Bounded States for Breathers–soliton and Breathers of sine–Gordon Equation

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Abstract The Wronskians solutions to the sine-Gordon (sG) equation that can provide interaction of different kinds of solutions are revisited. A novel expression N-soliton solution with a nonzero background is presented which is used to construct the soliton and breather solutions. Due to the existences of abundant structures of the solitons and breathers, it is possible to search for the coherent structures, or bounded states of solitons and breathers. By introducing the velocity resonant conditions, the sG equation is proved to possess the bounded state for breathers-soliton, or breather-soliton molecules (BSMs) and the bounded state for breathers, or breather molecules (BM) by different parameters selections. In addition, an approximately bounded state for solitons is demonstrated. The interesting thing is the interactions among the BSMs, BMs and solitons, breathers may be nonelastic by the particular meaning the sizes of the BSMs and BMs change.

Keywords Sine-Gordon equation · Breathers–soliton molecules · breather molecules · unelastic interaction

1 Introduction

The sine-Gordon (sG) equation first arose in the theory of differential geometry and later is used to describe the charged particles in electromagnetic field with characteristics coordinates [2], [3]. Also, sG equation arises widely in many physical systems, such as the kinks bounded by fermions [4], the one-dimensional domain wall motion in biaxial ferromagnets [5], the propagation of magnetic flux in Josephson-junction [6], [7], the problem of observer-based boundary control [8], the quantum field theories [9], etc. Due to its widely applications, the sG equation has attracted great interests in both physics

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and mathematics. It has been proved to possess Hamiltonian formalism, the bilinear form, Darboux transformations and Bäcklund transformations, Lax pairs, etc. The sG equation is found to possess not only the kink solutions and breather solutions, but also multiply periodic wave solutions by means of the new deformation relations [10].

Recently, the studies on soliton molecules have been a popular topic in nonlinear systems. SMs are considered as a special coherent states or bounded states for multiple-soliton solitons. Many theories on SMs, BMs have been presented and some experiments have been conducted to search for the special states, especially in nonlinear optics. Some experiments have successfully observed the special states. It is particular that the BSM is also reported theoretically in the mKdV-sG system describing few-optical-cycle solitons [11] under some approximation. The kink bounded states by fermions have been found in ϕ^4 , ϕ^6 and sG models [4]. Actually, this static multisoliton bound states are just the coherent state, or bounded state for the soliton solutions. Because the sG equation possesses abundant kinks (solitons) and breathers structures, it is possible for us to search for more coherent and bounded state of the solitons.

In our manuscript, we first revisit the Wronskians and solutions to the s-G equation, then present a novel expression for the multiple soliton solution to the sG equation. The novel multiple soliton solution is different from other results because it has a nonzero background term. The background term, has been proved to play an important role in some nonlinear phenomenon [12], [13]. Then, to generate the bounded states for solitons and breather, we introduce some velocity resonant conditions. The sG equation, is found to possess the two bounded states, the bounded state for breathers-soliton, named as breather-soliton molecules (BSMs) and the bounded state for breathers, named as breather molecules (BM). Though sG equation can be considered as a special case of mKdV-sG equation, the results are different from the mKdV-sG equation that possesses bounded states for soliton molecules (BM-s), BMs and BSMs. In addition, an approximate bounded state for solitons are also demonstrated if the parameters are nearly the same. Further studies on the interactions among BSMs, BMs, solitons and breathers show the nonelastic properties for the interactions.

2 Wronskians and solutions to sG equation

The Lagrangian for the sG equation in $(1 + 1)$ -dimensional field theory is defined by

$$\mathbb{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} [i \gamma^\mu \partial_\mu - m - g \phi] \psi - U(\phi), \quad (1)$$

where $U(\phi)$ represents the periodic potential of the self-interacting real scalar field ϕ , ψ is the Dirac spinor, m refers to the mass of the fermions and g is a constant called the Yukawa coupling constant. γ_μ are matrices with $\gamma_0 = \sigma_1$,

$\gamma_1 = i\sigma_3$, where σ_i are the Pauli matrices and $\bar{\psi} = \psi^\dagger \gamma^0$. With an infinite number of degenerate vacua $\phi_0 = 2n\pi$, $n \in \mathbb{Z}$ for the potential $U(\phi) = 1 - \cos \phi$ and the decoupled limit $g = 0$, the dynamical equation of Eq. (1) eventually leads to the famous sG equation [4], [14], [15]

$$u_{xt} = \sin u, \quad (2)$$

where we have replaced ψ by u .

Mathematicians [11], [16], [17] have connected the N -soliton solutions of nonlinear evolution systems obtained from inverse scattering theory to Wronskian forms. They believe the Wronskian forms are one of the most effective methods to search for the interactions of various types of solutions.

It is known the N th order Wronskian is defined as a $N \times N$ determinant

$$W(\phi_1, \phi_2, \dots, \phi_N) = |\phi, \phi^{(1)}, \dots, \phi^{(N)}| = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \dots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \dots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \dots & \phi_N^{(N-1)} \end{vmatrix}, \quad (3)$$

where ϕ is named the entry vector of the Wronskian with $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ and $\phi_m^{(n)} = \frac{\partial^n \phi_m}{\partial x^n}$. For simplicity, the Wronskian Eq. (3) is generally rewritten in the compact form [17]

$$W(\phi) = |\phi, \phi^{(1)}, \dots, \phi^{(N)}| = |0, 1, \dots, N-1| = |\widehat{N-1}|, \quad (4)$$

where $\widehat{N-m}$ denotes the set of consecutive columns $0, 1, \dots, N-m$.

It is well known that the transformation [18]

$$u = 2i \ln \frac{f^*}{f}, \quad (5)$$

where f^* means the complex conjugate of f transforms the sG equation Eq. (2) into the bilinear form

$$(D_x D_t) f \cdot f^* - \frac{1}{2}(f^2 - f^{*2}) = 0. \quad (6)$$

Here D_t and D_x are the famous bilinear operators first introduced by Hirota [19] with the definitions being

$$D_t^m D_x^n f(t, x) \cdot g(t, x) = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} f(t+s, x+y) \cdot g(t-s, x-y)|_{s=0, y=0}.$$

Eq. (6) is completely equivalent to the sG equation, so once f is known, the solution to sG equation is obtained by the transformation Eq. (5). According to [11], the solution f takes the following Wronskian form

$$f = W(\phi) = |\widehat{N-1}|, \quad (7)$$

where the entry vector ϕ is determined by

$$\phi_x = \mathbb{B}\phi^*, \quad \phi_t = \frac{1}{4}\partial^{-1}\phi, \quad (8)$$

with $\mathbb{B}$ being an N th order nonsingular complex constant matrix and $|\mathbb{B}| \in \mathbb{R}$ and $\partial^{-1}\partial = \partial\partial^{-1} = 1$.

Upon different choice of $\mathbb{B}$, some exact solutions of the sG equation Eq. (2) can be constructed. Here we just show some special known results to the sG equation Eq. (2) according to the complex constant matrix $\mathbb{B}$.

1. Soliton solution

The soliton solution of the sG equation can be obtained by selecting

$$\mathbb{B} = \text{Diag}(k_1, k_2, \dots, k_N), \quad (9)$$

where k_j is real and $|k_i| \neq |k_j|$ ($i \neq j$), with the solution $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ to Eq. (8) being expressed by

$$\phi_j = a_j^+ e^{\eta_j} + i a_j^- e^{-\eta_j}, \quad \eta_j = k_j x - \frac{t}{k_j} + \eta_j^{(0)}, \quad (10)$$

with a_j^+ , a_j^- , k_j and $\eta_j^{(0)}$ are all arbitrary real constants. Particularly, if $a_j^+ = (-1)^{j-1}$, $a_j^- = -1$, the Wronskian solution f is simplified to the following form [11]

$$f = \sum_{\nu=0,1} \exp \left[\sum_{j=1}^N \nu_j \left(\eta_j + i \frac{\pi}{2} \right) + \sum_{1 \leq j < l}^N \nu_j \nu_l A_{jl} \right], \quad (11)$$

where the sum over $\nu = 0, 1$ refers to each of $\nu = 0, 1$ for $j = 1, 2, \dots, N$ and

$$e^{A_{jl}} = \left(\frac{k_l - k_j}{k_l + k_j} \right)^2. \quad (12)$$

The solutions Eqs. (11)–(12) are equivalent to the results obtained from the Hirota bilinear form [20].

2. Breather solution

When the N th order nonsingular complex constant matrix $\mathbb{B}$ is written as

$$\mathbb{B} = \text{Diag}(\theta_1, \theta_2, \dots, \theta_N)_{2N}, \quad \theta_j = \begin{pmatrix} 0 & k_j \\ k_j^* & 0 \end{pmatrix}, \quad (13)$$

with $k_j \in \mathbb{C}$ and k_j^* being the complex conjugate of k_j , the Wronskian entry vector $\phi = (\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}, \phi_{N1}, \dots, \phi_{N2})^T$ has the solution

$$\phi_{j1} = a_j e^{\eta_j} + b_j e^{-\eta_j}, \quad \phi_{2j} = a_j^* e^{\eta_j^*} - b_j^* e^{-\eta_j^*}, \quad \eta_j = k_j x - \frac{t}{k_j} + \eta_j^{(0)}, \quad (14)$$

where $a_j, b_j, \eta_j^{(0)}$ are all complex. The structure of $\mathbb{B}$ and θ_j for breathers in Eq. (13) was early found in [16] and [21]. From Eqs. (7) and (13)–(14), the breathers solution to sG equation is thus constructed.

Furthermore, the limit soliton solutions and limit solution of breathers can also be obtained by different N th order nonsingular complex constant matrix of $\mathbb{B}$. Based on the eigenfunctions of the matrix $A = \mathbb{B}\mathbb{B}$, a full classification for the solutions to the mKdV equation has been done [16].

3 A Novel soliton solution to sG equation

In this section, we present a novel expression for soliton solution to sG equation by utilizing the expression Eq. (11). The N -soliton solutions to sG equation Eq. (2) on a nonzero background is given by

$$\begin{aligned} u &= N\pi + 2i \left[\ln \frac{\sum_{\nu} K_{\nu} \cosh(\frac{1}{2} \sum_{j=1}^N \nu_j \eta_j^-)}{\sum_{\nu} K_{\nu} \cosh(\frac{1}{2} \sum_{j=1}^N \nu_j \eta_j^+)} \right] \\ &= N\pi + 4 \arctan \left[\frac{\sum_{\nu_e} K_{\nu} \sinh(\frac{1}{2} \sum_{j=1}^N \nu_j \eta_j)}{\sum_{\nu_o} K_{\nu} \cosh(\frac{1}{2} \sum_{j=1}^N \nu_j \eta_j)} \right], \\ \eta_j^{\pm} &= k_j x + \frac{t}{k_j} + \eta_{j0} \pm i \frac{\pi}{2} \equiv \eta_j \pm i \frac{\pi}{2}, \end{aligned} \quad (15)$$

where the summation of ν should be done to all the permutations of $\{\nu_1, \nu_2, \dots, \nu_N\}$ for $\nu = 1, -1$, the summation of ν_e means all non-dual even permutations of $\{\nu_1, \nu_2, \dots, \nu_N\}$ for $\nu = 1, -1$ and the summation of ν_o means all non-dual odd permutations of $\{\nu_1, \nu_2, \dots, \nu_N\}$ for $\nu = 1, -1$, with

$$K_{\nu} \equiv \prod_{i>j} (k_i - \nu_i \nu_j k_j), \quad (16)$$

η_{j0} being an arbitrary real constant denoting the initial positions of every solitons.

One can directly check the results by substituting the solution Eq. (15) into the sG equation Eq. (2) by using the expression Eq. (16).

With Eq. (15), the one soliton solution to sG equation is written down

$$u = \pi + 4 \arctan \left[\tanh \left(\frac{k_1 x + t/k_1 + \eta_{10}}{2} \right) \right], \quad (17)$$

which is a kink wave or an anti-kink wave (actually, a soliton by u_x) for different selection of k_1 with the propagation speed provided by

$$v_s = \frac{1}{k_1^2}. \quad (18)$$

The character of the kink wave can be made by examining the asymptotic behaviour of u as $t \rightarrow \infty$ [22]. Here we don't show it in detail.

Fig. 1 A single breather presented by Eq. (20) with selections of the parameters being $\lambda_1 = \mu_1 = 1$, $x_0 = t_0 = 0$ for (a) the evolution plot and (b) the density plot.

For $N = 2$, the soliton solution is expressed as

$$u = 2\pi + 4 \arctan \left[\frac{(k_2 - k_1) \sinh \frac{\eta_1 + \eta_2}{2}}{(k_2 + k_1) \cosh \frac{\eta_1 - \eta_2}{2}} \right], \quad (19)$$

with $\eta_j = k_j x + \frac{t}{k_j} + \eta_{j0}$, $j = 1, 2$. This solution Eq. (19) describes the interactions among kink (or anti-kink) and kink (or anti-kink).

The breather solution is obtained from the two-soliton solution Eq. (19) directly by selecting $k_1 = -k_2^* = \lambda_1 + i\mu_1$ and $\eta_{10} = -\eta_{20}^* = x_1 + it_1$,

$$u = 2\pi + 4 \arctan \left[\frac{\lambda_1 \sin \left(\mu_1 x - \frac{\mu_1 t}{\lambda_1^2 + \mu_1^2} + t_1 \right)}{\mu_1 \cosh \left(\lambda_1 x + \frac{\lambda_1 t}{\lambda_1^2 + \mu_1^2} + x_1 \right)} \right]. \quad (20)$$

The breather solution Eq. (20) shows an oscillating periodically localized wave propagating along a straight line with the speed

$$v_B = \frac{1}{\lambda_1^2 + \mu_1^2}. \quad (21)$$

The behaviour of the breather has also been studied in [22] by using the dressing method. A sketch of breather is exhibited in Fig. 1 with the parameters selected as $\lambda_1 = \mu_1 = 1$, $x_0 = t_0 = 0$.

It can be concluded that there are abundant different solution structures according to the choice of the parameters k_i . For example, for $N = 3$ in Eq. (15), if $k_i \in \mathbb{R}$, $i = 1, 2, 3$, the solution denotes three solitons interactions by u_x . If any two k_i is complex, the solution u_x stands for the interactions between a soliton and breathers. The density plot for the interaction between a soliton and a breather is given in Fig. 2 where the parameters are $k_1 = k_2^* = 4/5 + 4i/5$, $k_3 = 3/5$, $\eta_{10} = \eta_{20} = \eta_{30} = 0$ in Eq. (15).

It should be pointed out that u_x is shown because u_x stands for different physical meaning. For example, u_x is the electric field intensity when the sG equation is used to describe the ultrashort optical pulse propagation in resonant medium [2] and the short-wave approximation regime [23].

Fig. 2 Density plot for u_x that exhibiting the interaction between soliton and breather.

4 Coherent Structures for Soliton-Breather Molecules and Breather Molecules

The existence of the abundant structures of the solutions to the sG equation Eq. (2) makes it possible to search for some special coherent structures or bounded states of solitons and breathers by using the velocity resonant conditions [12], [24], [25]. The velocity resonance condition means the solitons propagate with the same speed, thus they will form some special structures that can be used to describe significant physical phenomena, such as the few-cycle-solitons in optics, the nonlinear nonintegrable optical model with dispersions and nonlinear effects including self-steeping, Raman scattering and nonlinear dispersion. The coherent structures are named as soliton molecules in nonlinear optical physics [26]. By now, the BMs, the SMs and the BSMs are found in most nonlinear systems, even in the case of nonintegrable systems [27].

For the sG equation, we find it possesses the bounded state for breathers and soliton, i.e., the BSMs, and the bounded state for breathers, i.e., the BMs by utilizing the velocity resonant conditions.

4.1 Bounded states for breathers and soliton

If m breathers and one-soliton share the same the velocity, say

$$v_s = v_{B_1} = \cdots = v_{B_m}, \quad \text{i.e.,} \quad \frac{1}{k_{2m+1}^2} = \frac{1}{\lambda_1^2 + \mu_1^2} = \cdots = \frac{1}{\lambda_m^2 + \mu_m^2}, \quad (22)$$

they form a special coherent structure of BSM by suitable parameters selections. In other words, the BSM structure contains one soliton and m breathers. This BSM structure is generally found in $(2m+1)$ -soliton solutions where the parameters are selected as $k_i = k_{i+1}^* = \lambda_i + i\mu_i$, $i = 1, 2, \dots, m$ and k_{2m+1} is real, and all the parameters are determined by the velocity resonant conditions Eq. (22).

Here we show two particular bounded states for BSMs. The first case is $m = 1$ for the three-soliton solution. With the choice of the parameters

$$k_1 = k_2^* = 1 + \frac{4}{5}i, \quad \eta_{10} = \eta_{20} = 0, \quad k_3 = \frac{\sqrt{41}}{5}, \quad \eta_{30} = 10, \quad (23)$$

Fig. 3 The BSM structure for $N = 3$ in Eq. (15) where the parameters are $\lambda_1 = 1$, $\mu_1 = 4/5$, $x_0 = t_0 = 0$, $k_3 = \sqrt{41}/5$ and $\xi_{30} = 10$.

Fig. 4 Density plot for a BSM constituted of two breathers and one soliton with the parameters being selected as Eq. (24).

being constrained by the velocity resonant condition Eq. (22), a BSM consisted of one breather and one soliton is thus observed. The corresponding density plot for u_x is illustrated in Fig. 3.

The second case is for $m = 2$ in the five-soliton solution to sG equation. It is not difficult to find that the soliton solution for $N = 5$ leads to a BSM contained with one soliton and two breathers by using Eq. (22). Such a BSM is exhibited in Fig. 4 by the choice of parameters being

$$\begin{aligned} k_1 = k_2^* &= \frac{3}{5} + \frac{1}{2}i, & k_3 = k_4^* &= \frac{61}{100} + \frac{\sqrt{2379}}{100}, & k_5 &= \frac{\sqrt{61}}{10}, \\ \eta_{10} = \eta_{20}^* &= 5 + 5i, & \eta_{30} = \xi_{40} &= 0, & \eta_{50} &= 20. \end{aligned} \quad (24)$$

In general, the bounded states for BSMs depend on not only the parameters selections k_i , but also the number of solitons, i.e., N .

4.2 Bounded states for breathers

The velocity resonance condition may also arise in breathers. The $2m$ -soliton solution to the sG equation becomes to m -breather interaction solution by the selections of $k_i = k_{i+1}^* = \lambda_i + i\mu_i$ with $\xi_{i0} = \xi_{i+10}^* = x_{i0} + it_{i0}$, $i = 1, 2, \dots, m$. If m breathers travel its own path with the same speed, they will form the

Fig. 5 Density plot of the BM for $N = 4$ with the parameters selected as Eq. (26).

Fig. 6 A BM consisted of three breathers for $N = 6$ with the parameters being Eq. (26)–(27) for $N = 6$.

bounded states called BMs. The generation of bounded states is also related to the velocity resonance condition, i.e.,

$$v_{B_1} = v_{B_2} = \cdots = v_{B_m}, \quad \text{i.e.,} \quad \frac{1}{\lambda_1^2 + \mu_1^2} = \frac{1}{\lambda_2^2 + \mu_2^2} = \cdots = \frac{1}{\lambda_m^2 + \mu_m^2} \quad (25)$$

Fig. 5 exhibits a special BM consisted with two breathers for the density plot with the parameters selected as

$$\begin{aligned} k_1 = k_2^* &= \frac{3}{5} + \frac{i}{2}, & k_3 = k_4^* &= \frac{61}{100} + \frac{i\sqrt{2379}}{100}, \\ \eta_{10} = \eta_{20}^* &= 5 + 5i, & \eta_{30} = \eta_{40}^* &= -5 - 5i, \end{aligned} \quad (26)$$

in the four-soliton solution Eq. (15). From the profile, we see the two breathers propagate their own straight line separately with same speed as expected.

Furthermore, by introducing another breather with the parameters

$$k_5 = k_6^* = \frac{11}{20} + \frac{\sqrt{123}}{20}i, \quad \eta_{50} = \eta_{60}^* = -1 - i, \quad (27)$$

for $N = 6$ in the solution Eq. (15), a BM contains three breathers is shown in Fig. 7.

In conclusion, the velocity resonant condition is an effective way to construct the bounded states for solitons and breathers. The parameters selections affect both the soliton shapes (soliton or breather) and the bounded states.

Fig. 7 An approximately bounded state for solitons for $N = 2$ at $t = 0$ where u_1 denotes u_x .

Unfortunately, the velocity resonant can't take place in solitons because the speed of the soliton is determined by Eq. (18). So the sG equation do not possess the bounded states for SMs which is significantly different from the mKdV-sG equation and the complex mKdV equation. The bounded state for kinks [4] in $(1+1)$ -dimensional scalar field theories, may approximately occur in some special areas for the parameters for k_i are nearly the same. As an example, if the parameters for $k_1 = 1$, $k_2 = 0.9999$ and $\eta_{10} = \eta_{20} = 0$, the two solitons are approximately bounded together. Fig. 7 shows the two-soliton in approximately bounded state for $t = 0$. Because the velocities are determined by Eq. (18), we know the two solitons will meet eventually after sometime. Clearly, the approximately bounded state occurs in some special time and areas for some physical systems.

5 Interactions among BSMs, BMs, solitons and breathers

5.1 Interactions among BSMs, solitons and breathers

It is interesting the multiple soliton solutions exhibit the interactions among BSMs, BMs solitons and breathers with suitable parameters selections. The interactions, in some particular senses may not be elastic.

For example, with the choice of $k_1 = k_2^* = \lambda_1 + i\mu_1$ and $\eta_{10} = \eta_{20}^* = x_{10} + it_{10}$ for $N = 4$, and k_3 being determined by the velocity resonant Eq. (22) in Eq. (15), an interaction between BSM and a soliton is therefore constructed. The profile of the special interaction is exhibited in Fig. 8.

From the profile, we see the distance between breather and soliton of the BSM becomes larger after the interaction with the other soliton. In other words, the size of the BSM has been changed after the interaction. From the point view of the BSM size changes, the interactions among BSM and soliton are considered as nonelastic. The same nonelastic phenomena have also been reported in [12] for a model describing the few-cycle-optical pulses beyond slowly varying envelope approximation.

The properties of this nonelastic interaction can be made evident by examining the asymptotic behavior of $u(x, t)$ as $t \rightarrow \pm\infty$ by using dressing method [22], the inverse scattering method [28], etc. According to [22], the positional

Fig. 8 A density profile of u_x for the interaction between BSM and soliton for $N = 4$ in the Eq. (15) with the parameters being $k_1 = k_2^* = 1 + i/2$, $k_3 = \sqrt{5}/2$, $k_4 = 1$, $\eta_{10} = \eta_{20} = 0$, $\eta_{30} = 10$, $\eta_{40} = 5$.

shift due to collisions in the soliton (kink) and breather for the breather is

$$\Delta_B = \frac{1}{\lambda_1} \ln \frac{(k_4 + \lambda_1)^2 + \mu_1^2}{(k_4 - \lambda_1)^2 + \mu_1^2},$$

while the position shift for the kink (soliton) in BSM caused by the interactions of solitons is

$$\Delta_s = \frac{2}{k_3} \ln \frac{k_4 - k_3}{k_4 + k_3},$$

thus the whole position shift for the BSM is obtained

$$\Delta_1 = \Delta_s + \Delta_B = \frac{2}{k_3} \ln \frac{k_4 - k_3}{k_4 + k_3} + \frac{1}{\lambda_1} \ln \frac{(k_4 + \lambda_1)^2 + \mu_1^2}{(k_4 - \lambda_1)^2 + \mu_1^2}. \quad (28)$$

By using the velocity resonant condition $k_3^2 = \lambda_1^2 + \mu_1^2$, the position shift generally holds $\Delta_1 \neq 0$. This will naturally lead to the size of BSM changes after the collision.

Similarly, for $N = 5$, the interaction between BSM and a breather can be constructed by suitable parameters selections. Fig. 9 shows the special interaction for BSM and breather where the parameters are selected based on the velocity resonant condition Eq. (22),

$$\begin{aligned} k_1 = k_2^* = \lambda_1 + i\mu_1 = 1 + i/2, \quad k_3 = k_4^* = \lambda_2 + i\mu_2 = \frac{\sqrt{3}}{5} + \frac{i}{3}, \\ k_5 = \frac{\sqrt{5}}{2}, \quad \eta_{10} = \eta_{20} = \eta_{30} = \eta_{40} = 0, \quad \eta_{50} = 5. \end{aligned} \quad (29)$$

The position shift caused by BSM and breather interaction is exactly determined by breather-breather interaction and breather-soliton interaction, with the result presented by

$$\begin{aligned} \Delta_2 = \frac{2}{\lambda_1} \ln \frac{[(\lambda_1 - \lambda_2) + (\mu_1 - \mu_2 i)][(\lambda_1 + \lambda_2) - (\mu_1 - \mu_2 i)]}{[(\lambda_1 + \lambda_2) + (\mu_1 + \mu_2 i)][(\lambda_1 + \lambda_2) - (\mu_1 + \mu_2 i)]} \\ + \frac{1}{\lambda_1} \ln \frac{(k_5 + \lambda_1)^2 + \mu_1^2}{(k_5 - \lambda_1)^2 + \mu_1^2}. \end{aligned} \quad (30)$$

Fig. 9 Density profile for BSM and breather interaction with parameters determined by Eq. (29).

Fig. 10 The nonelastic interaction between BSMs for $N = 6$ by selecting the parameters as Eq. (31).

Thus the size of the BSM is also changed after the interaction of BSM and breather. Same as BSM and soliton interaction, the interaction among BSM and breather, is still nonelastic by the meaning the size of the BSM changes.

Moreover, there exist BSMs interactions if the velocities of the BSMs are different. By selecting $k_i = k_{i+1}^* = \lambda_i + i\mu_i$ with $\xi_{i0} = \xi_{i+10}^* = x_{i0} + it_{i0}$, $i = 1, 2$ in the six-soliton solution, where $\{\lambda_1, \mu_1, k_5\}$ and $\{\lambda_2, \mu_2, k_6\}$ are given by the velocity resonant condition, respectively, the interactions between BSMs are constructed and demonstrated in Fig. 10 with the choice of the parameters

$$\begin{aligned}
 k_1 = k_2^* &= 1 + \frac{4\sqrt{3}}{15}i, & k_3 = k_4^* &= \frac{6}{5} + \frac{\sqrt{21}}{10}i, & k_5 &= \frac{3}{5}, \\
 k_6 &= -\frac{9}{10}, & \eta_{10} = \eta_{20}^* &= 1 + i, & \eta_{30} = \eta_{40}^* &= -1 + i, \\
 \eta_{50} &= 5, & \eta_{60} &= -10.
 \end{aligned} \tag{31}$$

5.2 Interactions among BMs, solitons and breathers

There exist other kinds of interactions, such as the interaction between BMs and solitons, the interactions between BMs and breathers. One of the examples

Fig. 11 Density plot for the interaction between BM and soliton for $N = 6$ in Eq. (15) with the parameters determined by Eq. (33).

Fig. 12 Density plot for the interaction between BM and soliton with the parameters being determined by Eq. (33).

is for the five-soliton. If the parameters are selected as

$$\begin{aligned} k_1 = k_2^* &= \frac{3}{5} + \frac{1}{2}i, & k_3 = k_4^* &= \frac{61}{100} + \frac{\sqrt{2379}}{100}i, & k_5 &= \frac{1}{3}, \\ \eta_{10} = \eta_{20}^* &= 5 + 5i, & \eta_{30} = \eta_{40}^* &= -5 - 5i, & \eta_{50} &= 0, \end{aligned} \quad (32)$$

the solution displays the interaction between a BM and a soliton which is demonstrated in Fig. 11. By examining the the distance between breathers that consisted of the BM, we believe the interaction between BM and soliton is nonelastic because the BM changes its size after interaction.

It is not difficult to find the six-soliton solution illustrate the interactions between BM and breather if the parameters are chosen to satisfy $v_{B_1} = v_{B_2} \neq v_{B_3}$. Fig. 12 is the density profile of a special interaction between BM and breather where the parameters are

$$\begin{aligned} k_1 = k_2^* &= \frac{3}{5} + \frac{1}{2}i, & k_3 = k_4^* &= \frac{61}{100} + \frac{\sqrt{2379}}{100}i, & k_5 = k_6^* &= \frac{1}{3} + \frac{1}{4}i, \\ \eta_{10} = \eta_{20}^* &= 5 + 5i, & \eta_{30} = \eta_{40}^* &= -5 - 5i, & \eta_{50} = \eta_{60}^* &= -2 - 2i. \end{aligned} \quad (33)$$

The multiple soliton solution to the sG equation makes it possible for us to search for more meaningful coherent structures.

6 Summary and discussions

In summary, we present a novel expression of the N -soliton solution to the sG equation with a nonzero background. The nonzero background term motivates us to search for more meaningful solutions. Recent studies [13], [29] show that some special soliton solutions related to the nonzero background to some nonlinear systems have been found. These soliton solutions, such as few-cycle-soliton solutions first obtained from the travelling wave method are excited by the background. Because almost all the integrable systems possess similar transformations and multisoliton solutions, further studies should be done to make clear descriptions for the background.

The new form of multiple soliton solution is used to construct abundant coherent structures of solitons and breathers by introducing the velocity resonant condition. The mechanism of generating bounded states is related to the velocities of the solitons and breathers. Two special bounded states for sG equation are successfully constructed. In addition, the approximately bounded states for solitons (kinks or anti-kinks) are also studied in detail.

Furthermore, the interactions we constructed among BSMs, BMs soliton and breathers are found to be nonelastic in some particular senses that the sizes of the BMs and BSMs change.

Results show that the mKdV-sG model describing few-optical-cycle solitons possesses abundant SMs, BMs and BSMs structures [12]. Though sG equation can be considered as a special case of the mKdV-sG equation, the sG equation does not possess SMs structures.

Because of the widely applications of sG equation and the observations of the bounded stated in laboratories, we believe it worth further studying to find more possible applications in a wide type of physical systems.

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Conflict of Interest

The authors declare that they have no conflict of interest.

Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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