid dark sector [130; 131]. The fact that dark matter particles could assume different "phases" according to the environment (gas, fluid or BECs) is very suitable to account for the plethora of dark matter observations at all scales, from galactic to cosmological ones. The realization of this scenario usually involves axions or ALPS (axion like particles) [132], which are light or ultralight (with masses from 10^{-24} to 1 eV in natural units) and capable of reproducing DM

halos properties: dark matter condensation and self-gravitating Bose-Einstein condensates have been extensively studied in [133; 134; 135].

In our scenario we hypothesize the main constituents of a possible dark fluid are broken- $G(2)$ massive gluons composite states. They are heavy, so that we cannot invoke the axion-like description, rather a nuclear/atomic approach, like for Rubidium condensates [136]. But still $G(2)$ glueballs could aggregate into extended objects: one can explore the possibility that heavy $G(2)$ gluon dark matter is capable of producing stellar objects. Many models of exotic stars made of unknown particles have been proposed, especially for sub-GeV masses, such as bosonic stars (where the particle is a scalar or pseudoscalar [137; 138; 139], most likely for a quartic order self-interaction [139]), Proca stars (for massive spin 1 bosons [140; 141; 142], if we invoke 3-gluons glueballs), BEC stars [143], or glueballs stars [116]. In these cases, DM candidates could be also studied from peculiar behaviors of compact astrophysical objects, characterizing physical observables useful to disentangle standard scenarios from exotic phenomenology. The key ingredient to try to build up stellar objects with non-light bosons of mass m_B is mainly the magnitude of the quartic order self-interaction λ_B of the constitutive boson, since $M_{star} \propto \sqrt{\lambda_B}/m_B^2$, as shown in [139; 143]. Even if it is fair that glueballs are strongly self interacting, it is quite hard to make precise estimates for their scattering process, given our general limited knowledge concerning strongly coupled theories.

Even more ambitious, one should consider the possibility to probe the $SU(3) - G(2)$ phase transition from black hole formation, as suggested in [144] for quark matter, if black holes are formed through exotic condensed matter stages, exploiting the theoretical framework of gauge/gravity dualities [145]. It could be worthwhile to speculate on the possibility that, in extremely high pressure and temperature quark matter phases, quarks can be seized by $G(2)$ gluons to form $qGGG$ screened states, rearranging QCD matter into color-singlet hybrids.

Concluding, for a few years we can take advantage of gravitational waves astronomy as a powerful probe to distinguish, in principle, a hypothetical binary *dark-colored gluons star* from a binary black hole system, due to possible differences in the gravitational wave frequency and amplitude, as demonstrated in [106].

So dark $G(2)$ glueballs can be very versatile and exploitable within existing theoretical speculations.

4 Conclusions

If Nature physical description is intrinsically mathematical, fundamental microscopic forces might be manifestation of the algebras that can be built via the Cayley-Dickson construction process. In other words, algebras can guide physics through the understanding of fundamental interactions. To translate the mathematical meaning into a physical language, one has to move from abstract algebras to groups of symmetry, through a correspondence here proposed as an *automorphism relation*. This leads to the discovery of a mismatch between $SU(3)$ strong force and octonions: the octonions automorphism group is the exceptional group $G(2)$, which contains $SU(3)$, but it is not exhausted by $SU(3)$ itself. In the difference between the physical content of $G(2)$ and $SU(3)$ new particles lie, in the form of six additional heavy bosons organized in composite states, disconnected by Standard Model dynamics: *the dark-colored $G(2)$ gluons*. These gluons cannot interact with SM particles, at least at the explored energy scales, due to their high mass and QCD string behavior. *Mathematical realism* is the guide and criterion to build a minimal extension of

the Standard Model.

Hence, for the first time in literature, $G(2)$ was treated and developed as a realistic symmetry to enlarge the Standard Model, and not only as a lattice QCD tool for computation. $G(2)$ is a good gauge group to describe a larger interaction, which operated in the Early Universe before the emergence of visible matter; when the Universe cooled down, reaching a proper far beyond TeV energy scale at which $G(2)$ is broken, usual $SU(3)$ QCD appeared, while an extra Higgs mechanism produced a secluded massive sector of cold dark-colored bosons. The SM is naturally embedded in this framework with a minimal additional particle content, *i.e.* a heavy scalar Higgs particle, responsible for a Higgs mechanism for the strong sector symmetrical w.r.t. the electroweak one, and a bunch of massive gluons, in principle with the same mass, whose composite states play the role of dark matter. Some *accidental* stability mechanisms for the dark glueballs have been proposed. The presence of a new Higgs field represents the usual need of a scalar sector to induce the symmetry breaking of a fundamental gauge symmetry. The extra Higgs might have visible decay channels, but both this Higgs and the dark $G(2)$ glueballs belong to a very high energy scale which is certainly beyond current LHC searches, *i.e.* a multi-TeV or tens/hundreds of TeV scale defined by the vacuum expectation value of the extra Higgs. Such a DM is certainly compatible with direct, indirect, collider searches and astrophysical observations, as it is extremely massive and almost collisionless.

In addition, if one tries to extend the correspondence between mathematical algebras and physical symmetries further beyond octonions, sedenions show an intriguing property: they still have $G(2)$ as a fundamental automorphism, but “tripled” by an S_3 factor, which resembles the three fermion families of the Standard Model and its S_3 -invariant extension. We know larger symmetries can be constructed using the products of octonions (sedenions) and the other division algebras, pointing towards subsequent exceptional groups, as illustrated by the Freudenthal–Tits magic square, which have been the subjects of string theories, but we did not want to push the dissertation in this direction. Indeed the choice of $G(2)$, as automorphism of octonions and minimally enlarged non-Abelian compact Lie algebra of rank 2, is the minimal exceptional extension of the Standard Model including a reliable dark sector, requiring no additional particle families nor extra fundamental forces. This fact reconciles the particle *desert* observed between the SM Higgs mass scale and the TeV scale. $G(2)$ could guarantee peculiar manifestations in extreme astrophysical compact objects, such as neutron stars and black holes, which can be observed studying gravitational waves and dynamics. In future, an interplay between condensed matter community and particle physicists could be necessary to deeply investigate dark matter properties.

The development of a definitive theory is beyond the purpose of the present phenomenological proposal, which is intended as a guideline for further speculations and future works.

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