Cubic-Quartic Optical Solitons in Bragg Gratings Fibers for Perturbed NLSE Having Parabolic-Nonlocal Law of Refractive Index Using Two Integration Schemes

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Cubic-quartic optical solitons in Bragg gratings fibers for perturbed NLSE having parabolic-nonlocal law of refractive index using two integration schemes

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Abstract

The optical solitons in Bragg gratings fibers for perturbed NLSE having cubic-quartic dispersive reflectivity with parabolic-nonlocal combo law of refractive index are studied. The extended auxiliary equation method and the addendum to Kudryashov's method are introduced. The existence criteria for such solitons are indicated.

Keywords: Jacobi elliptic function solutions; The extended auxiliary equation method; The addendum to Kudryashov's method ; Fiber Bragg gratings; Cubic-quartic; Parabolic law; Nonlocal law.

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1 Introduction

In the past few decades [1-22], the study of optical solitons in optical fibers, PCF, metamaterials, optical couplers, fibers with Bragg gratings is going on strong and steady. Lately, when the chromatic dispersion (CD) gets low, the definition of cubic-quartic (CQ) solitons have emerged when it was realized that the delicate balance between CD and self-phase modulation (SPM) grows feeble. In that case, to boost up the much-needed balance, it is necessary to compensate for the low count of CD by introducing third-order dispersion (3OD) and fourth-order dispersion (4OD) terms. Together, these constitute CQ dispersive effects. For Bragg gratings fibers, it is the dispersive reflectivity, which originally stems from CD, now comes from CQ dispersion. In this paper, the coupled system of cubic-quartic nonlinear Schrödinger equation (CQ-NLSE) in Bragg gratings fibers with parabolic-nonlocal combo law of refractive index will be discussed for the first time by using two different methods mentioned in the abstract. Today's work will address the same problem with two fresh approaches. These are the extended auxiliary equation method and the

addendum to Kudryashov's method. These methods will lead to the emergence of Jacobi elliptic function solutions, bright, dark and singular solitons from the governing model. The essential details are jotted with the existence criteria for such solitons, after a quick into to the model.

1.1 Governing model

The coupled cubic-quartic perturbed NLSE in Bragg gratings fibers with parabolic-nonlocal law nonlinearity is written, for the first time, as

$$\begin{aligned}
& iu_t + ia_1 v_{xxx} + b_1 v_{xxxx} + (c_1 |u|^2 + d_1 |v|^2) u + (\xi_1 |u|^4 + \eta_1 |u|^2 |v|^2 + \zeta_1 |v|^4) u \\
& + [f_1 (|u|^2)_{xx} + g_1 (|v|^2)_{xx}] u + i\alpha_1 u_x + \beta_1 v + \sigma_1 u^* v^2 = i [\gamma_1 (|u|^2 u)_x + \theta_1 (|u|^2)_x u + \rho_1 |u|^2 u_x],
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
& iv_t + ia_2 u_{xxx} + b_2 u_{xxxx} + (c_2 |v|^2 + d_2 |u|^2) v + (\xi_2 |v|^4 + \eta_2 |v|^2 |u|^2 + \zeta_2 |u|^4) v \\
& + [f_2 (|v|^2)_{xx} + g_2 (|u|^2)_{xx}] v + i\alpha_2 v_x + \beta_2 u + \sigma_2 v^* u^2 = i [\gamma_2 (|v|^2 v)_x + \theta_2 (|v|^2)_x v + \rho_2 |v|^2 v_x],
\end{aligned} \tag{2}$$

where $a_l, b_l, c_l, d_l, \xi_l, \eta_l, \zeta_l, f_l, g_l, \alpha_l, \beta_l, \sigma_l, \gamma_l, \theta_l$ and ρ_l , ($l = 1, 2$) are real parameters such that $i = \sqrt{-1}$. Here, $u(x, t)$, $v(x, t)$ are the complex wave profiles. The coefficients of a_l, b_l are third order dispersion (3OD) and fourth order dispersion (4OD), respectively. The parameters c_l, ξ_l, f_l represent the self-phase modulation (SPM) coefficients, while the cross-phase modulation (XPM) effect comes from the coefficients d_l, η_l, ζ_l and g_l . The parameters α_l, β_l and σ_l are the coefficients of inter-modal dispersion (IMD), detuning parameter and four-wave mixing effect (4WM) for Kerr part of the nonlinearity, respectively, while, γ_l is the coefficient of self-steepening (SS) term. Lastly, the parameters θ_l, ρ_l are the coefficients of nonlinear dispersion terms. The system (1) and (2) is a manifested version of the standard model. That is the well-known NLSE model in fiber Bragg gratings with chromatic dispersion (CD), that is structured as [22]:

$$\begin{aligned}
& iu_t + E_1 v_{xx} + (c_1 |u|^2 + d_1 |v|^2) u + (\xi_1 |u|^4 + \eta_1 |u|^2 |v|^2 + \zeta_1 |v|^4) u + [f_1 (|u|^2)_{xx} + g_1 (|v|^2)_{xx}] u \\
& + i\alpha_1 u_x + \beta_1 v + \sigma_1 u^* v^2 = 0,
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
& iv_t + E_2 u_{xx} + (c_2 |v|^2 + d_2 |u|^2) v + (\xi_2 |v|^4 + \eta_2 |u|^2 |v|^2 + \zeta_2 |u|^4) v + [f_2 (|v|^2)_{xx} + g_2 (|u|^2)_{xx}] v \\
& + i\alpha_2 v_x + \beta_2 u + \sigma_2 v^* u^2 = 0,
\end{aligned} \tag{4}$$

where E_1 and E_2 are the coefficients of chromatic dispersion (CD). In the system (1) and (2), it is this CD that is replaced by 3OD and 4OD, which formulate the dispersion effects.

The objective of this paper is to apply the extended auxiliary equation method and the addendum to Kudryashov's method to find the bright, dark and singular solitons solutions as well as the

Jacobi elliptic function solutions of the coupled system (1) and (2).

The organization of this article can be written as: the mathematical preliminaries are discussed in Section 2. The extended auxiliary equation method is applied to the coupled system (1) and (2) in Section 3. The addendum to Kudryashov's method is applied to the same coupled system in Section 4. Lastly, conclusions are given in Section 5.

2 Mathematical preliminaries

In order to recover solitons of the CQ-NLSE in fiber Bragg gratings with parabolic non-local law nonlinearity, we set

$$u(x, t) = P_1(\xi) \exp [i\phi(x, t)], \quad (5)$$

$$v(x, t) = P_2(\xi) \exp [i\phi(x, t)],$$

and

$$\xi = x - Ct, \quad \phi(x, t) = -\kappa x + \omega t + \theta_0, \quad (6)$$

where C, κ, ω and θ_0 are all non zero parameters. Here, C is the velocity of soliton, κ is the frequency of soliton, ω is the wave number of the soliton and finally, θ_0 is the phase parameter, while $P_1(\xi)$, $P_2(\xi)$ and $\phi(x, t)$ are real functions representing the amplitude portion of the soliton and the phase component of the soliton, respectively. If we substitute (5) and (6) into Eqs. (1) and (2) and separate the real and imaginary parts, we deduce that the real parts are

$$\begin{aligned} & b_1 P_2^{(4)} + 3\kappa (a_1 - 2b_1\kappa) P_2'' + (\alpha_1\kappa - \omega) P_1 + [\kappa^3 (b_1\kappa - a_1) + \beta_1] P_2 + (c_1 - \gamma_1\kappa - \rho_1\kappa) P_1^3 \\ & + 2[f_1 (P_1 P_1'^2 + P_1^2 P_1'') + g_1 (P_1 P_2'^2 + P_1 P_2 P_2'')] + (d_1 + \sigma_1) P_1 P_2^2 + \xi_1 P_1^5 + \eta_1 P_1^3 P_2^2 + \zeta_1 P_1 P_2^4 = 0, \end{aligned} \quad (7)$$

$$\begin{aligned} & b_2 P_1^{(4)} + 3\kappa (a_2 - 2b_2\kappa) P_1'' + (\alpha_2\kappa - \omega) P_2 + [\kappa^3 (b_2\kappa - a_2) + \beta_2] P_1 + (c_2 - \gamma_2\kappa - \rho_2\kappa) P_2^3 \\ & + 2[f_2 (P_2 P_2'^2 + P_2^2 P_2'') + g_2 (P_2 P_1'^2 + P_2 P_1 P_1'')] + (d_2 + \sigma_2) P_2 P_1^2 + \xi_2 P_2^5 + \eta_2 P_2^3 P_1^2 + \zeta_2 P_2 P_1^4 = 0, \end{aligned} \quad (8)$$

and the imaginary parts are

$$(a_1 - 4b_1\kappa) P_2''' - (3a_1 - 4b_1\kappa) \kappa^2 P_2' + (\alpha_1 - C) P_1' - (3\gamma_1 + 2\theta_1 + \rho_1) P_1^2 P_1' = 0, \quad (9)$$

$$(a_2 - 4b_2\kappa) P_1''' - (3a_2 - 4b_2\kappa) \kappa^2 P_1' + (\alpha_2 - C) P_2' - (3\gamma_2 + 2\theta_2 + \rho_2) P_2^2 P_2' = 0. \quad (10)$$

Setting

$$P_2(\xi) = \chi P_1(\xi), \quad (11)$$

where χ is a constant such that $\chi \neq 0, 1$. Now, Eqs.(7) - (10) become

$$\begin{aligned} & b_1 \chi P_1^{(4)} + 3\kappa \chi (a_1 - 2b_1\kappa) P_1'' + 2(f_1 + \chi^2 g_1) (P_1 P_1'^2 + P_1^2 P_1'') \\ & + [\alpha_1\kappa - \omega + \chi\kappa^3 (b_1\kappa - a_1) + \chi\beta_1] P_1 + [c_1 - \gamma_1\kappa - \rho_1\kappa + \chi^2 (d_1 + \sigma_1)] P_1^3 \\ & + (\xi_1 + \eta_1 \chi^2 + \chi^4 \zeta_1) P_1^5 = 0, \end{aligned} \quad (12)$$

$$\begin{aligned}
& b_2 P_1^{(4)} + 3\kappa (a_2 - 2b_2\kappa) P_1'' + 2\chi (\chi^2 f_2 + g_2) (P_1 P_1'^2 + P_1^2 P_1'') \\
& + [\chi (\alpha_2\kappa - \omega) + \kappa^3 (b_2\kappa - a_2) + \beta_2] P_1 + \chi [\chi^2 (c_2 - \gamma_2\kappa - \rho_2\kappa) + d_2 + \sigma_2] P_1^3 \\
& + \chi (\chi^4 \xi_2 + \chi^2 \eta_2 + \zeta_2) P_1^5 = 0,
\end{aligned} \tag{13}$$

and

$$\chi (a_1 - 4b_1\kappa) P_1''' + [\alpha_1 - C - \chi (3a_1 - 4b_1\kappa) \kappa^2] P_1' - (3\gamma_1 + 2\theta_1 + \rho_1) P_1^2 P_1' = 0, \tag{14}$$

$$(a_2 - 4b_2\kappa) P_1''' + [\chi (\alpha_2 - C) - (3a_2 - 4b_2\kappa) \kappa^2] P_1' - \chi^3 (3\gamma_2 + 2\theta_2 + \rho_2) P_1^2 P_1' = 0. \tag{15}$$

Integrating Eqs.(14) and (15) with zero-integration constants, we have

$$\chi (a_1 - 4b_1\kappa) P_1'' + [\alpha_1 - C - \chi (3a_1 - 4b_1\kappa) \kappa^2] P_1 - \frac{1}{3} (3\gamma_1 + 2\theta_1 + \rho_1) P_1^3 = 0, \tag{16}$$

$$(a_2 - 4b_2\kappa) P_1'' + [\chi (\alpha_2 - C) - (3a_2 - 4b_2\kappa) \kappa^2] P_1 - \frac{1}{3} \chi^3 (3\gamma_2 + 2\theta_2 + \rho_2) P_1^3 = 0. \tag{17}$$

Setting the coefficients of the linearly independent functions of Eqs. (16) and (17) to zero, yields

$$\kappa = \frac{a_l}{4b_l}, \quad l = 1, 2, \quad a_1 b_2 = a_2 b_1, \tag{18}$$

$$C = \alpha_1 - \chi (3a_1 - 4b_1\kappa) \kappa^2, \tag{19}$$

$$C = \frac{\chi \alpha_2 - (3a_2 - 4b_2\kappa) \kappa^2}{\chi}, \tag{20}$$

and

$$3\gamma_1 + 2\theta_1 + \rho_1 = 0, \tag{21}$$

$$3\gamma_2 + 2\theta_2 + \rho_2 = 0. \tag{22}$$

From (19) and (20), one gets the following constraint condition:

$$\chi (\alpha_1 - \alpha_2) - 3\kappa^2 (a_1 \chi^2 - a_2) + 4\kappa^3 (b_1 \chi^2 - b_2) = 0. \tag{23}$$

Eqs. (12) and (13) have the same form under the constraint conditions:

$$b_1 \chi = b_2, \tag{24}$$

$$\chi (a_1 - 2b_1\kappa) = a_2 - 2b_2\kappa, \tag{25}$$

$$f_1 + \chi^2 g_1 = \chi (\chi^2 f_2 + g_2), \tag{26}$$

$$\alpha_1 \kappa - \omega + \chi \kappa^3 (b_1 \kappa - a_1) + \chi \beta_1 = \chi (\alpha_2 \kappa - \omega) + \kappa^3 (b_2 \kappa - a_2) + \beta_2, \tag{27}$$

$$c_1 - \gamma_1 \kappa - \rho_1 \kappa + \chi^2 (d_1 + \sigma_1) = \chi [\chi^2 (c_2 - \gamma_2 \kappa - \rho_2 \kappa) + d_2 + \sigma_2], \tag{28}$$

$$\xi_1 + \eta_1 \chi^2 + \chi^4 \zeta_1 = \chi (\chi^4 \xi_2 + \chi^2 \eta_2 + \zeta_2). \tag{29}$$

Consequently, we deduce that

$$\omega = \frac{(\alpha_1 - \chi\alpha_2)\kappa + \chi\beta_1 - \beta_2}{1 - \chi},$$

$$\chi = \frac{b_2}{b_1} = \frac{a_2 - 2b_2\kappa}{a_1 - 2b_1\kappa},$$
(30)

provided $a_1 \neq a_2$ and $b_1 \neq b_2$. Eq. (12) can be rewritten in the form:

$$P_1^{(4)} + L_1 P_1'' + L_2 (P_1 P_1'^2 + P_1^2 P_1'') + L_3 P_1 + L_4 P_1^3 + L_5 P_1^5 = 0, \quad (31)$$

where

$$L_1 = 6\kappa^2, L_2 = \frac{2(f_1 + \chi^2 g_1)}{b_1 \chi}, L_3 = \frac{\alpha_1 \kappa - \omega + \chi \kappa^3 (b_1 \kappa - a_1) + \chi \beta_1}{b_1 \chi},$$

$$L_4 = \frac{c_1 - \gamma_1 \kappa - \rho_1 \kappa + \chi^2 (d_1 + \sigma_1)}{b_1 \chi}, L_5 = \frac{\xi_1 + \eta_1 \chi^2 + \chi^4 \zeta_1}{b_1 \chi},$$
(32)

provided $b_1 \chi \neq 0$ and $L_1 > 0$. Balancing $P_1^{(4)}$ with P_1^5 in (31), yields the balance number $N = 1$. The problem now is to solve Eq. (31) using the following two schemes.

3 The extended auxiliary equation method

According to this method [15-18], we assume that Eq.(31) has the formal solution

$$P_1(\xi) = A_0 + A_1 F(\xi) + A_2 F^2(\xi), \quad (33)$$

where A_0, A_1 and A_2 are constants to be determined, such that $A_2 \neq 0$, while the function $F(\xi)$ satisfies the following first order equation:

$$F'^2(\xi) = C_0 + C_2 F^2(\xi) + C_4 F^4(\xi) + C_6 F^6(\xi), \quad (34)$$

where C_j ($j = 0, 2, 4, 6$) are constants to be determined. It is well known that Eq.(34) has the following solution:

$$F(\xi) = \frac{1}{2} \left[-\frac{C_4}{C_6} (1 \pm f(\xi)) \right]^{\frac{1}{2}}, \quad (35)$$

where $f(\xi)$ could be expressed through the Jacobi elliptic functions $\text{sn}(\xi, m)$, $\text{cn}(\xi, m)$, $\text{dn}(\xi, m)$ and so on. Here $0 < m < 1$ is the modulus of the Jacobi elliptic functions. Substituting (33) along with (34) into Eq.(31), collecting the coefficients of each power $F^l(\xi)$ ($F'(\xi)$) ^{j} , ($l = 0, 1, 2, \dots, 10$, $j = 0, 1$) and setting these coefficients to zero, we have a set of algebraic equations which can be solved using the Maple to obtain the following results:

$$L_2 = -\frac{5\sqrt{6L_5}}{3}, L_3 = -A_0^4 L_5 + \frac{1}{3} A_0 (L_1 A_0 + 4A_2 C_0) \sqrt{6L_5}, L_4 = -\frac{L_1 \sqrt{6L_5}}{3}, C_0 = C_0,$$

$$C_2 = \frac{A_0^2 \sqrt{6L_5} - L_1}{4}, C_4 = \frac{A_0 A_2 \sqrt{6L_5}}{6}, C_6 = \frac{A_2^2 \sqrt{6L_5}}{24}, A_0 = A_0, A_1 = 0, A_2 = A_2,$$
(36)

provided $L_5 > 0$. From (33), (35) and (36), then we have the solutions:

$$P_1(\xi) = \mp A_0 f(\xi). \quad (37)$$

We have the following families of Jacobi elliptic functions solutions of Eqs.(1) and (2):

Family-1. If $C_0 = \frac{C_4^3(m^2 - 1)}{32C_6^2m^2}$, $C_2 = \frac{C_4^2(5m^2 - 1)}{16C_6m^2}$, $C_6 > 0$, then

$$u(x, t) = \left(\frac{6m^4 L_1^2}{L_5(m^2 + 1)^2} \right)^{\frac{1}{4}} \operatorname{sn} \left(\sqrt{\frac{L_1}{m^2 + 1}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (38)$$

and

$$v(x, t) = \chi \left(\frac{6m^4 L_1^2}{L_5(m^2 + 1)^2} \right)^{\frac{1}{4}} \operatorname{sn} \left(\sqrt{\frac{L_1}{m^2 + 1}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (39)$$

or

$$u(x, t) = \left(\frac{6L_1^2}{L_5(m^2 + 1)^2} \right)^{\frac{1}{4}} \operatorname{ns} \left(\sqrt{\frac{L_1}{m^2 + 1}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (40)$$

and

$$v(x, t) = \chi \left(\frac{6L_1^2}{L_5(m^2 + 1)^2} \right)^{\frac{1}{4}} \operatorname{ns} \left(\sqrt{\frac{L_1}{m^2 + 1}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}. \quad (41)$$

In particular, if $m \rightarrow 1^-$, then we have the dark soliton solutions:

$$u(x, t) = \left(\frac{3L_1^2}{2L_5} \right)^{\frac{1}{4}} \tanh \left(\sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (42)$$

and

$$v(x, t) = \chi \left(\frac{3L_1^2}{2L_5} \right)^{\frac{1}{4}} \tanh \left(\sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (43)$$

as well as the singular soliton solution:

$$u(x, t) = \left(\frac{3L_1^2}{2L_5} \right)^{\frac{1}{4}} \coth \left(\sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (44)$$

and

$$v(x, t) = \chi \left(\frac{3L_1^2}{2L_5} \right)^{\frac{1}{4}} \coth \left(\sqrt{\frac{L_1}{2}} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (45)$$

while if $m \rightarrow 0^+$, then we have the periodic solutions:

$$u(x, t) = \left(\frac{6L_1^2}{L_5} \right)^{\frac{1}{4}} \operatorname{csc} \left(\sqrt{L_1} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (46)$$

and

$$v(x, t) = \chi \left(\frac{6L_1^2}{L_5} \right)^{\frac{1}{4}} \csc \left(\sqrt{L_1} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (47)$$

provided $L_5 > 0$.

Figure-1

Figure-1. Plot of the dark soliton solutions (42) and (43) with $a_1 = 0.4$, $b_1 = 0.1$, $\kappa = 1$, $\chi = 2$, $\xi_1 = 0.2$, $\eta_1 = 0.3$, $\zeta_1 = 0.22$, $v = -1$ and $-10 \leq x, t \leq 10$.

Family-2. If $C_0 = \frac{C_4^3}{32C_6^2(1-m^2)}$, $C_2 = \frac{C_4^2(4m^2-5)}{16C_6(m^2-1)}$, $C_6 > 0$, then

$$u(x, t) = \left(\frac{6L_1^2(m^2-1)^2}{L_5(2m^2-1)^2} \right)^{\frac{1}{4}} \text{nc} \left(\sqrt{\frac{L_1}{1-2m^2}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (48)$$

and

$$v(x, t) = \chi \left(\frac{6L_1^2(m^2-1)^2}{L_5(2m^2-1)^2} \right)^{\frac{1}{4}} \text{nc} \left(\sqrt{\frac{L_1}{1-2m^2}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (49)$$

or

$$u(x, t) = \left(\frac{6L_1^2}{L_5(2m^2-1)^2} \right)^{\frac{1}{4}} \text{ds} \left(\sqrt{\frac{L_1}{1-2m^2}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (50)$$

and

$$v(x, t) = \chi \left(\frac{6L_1^2}{L_5(2m^2-1)^2} \right)^{\frac{1}{4}} \text{ds} \left(\sqrt{\frac{L_1}{1-2m^2}} \xi, m \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (51)$$

where $0 < (1-2m^2) < 1$ and $L_5 > 0$. In particular, if $m \rightarrow 0^+$, in (48) and (49), then we have the periodic solutions:

$$u(x, t) = \left(\frac{6L_1^2}{L_5} \right)^{\frac{1}{4}} \sec \left(\sqrt{L_1} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (52)$$

and

$$v(x, t) = \chi \left(\frac{6L_1^2}{L_5} \right)^{\frac{1}{4}} \sec \left(\sqrt{L_1} \xi \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (53)$$

while if $m \rightarrow 0^+$, in (50) and (51), then we have the same periodic solutions (46) and (47).

4 The addendum to Kudryashov's method

According to this method [2, 19-21], we balance $P_1^{(4)}$ with P_1^5 in Eq. (31), we get

$$M + 4p = 5M \implies M = p. \quad (54)$$

Now, the following cases can be considered.

Case-1. Choose $p = 1$, then $M = 1$. Now, we have the formal solution

$$P_1(\xi) = B_0 + B_1 R(\xi), \quad (55)$$

where B_0 and B_1 are parameters, provided $B_1 \neq 0$. Here $R(\xi)$ satisfies the differential equation:

$$R'^2(\xi) = R^2(\xi) [1 - \lambda R^2(\xi)] \ln^2 K, \quad 0 < K \neq 1, \quad (56)$$

where λ is a constant. Substituting (55), (56) in (31) and setting all the coefficients of $[R(\xi)]^{u_1}$ $[R'(\xi)]^{u_2}$, ($u_1 = 0, \dots, 5$, $u_2 = 0, 1$) to zero, we have the results

$$B_0 = 0, B_1 = \sqrt{\frac{2\lambda (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}}, \quad (57)$$

and

$$L_3 = -(L_1 + \ln^2 K) \ln^2 K, L_5 = \frac{3 (L_1 L_2 + 2L_2 \ln^2 K - 4L_4) (L_4 + 2L_2 \ln^2 K)}{2 (L_1 + 10 \ln^2 K)^2}, \quad (58)$$

provided $\lambda (L_4 + 2L_2 \ln^2 K) > 0$. Substituting (57) along with the well-known solution of Eq. (56) obtained in [2, 19-21] in (55), we have the solutions:

$$u(x, t) = \sqrt{\frac{2\lambda (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left(\frac{4S}{4S^2 K^\xi + \lambda K^{-\xi}} \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (59)$$

and

$$v(x, t) = \chi \sqrt{\frac{2\lambda (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left(\frac{4S}{4S^2 K^\xi + \lambda K^{-\xi}} \right) e^{i(-\kappa x + \omega t + \theta_0)}. \quad (60)$$

The solutions (59), (60) can be rewritten in the form:

$$u(x, t) = \sqrt{\frac{2\lambda (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left[\frac{4S}{(4S^2 + \lambda) \cosh(\xi \ln K) + (4S^2 - \lambda) \sinh(\xi \ln K)} \right] e^{i(-\kappa x + \omega t + \theta_0)}, \quad (61)$$

and

$$v(x, t) = \chi \sqrt{\frac{2\lambda (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \left[\frac{4S}{(4S^2 + \lambda) \cosh(\xi \ln K) + (4S^2 - \lambda) \sinh(\xi \ln K)} \right] e^{i(-\kappa x + \omega t + \theta_0)}, \quad (62)$$

which represent the combo bright-singular soliton solution. In particular, when $\lambda = 4S^2$, we have the bright soliton solution:

$$u(x, t) = \sqrt{\frac{2 (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \operatorname{sech}(\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (63)$$

and

$$v(x, t) = \chi \sqrt{\frac{2 (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \operatorname{sech}(\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (64)$$

provided $(L_4 + 2L_2 \ln^2 K) > 0$, while if $\lambda = -4S^2$, we have the singular soliton solutions:

$$u(x, t) = \sqrt{-\frac{2 (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \operatorname{csch}(\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (65)$$

and

$$v(x, t) = \chi \sqrt{-\frac{2 (L_1 + 10 \ln^2 K) \ln^2 K}{(L_4 + 2L_2 \ln^2 K)}} \operatorname{csch}(\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (66)$$

provided $(L_4 + 2L_2 \ln^2 K) < 0$. The solutions (59)-(66) exist under the conditions (58).

Figure-2

Figure-2. Plot of the bright soliton solutions (63) and (64) with $a_1 = 0.4$, $b_1 = 0.1$, $\kappa = 1$, $\chi = 2$, $\alpha_1 = 0.6$, $d_1 = 0.25$, $c_1 = 0.25$, $\gamma_1 = 0.2$, $\rho_1 = -0.5$, $\sigma_1 = 0.33$, $v = -1$, $K = 3$, $f_1 = 0.22$, $g_1 = 0.1$,

$v = -1$ and $-10 \leq x, t \leq 10$.

Case-2. Choose $p = 2$, then $M = 2$. Now, we have the formal solution

$$P_1(\xi) = B_0 + B_1 R(\xi) + B_2 R^2(\xi), \quad (67)$$

where B_0, B_1 and B_2 are parameters, provided $B_2 \neq 0$. Here $R(\xi)$ satisfies the differential equation:

$$R'^2(\xi) = R^2(\xi) [1 - \lambda R^4(\xi)] \ln^2 K, \quad 0 < K \neq 1, \quad (68)$$

where λ is a constant. Substituting (67) and (68) into (31) and setting all the coefficients of $[R(\xi)]^{u_1} [R'(\xi)]^{u_2}$, ($u_1 = 0, \dots, 10$, $u_2 = 0, 1$) to zero, we have the results

$$B_0 = 0, B_1 = 0, B_2 = \sqrt{\frac{8\lambda (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}}, \quad (69)$$

and

$$L_3 = -4 \ln^2 K (L_1 + 4 \ln^2 K), L_5 = \frac{3 (L_1 L_2 + 8L_2 \ln^2 K - 4L_4) (L_4 + 8L_2 \ln^2 K)}{2 (L_1 + 40 \ln^2 K)^2}, \quad (70)$$

provided $\lambda (L_4 + 8L_2 \ln^2 K) > 0$. Now, we have the results:

$$u(x, t) = \sqrt{\frac{8\lambda (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \left(\frac{4S}{4S^2 K^{2\xi} + \lambda K^{-2\xi}} \right) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (71)$$

and

$$v(x, t) = \chi \sqrt{\frac{8\lambda (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \left(\frac{4S}{4S^2 K^{2\xi} + \lambda K^{-2\xi}} \right) e^{i(-\kappa x + \omega t + \theta_0)}. \quad (72)$$

The solutions (71), (72) can be rewritten in the form

$$u(x, t) = \sqrt{\frac{8\lambda (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \left[\frac{4S}{(4S^2 + \lambda) \cosh(2\xi \ln K) + (4S^2 - \lambda) \sinh(2\xi \ln K)} \right] e^{i(-\kappa x + \omega t + \theta_0)}, \quad (73)$$

and

$$v(x, t) = \chi \sqrt{\frac{8\lambda (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \left[\frac{4S}{(4S^2 + \lambda) \cosh(2\xi \ln K) + (4S^2 - \lambda) \sinh(2\xi \ln K)} \right] e^{i(-\kappa x + \omega t + \theta_0)}, \quad (74)$$

which represent the combo bright-singular soliton solution. In particular, when $\lambda = 4S^2$, we have the bright soliton solution:

$$u(x, t) = \sqrt{\frac{8 (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \operatorname{sech}(2\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (75)$$

and

$$v(x, t) = \chi \sqrt{\frac{8 (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \operatorname{sech}(2\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (76)$$

provided $(L_4 + 8L_2 \ln^2 K) > 0$, while if $\lambda = -4S^2$, we have the singular soliton solution:

$$u(x, t) = \chi \sqrt{-\frac{8 (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \operatorname{csch}(2\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (77)$$

and

$$v(x, t) = \chi \sqrt{-\frac{8 (L_1 + 40 \ln^2 K) \ln^2 K}{(L_4 + 8L_2 \ln^2 K)}} \operatorname{csch}(2\xi \ln K) e^{i(-\kappa x + \omega t + \theta_0)}, \quad (78)$$

provided $(L_4 + 8L_2 \ln^2 K) < 0$. The solutions (71)-(78) exist under the conditions (70).

Similarly, we can find many other solutions by choosing other values for p and M .

5 Conclusions

Many exact solutions have been obtained for the coupled system of cubic-quartic nonlinear Schrödinger equation (CQ-NLSE) in fiber Bragg gratings with parabolic-nonlocal combo law of refractive index. The two integrated schemes, namely the extended auxiliary equation method and the addendum to Kudryashov's method are applied. The Jacobi elliptic function solutions, bright, dark and singular solitons from the governing model are obtained. These soliton solutions depend on certain constraint conditions and are given together with their existence criteria. Finally, our solutions have been checked using the Maple by putting them back into the original equations.

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