Rule Reductions of Decision Formal Context Based on Mixed Information

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Rule Reductions of Decision Formal Context Based on Mixed Information

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Abstract Three-way concept lattice extracts knowledge from positive and negative information separately. However, some applications require to mix positive and negative information for management and representation explicitly. In this paper, rule acquisition of FCA is extended based on mixed information. Firstly, two types of mixed concept lattices are studied. Then the relationships between mixed concept and classical concept, three-way concept are explored in depth. Secondly, mixed decision rules are investigated thoroughly. And the weak-basis is put forward to approximate to the basis of mixed decision rules. Finally, the comparison of mixed decision rules and three-way decision rules is perfectly discussed.

Keywords Decision formal contexts · Mixed information · Rule acquisition · Three-way concept lattices

1 Introduction

Formal concept analysis (FCA) is turned in the direction of a useful mathematical method to knowledge representation and knowledge discovery. Nowadays, the theory of FCA has shown strong vitality and broad prospects. It is a successful tool to analyze data aroused scholars' wide concern [1, 2, 7, 8, 17, 23, 33].

In FCA, the formal contexts are analysed by taking usually care of the positive information. Namely,

the presence of an attribute in an object. Few studies paid attention to negative information[4, 9, 10], from which the negation in the formal contexts also bears a lot of useful information. Incorporating with three-way decisions [28–30] and FCA, Qi et al. in 2014 [13] proposed three-way concept analysis (3WCA), in which not only the positive information but also the negative information hidden in the formal context were took into account. Later, Qi et al. [14] introduced the relationship between 3WCA and FCA systematically. To make the given formal concept easily be understood, Ren et al. [21] put subsequently forward the attribute reductions in the framework of 3WCA. Based on three-way analysis, the conflict analysis model was furthermore formulated by Zhi et al. [34].

An important task in FCA is knowledge discovering, which concerns on the tools and techniques to excavate useful information to improve decision making. Rule acquisition (or rule association) is one of the main patterns. From logical viewpoint, the complete theorem of implications as well as the redundances from a formal context were proposed in [15]. Then, Zhai et al. [16, 31, 32] paid attention to decision implication canonical basis, inference rule and the semantic interpretation for decision formal context. To devote to rule acquisition in decision formal context, granular rules [27] and decision rules [5, 6, 22] have been available. In addition, [5] and [6] have further served to derive all the non-redundant ones methodologically.

However, most of the above studies were disable to negative information of contexts actually. And 3WCA thinks over positive and negative information independently. A few works [12, 18] have recently been served to rule acquisition or implications by integrating with negative information. Although Wei et al. [25] studied rules acquisition of decision formal context based

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on three-way concept lattices, from which the rules are the forms of “If A , then C ” and “If not B , then not D ”. And these “If A then not B ” and “If not A then B ” are definitely not turned in 3WCA. In these cases, positive and negative information is took into account separately. Therefore, per se, the information of formal context is also not well handled completely by 3WCA. For instance, a rule that “cyclist with short and sharp accelerations are not great climbers” [17, 18] will not be gained typically from no matter classical formal concept lattice but also three-way formal concept lattice. For this reason, we will address rule acquisition from decision formal context with making a mixed consideration of positive and negative information.

In reality, driven by requirements from practical applications, except for 3WCA, people had made different attempts to extend FCA with negative information. As early as 2000, Wille [26] had put forward diverse types of negation. Then positive and negative implications were dealt with to generate mixed implications in [11]. Subsequently, Missaoui et al. [12] defined a method for generating implications with negation from a provided formal context by a minimal generator. Recently, instead of using the large context, Rodriguez et al. [18] formulated a novel approach to gain mixed implications from a formal context. Furthermore, they [19] studied in depth mixed concept lattices (MCA) and obtained a characterization theorem. To devote to breast cancer, data mining algorithms to compute mixed concepts were developed [20]. In the same year, Bartl et al. [1] introduced formal fuzzy concept analysis by using positive and negative attributes. Nevertheless, few works have recently been paid attention to the mixed concepts and the rule acquisition with mixed information in decision formal contexts. To our best knowledge, there is rule acquisition based on 3WCA [25], which dealt with positive and negative decision rules, respectively. To address this deficiency, in this article we focus on implementing mixed decision rules by using mixed concept lattices.

In this paper, we first recall several basic notions in Section 2. Subsequently, two kinds of mixed concept are investigated. And then the relationships between three types of concept lattices are explored in Section 3. Moreover, in Section 4, two types of mixed decision rules acquisition in decision formal context are investigated, respectively. With the framework of non-redundant mixed decision rules, rule classification is proposed. In Section 5, the connection between mixed decision rules and three-way rules is studied in depth.

2 Preliminaries

This section refers to [3, 13, 17–20] for the basic notions about FCA, 3WCA and MCA.

2.1 Basic notations in FCA

A formal context is recorded as $F = (G, M, I)$, in which G is the set of considered objects, M is the collection of attributes and $I \subseteq G \times M$. In F , the following maps were specified from Reference [3].

Take $X \subseteq G$ and $B \subseteq M$. Maps $*$: $\mathcal{P}(G) \rightarrow \mathcal{P}(M)$ and $*$: $\mathcal{P}(M) \rightarrow \mathcal{P}(G)$ are respectively specified by:

$$X^* = \{a \in M | \forall x \in X, (x, a) \in I\}, \quad (1)$$

$$B^* = \{x \in G | \forall a \in B, (x, a) \in I\}. \quad (2)$$

The tuple (X, B) is called a formal concept derived from F whenever $X^* = B$ and $B^* = X$, in which the former and the latter are said to be extent and intent, respectively. The relation “ $\leq$ ” between any two concepts (Y, C) and (X, B) is:

$$(X, B) \leq (Y, C) \Leftrightarrow X \subseteq Y \Leftrightarrow C \subseteq B. \quad (3)$$

Furthermore, the operators “ $\wedge$ ” and “ $\vee$ ” are given as follows:

$$(X, B) \wedge (Y, C) = (X \cap Y, (B \cup C)^*), \quad (4)$$

$$(X, B) \vee (Y, C) = ((X \cup Y)^*, B \cap C), \quad (5)$$

It is easy to see both right sides of items (4) and (5) are again concepts. In this case, it is conceivable that the set of all the concepts derived from F specifies the concept lattice of F and denotes as $L(G, M, I)$. Simultaneously, we record $L_G(G, M, I)$ as the collection of the extensions of all the concepts as well as $L_M(G, M, I)$ the family of the intensions of all concepts.

As illustrated by [13], the two defined maps $*$ are called positive operators. To save confusion, a concept induced by the positive operators with respect to F is called a P-concept. In the similar fashion, Qi et al. [13] discussed a pair of negative operators in the following.

Maps $\bar{*}$: $\mathcal{P}(G) \rightarrow \mathcal{P}(M)$ and $\bar{*}$: $\mathcal{P}(M) \rightarrow \mathcal{P}(G)$ for any $X \subseteq G$ and $B \subseteq M$ w.r.t. F called negative operators are defined by

$$X^{\bar{*}} = \{a \in M | \forall x \in X, (x, a) \notin I\}, \quad (6)$$

$$B^{\bar{*}} = \{x \in G | \forall a \in B, (x, a) \notin I\}. \quad (7)$$

In this regard, if $X^{\bar{*}} = B$ as well as $B^{\bar{*}} = X$, then (X, B) is an N-concept of F . And “ $\bar{*}$ ” has properties similar to “ $*$ ” clearly. Similarly, all the N-concepts form a concept lattice of F and is denoted by $L(G, M, \bar{I})$,

where $\bar{I}$ means the complementary of I . Meanwhile, we denote $L_G(G, M, \bar{I})$ and $L_M(G, M, \bar{I})$ the collection of extensions and the set of intensions of all N-concepts of F , respectively.

2.2 Basic concepts of 3WCA

The model of 3WCA is a newly proposed model based three-way decisions. Integrating positive and negative operators, Qi et al. [13] proposed the following three-way operators.

Take $X \subseteq G$ and $B, C \subseteq M$ from a formal context F . The definitions of object-induced three-way maps $< : \mathcal{P}(G) \rightarrow \mathcal{DP}(M)$ and $> : \mathcal{DP}(M) \rightarrow \mathcal{P}(G)$ are presented as follows:

$$X^< = (X^*, X^{\bar{*}}), \quad (8)$$

$$(B, C)^> = \{x \in G | x \in B^* \text{ and } x \in C^{\bar{*}}\}, \quad (9)$$

where $\mathcal{DP}(M) = \mathcal{P}(M) \times \mathcal{P}(M)$. The pair $(X, (B, C))$ meeting $X^< = (B, C)$ and $(B, C)^> = X$ is named an OE-concept, in which X and (B, C) are the extension and intension, respectively. The object-induced three-way concept lattice formed by all the OE-concepts of F is recorded as $OEL(G, M, I)$. In addition, we denote $OEL_G(G, M, I)$, $OEL_M^+(G, M, I)$, $OEL_M^-(G, M, I)$ as the collections of extensions, the first and second components of intensions of $OEL(G, M, I)$, respectively. For this reason, although the positive and negation are involved in three-way concept lattice, they are separated discussed.

2.3 Negative attributes

A bit more technically, from a formal context, knowledge discovery, association and implication rules are usually constructed via only positive information. Nevertheless, this is incomplete in some requirements from practical situations.

Table 1: A formal context $F = (G, M, I)$

I	m_1	m_2	m_3	m_4	m_5
g_1	1	1	0	0	0
g_2	0	0	1	1	0
g_3	0	1	1	0	1
g_4	0	1	1	0	1

For example, from the formal context shown on Table 1, one can easily obtain the following implication basis $\Sigma = \{m_5 \leftrightarrow m_2 m_3, m_4 \rightarrow m_3, m_1 \rightarrow m_2\}$. In this case, any implication rule holds in F if and only if it can

be derived from Σ . Thus, the implications $m_2 \rightarrow m_3$ as well as $m_2 \rightarrow m_4$ do not hold. However, this two implications have different reasons for failure according to Table 1. For the former, any object with the attribute m_2 does not possess the attribute m_3 . However, the latter implication implies that such objects shared by m_2 can be distinguished based on the presence or absence of m_4 . Both them are not available without negative information. Mathematically, $m_2 \rightarrow \bar{m}_4$ holds in Table 1 combining with positive and negative information, where $\bar{m}_4$ means the absence of m_4 (negative attribute). Simultaneously, neither $m_2 \rightarrow m_3$ nor $m_2 \rightarrow \bar{m}_3$ holds. Therefore, it is more meaningful to combine with positive and negative attributes in data mining, knowledge discovery and rule extraction.

To overcome this problem, Rodriguez-Jimenez et al. [18–20] developed mixed concept lattice with mixing the positive and negation. In the first place, we emphasize an extended notation of negative attribute [18–20]. Considering a formal context $F = (G, M, I)$, from now on, any element m in M is called positive attribute and $\bar{m}$ identified with negative attribute means that those objects have not m . Meanwhile, let $\bar{M}$ represents $\{\bar{m} | m \in M\}$. The members in $M \cup \bar{M}$ will be recorded as alphabet letters: $a, b, c, \dots$. Thus, $a, b, c, \dots$ could present either positive attributes or negative attributes. The subsets of $M \cup \bar{M}$ are denoted by capital letters $A, B, C, \dots$. Especially, for any $A \subseteq M \cup \bar{M}$, $\bar{A}$ implies the opposite of A . That is, $\bar{A} = \{\bar{a} | a \in A\}$, where $\bar{\bar{a}} = a$. In addition, we have the following symbols used in this paper: for $A \subseteq M \cup \bar{M}$,

$$Pos(A) = \{m \in M | m \in A\}; \quad (10)$$

$$Neg(A) = \{m \in M | \bar{m} \in A\}; \quad (11)$$

$$Tot(A) = Pos(A) \cup Neg(A). \quad (12)$$

Obviously, $Pos(A) = A \cap M$, $Neg(A) = \bar{A} \cap M$ and $A = Pos(A) \cup Neg(A)$. With these basic symbols, the extended definitions and formal concepts are presented in the following.

Definition 1 [18–20] The mixed derivation maps $\Uparrow : \mathcal{P}(G) \rightarrow \mathcal{P}(M \cup \bar{M})$ and $\Downarrow : \mathcal{P}(M \cup \bar{M}) \rightarrow \mathcal{P}(G)$ in $F = (G, M, I)$ are defined by: for $X \subseteq G$ and $B \subseteq M \cup \bar{M}$

$$X^\Uparrow = \{m \in M | \forall g \in X, (g, m) \in I\} \cup \{\bar{m} \in \bar{M} | \forall g \in X, (g, m) \notin I\},$$

$$B^\Downarrow = \{g \in G | \forall m \in B \cap M, (g, m) \in I\} \cap \{g \in G | \forall \bar{m} \in B \cap \bar{M}, (g, m) \notin I\}.$$

The pair (X, B) is named a mixed concept of F whenever $X^\Uparrow = B$ and $B^\Downarrow = X$.

Especially, $\emptyset^\uparrow = M \cup \overline{M}$ and $(M \cup \overline{M})^\downarrow = \emptyset$. At the same time, $\emptyset^\downarrow = G$. Conveniently, we record $\{x\}^\uparrow$ as $x^\uparrow$ and $\{a\}^\downarrow$ as $a^\downarrow$ for $x \in G$ and $a \in M \cup \overline{M}$. In addition, the partial relationship between any two mixed concepts (X, B) and (Y, C) is expressed as:

$$(X, B) \leq (Y, C) \text{ iff } X \subseteq Y \text{ (equivalently } C \subseteq B). \quad (13)$$

Table 2: A formal context $F = (G, M, I)$

I	m_1	m_2	m_3	m_4	m_5
g_1	1	1	0	1	1
g_2	1	1	1	0	0
g_3	0	0	0	1	0
g_4	1	1	1	0	0

Example 1 From Table 2, the derived mixed formal context with respect to objects is shown on Table 3. Take $X = \{g_2, g_3, g_4\}$ and $B = \{m_3, \overline{m}_1\}$. Then $Pos(B) = \{m_3\}$ and $Neg(B) = \{m_1\}$. Obviously, (X, B) is a mixed concept.

3 Mixed concept lattices

3.1 Object induced mixed concepts and attribute induced mixed concepts

The target of this subsection is to enrich the theory of MCA. First of all, we discuss the properties of mixed maps in the following.

Theorem 1 For $X, Y \subseteq G$ and $B, C \subseteq M \cup \overline{M}$ in a formal context F , we have

1. $X \subseteq Y \Rightarrow Y^\uparrow \subseteq X^\uparrow$, $B \subseteq C \Rightarrow Pos(C)^* \subseteq Pos(B)^*$ and $Neg(C)^* \subseteq Neg(B)^* \Rightarrow C^\downarrow \subseteq B^\downarrow$,
2. $B^\downarrow = Pos(B)^\downarrow \cap \overline{Neg(B)}^\downarrow$,
3. $X \subseteq X^{\uparrow\downarrow}$ and $B \subseteq Pos(B)^{\downarrow\uparrow} \cup \overline{Neg(B)}^{\downarrow\uparrow} \subseteq B^{\downarrow\uparrow}$,
4. $X^\uparrow = X^{\uparrow\downarrow\uparrow}$ and $B^\downarrow = B^{\downarrow\uparrow\downarrow}$,
5. $X \subseteq B^\downarrow \Leftrightarrow B \subseteq X^\uparrow$,
6. $(X \cup Y)^\uparrow = X^\uparrow \cap Y^\uparrow$ and $(B \cup C)^\downarrow = B^\downarrow \cap C^\downarrow$,
7. $X^\uparrow \cup Y^\uparrow \subseteq (X \cap Y)^\uparrow$ and $B^\downarrow \cup C^\downarrow \subseteq (B \cap C)^\downarrow$,
8. Both $(X^{\uparrow\downarrow}, X^\uparrow)$ and $(B^\downarrow, B^{\downarrow\uparrow})$ are mixed concepts.

Evidently, $X^\uparrow = \bigcap_{x \in X} x^\uparrow$ and $B^\downarrow = Pos(B)^* \cap Neg(B)^* = \bigcap_{a \in B} a^\downarrow$. Actually, $Pos(x^\uparrow) \cap Neg(x^\uparrow) = \emptyset$ for $x \in G$ and $Tot(x^\uparrow) = M$ [19], thus $Pos(X^\uparrow) \cap Neg(X^\uparrow) = \emptyset$ for $X \subseteq G$.

We record $OML(G, M, I)$ as the collection of all mixed concepts of F . Furthermore, take $OML(G, M, I)$ and $OML_M(G, M, I)$ be the set of extensions and the family of intensions of all mixed concepts in F , respectively.

The “ $\wedge$ ” and “ $\vee$ ” between any mixed concepts (X, B) and (Y, C) are presented as follows:

$$(X, B) \wedge (Y, C) = (X \cap Y, (B \cup C)^{\downarrow\uparrow}), \quad (14)$$

$$(X, B) \vee (Y, C) = ((X \cup Y)^{\uparrow\downarrow}, B \cap C). \quad (15)$$

With partial order relation “ $\leq$ ” from Eq.(13) and operations “ $\wedge$ ” and “ $\vee$ ”, all mixed concepts forms a complete lattice, hence $OML(G, M, I)$ is called the mixed concept lattice of F . In addition,

$$\begin{aligned} OML(G, M, I) &= \{(X^{\uparrow\downarrow}, X^\uparrow) | X \subseteq G\} \\ &= \{(B^\downarrow, B^{\downarrow\uparrow}) | B \subseteq M \cup \overline{M}\}. \end{aligned}$$

Such mixed concepts focus on positive and negative attributes are called object-induced mixed concepts (O-mixed concepts). It is easy to see that

$$(X, B) = \bigvee_{g \in X} (g^{\uparrow\downarrow}, g^\uparrow) \quad (16)$$

for $(X, B) \in OML(G, M, I)$. Thus, any join irreducible element has the form of $(g^{\uparrow\downarrow}, g^\uparrow)$. Then we have the following conclusion.

Lemma 1 For $g \in G$ from F , if the cardinality $|g^{\uparrow\downarrow}| \geq 2$, then $g_0^\uparrow = g^\uparrow$ for any $g_0 \in g^{\uparrow\downarrow}$.

Proof Evidently, for each $g \in G$, $Tot(g^\uparrow) = M$ and $Pos(g^\uparrow) \cap Neg(g^\uparrow) = \emptyset$. If $|g^{\uparrow\downarrow}| \geq 2$, assume there is $g_0 \in g^{\uparrow\downarrow}$ such that $g_0^\uparrow \neq g^\uparrow$, then $g^{\uparrow\downarrow\uparrow} \subseteq g_0^\uparrow$. That is to say $g^\uparrow \subseteq g_0^\uparrow$. This implies $Pos(g^\uparrow) \subseteq Pos(g_0^\uparrow)$ as well as $Neg(g^\uparrow) \subseteq Neg(g_0^\uparrow)$. Then $Tot(g^\uparrow) \subseteq Tot(g_0^\uparrow)$, which is a contradiction.

Theorem 2 For any $g \in G$ from F , $(g^{\uparrow\downarrow}, g^\uparrow)$ is a join irreducible in $OML(G, M, I)$.

Proof On the contrary, suppose $(g^{\uparrow\downarrow}, g^\uparrow)$ is join reducible, then there exists an index set T such that $(g^{\uparrow\downarrow}, g^\uparrow) = \bigvee_{i \in T} (g_i^{\uparrow\downarrow}, g_i^\uparrow)$. Thus $g^\uparrow = \bigcap_{i \in T} g_i^\uparrow$. Since $Tot(g_0^\uparrow) = M$ and $Pos(g_0^\uparrow) \cap Neg(g_0^\uparrow) = \emptyset$ for any $g_0 \in G$, there can be only one conclusion that $g^\uparrow = g_i^\uparrow$ for $i \in T$. This indicates $(g^{\uparrow\downarrow}, g^\uparrow) = (g_i^{\uparrow\downarrow}, g_i^\uparrow)$ for $i \in T$, which contradicts to the hypothesis.

Thus each $(g^{\uparrow\downarrow}, g^\uparrow)$ for $g \in G$ is join irreducible element in $OML(G, M, I)$. In addition, we denote $(F|\overline{F}) = (G, M \cup \overline{M}, I \cup \overline{I})$ the mixed formal context with respect to objects.

Likewise, we can also consider the negative objects, the opposite of objects. These situation prevails in real life. For instance, in personnel management, the employer decides whether to recruit or not according to whether the position is short of people. In data mining and knowledge discovery, only considering positive object without its opposite also may lead to the information loss. In this case, it is also meaningful to search the negative information with respect to attributes.

Table 3: The mixed formal context of F with respect to objects

$I \cup \bar{I}$	m_1	m_2	m_3	m_4	m_5	$\bar{m}_1$	$\bar{m}_2$	$\bar{m}_3$	$\bar{m}_4$	$\bar{m}_5$
g_1	1	1	0	0	0	0	0	1	1	1
g_2	0	0	1	1	0	1	1	0	0	1
g_3	0	1	1	0	1	1	0	0	1	0
g_4	0	1	1	0	1	1	0	0	1	0

Definition 2 Denote $F|\bar{F} = (G \cup \bar{G}, M, I \cup \bar{I})$ the mixed formal context of F with respect to attributes. Take $X \subseteq G \cup \bar{G}$ and $B \subseteq M$, $\uparrow: \mathcal{P}(G \cup \bar{G}) \rightarrow \mathcal{P}(M)$ and $\downarrow: \mathcal{P}(M) \rightarrow \mathcal{P}(G \cup \bar{G})$ are respectively defined by

$$X^\uparrow = \{m \in M | \forall g \in X, (g, m) \in I\} \cap \{m \in M | \forall \bar{g} \in X, (g, m) \notin I\},$$

$$B^\downarrow = \{g \in G | \forall a \in B, (g, a) \in I\} \cup \{\bar{g} \in \bar{G} | \forall m \in B, (g, m) \notin I\}.$$

Then (X, B) is called an attribute-induced mixed formal concept of F (A-mixed concept for short) whenever $X^\uparrow = B$ and $B^\downarrow = X$, in which X is the extension and B is the intention.

Table 4: The mixed formal context $F|\bar{F}$ of Table 2

I	m_1	m_2	m_3	m_4	m_5
g_1	1	1	0	1	1
g_2	1	1	1	0	0
g_3	0	0	0	1	0
g_4	1	1	1	0	0
$\bar{g}_1$	0	0	1	0	0
$\bar{g}_2$	0	0	0	1	1
$\bar{g}_3$	1	1	1	0	1
$\bar{g}_4$	0	0	0	1	1

As shown by Table 4, the “ $\leq$ ” between any A-mixed concepts (X, B) and (Y, C) is similarly defined by

$$(X, B) \leq (Y, C) \Leftrightarrow X \subseteq Y \Leftrightarrow C \subseteq B. \quad (17)$$

All the A-mixed concepts of F form a concept lattice is recorded as $AML(G, M, I)$. And let $AML_G(G, M, I)$ be the family of extensions and $AML_M(G, M, I)$ be the set of intentions of all A-mixed concepts of F , respectively.

The following properties hold for the mixed operators with respect to attributes.

1. $B \subseteq C \Rightarrow C^\downarrow \subseteq B^\downarrow$, $X \subseteq Y \Rightarrow Pos(Y)^* \subseteq Pos(X)^*$ and $Neg(Y)^* \subseteq Neg(X)^* \Rightarrow Y^\uparrow \subseteq X^\uparrow$,
2. $X^\uparrow = Pos(X)^\uparrow \cap \overline{Neg(X)}^\uparrow$,
3. $B \subseteq B^{\downarrow\uparrow}$ and $X \subseteq Pos(X)^{\uparrow\downarrow} \cup \overline{Neg(X)}^{\uparrow\downarrow} \subseteq X^{\uparrow\downarrow}$,
4. $X^\uparrow = X^{\uparrow\downarrow\uparrow}$ and $B^\downarrow = B^{\downarrow\uparrow\downarrow}$,
5. $X \subseteq B^\downarrow \Leftrightarrow B \subseteq X^\uparrow$,

6. $(X \cup Y)^\uparrow = X^\uparrow \cap Y^\uparrow$ and $(B \cup C)^\downarrow = B^\downarrow \cap C^\downarrow$,
7. $X^\uparrow \cup Y^\uparrow \subseteq (X \cap Y)^\uparrow$ and $B^\downarrow \cup C^\downarrow \subseteq (B \cap C)^\downarrow$,
8. Both $(X^{\uparrow\downarrow}, X^\uparrow)$ and $(B^\downarrow, B^{\downarrow\uparrow})$ are A-mixed concepts.

It is also worth pointing out that $Tot(m^\downarrow) = G$, $Pos(m^\downarrow) \cap Neg(m^\downarrow) = \emptyset$ and $m^\downarrow = Pos(m^\downarrow) \cup Neg(m^\downarrow)$ for $m \in M$.

Lemma 2 For any $m \in M$ from F , $m_0 \in m^\downarrow$ if and only if $m_0^\downarrow = m^\downarrow$ for any $m_0 \in m^{\downarrow\uparrow}$.

Proof Similar to Lemma 1.

Theorem 3 Any $(m^\downarrow, m^{\downarrow\uparrow})$, $m \in M$, is a meet irreducible element in $AML(G, M, I)$.

Proof Similar to Theorem 2.

Example 2 From Tables 2 and 3, $OML(G, M, I)$ and $AML(G, M, I)$ are presented in Figures 1 and 2, respectively.

3.2 The connections between FCA, 3WCA and MCA

This subsection focuses on the connection between MCA and FCA, and the relationship between MCA and 3WCA. First of all, the link between O-mixed concept lattice and concept lattice is discussed as follows.

Theorem 4 Let $F = (G, M, I)$ be a formal context.

1. $(X, B \cup X^*)$ and $(Y, C \cup Y^*)$ are both O-mixed concepts whenever (X, B) and (Y, C) are a P-concept and an N-concept, respectively.
2. When (X, B) is an O-mixed concept, $(X, Pos(B))$ is P-concept if $Pos(B) \neq \emptyset$, and $(X, Neg(B))$ is N-concept if $Neg(B) \neq \emptyset$.

Proof We only illustrate $(X, B \cup X^*)$ is an O-mixed concept. Actually, $X^\uparrow = X^* \cup X^* = B \cup X^*$. And $(B \cup X^*)^\downarrow = Pos(B \cup X^*)^* \cap Neg(B \cup X^*)^* = B^* \cap X^{**} = X$. Therefore, $(X, B \cup X^*)$ is an O-mixed concept.

When discussing the O-mixed lattice and classical concept lattice, conclusions can be developed as follows.

Theorem 5 The relationships between P-concept, N-concept and O-mixed concept from a formal context F are

Fig. 1: The $OML(G, M, I)$ of Example 2 Fig. 2: The $AML(G, M, I)$ of Example 2

1. $L_G(G, M, I) \cup L_G(G, M, \bar{I}) = OML_G(G, M, I)$;
2. $\forall B \in L_M(G, M, I)$, there is $B_0 \in OML_M(G, M, I)$ satisfying $B = Pos(B_0)$ and vice versa;
3. $\forall B \in L_M(G, M, \bar{I})$, there is $B_0 \in OML_M(G, M, I)$ satisfying $B = Neg(B_0)$ and vice versa.

Proof (1) It is only needed to formulate items (1) and (2). For any $X \in L_G(G, M, I)$, we have $(X, X^*) \in L(G, M, I)$, which implies $(X, X^* \cup X^{\bar{*}}) \in OML(G, M, I)$. Then $X \in OML_G(G, M, I)$, that is, $L_G(G, M, I) \subseteq OML_G(G, M, I)$. The same procedure may be easily adapted to obtain $L_G(G, M, \bar{I}) \subseteq OML_G(G, M, I)$. That is, $L_G(G, M, I) \cup L_G(G, M, \bar{I}) \subseteq OML_G(G, M, I)$. Whenever $X \in OML_G(G, M, I)$, $(X, X^\dagger) \in OML(G, M, I)$ means that $(X, Pos(X^\dagger)) \in L(G, M, I)$ if $Pos(X^\dagger) \neq \emptyset$ and $(X, Neg(X^\dagger)) \in L(G, M, \bar{I})$ when $Neg(X^\dagger) \neq \emptyset$. Thereby, $X \in L_G(G, M, I) \cup L_G(G, M, \bar{I})$. Consequently, item (1) holds.

(2) $\forall B \in L_M(G, M, I)$, we have $(B^*, B \cup B^{**}) \in OML(G, M, I)$. Take that $B_0 = B \cup B^{**}$. Then $B_0 \in OML(G, M, I)$ and $Pos(B_0) = B$. Conversely, for all

$B_0 \in OML(G, M, I)$ we have $(B_0^\downarrow, B_0) \in L(G, M, I)$. It is easily checked that $(B_0^\downarrow, Pos(B_0))$.

Now, we discuss $(F|\bar{F})$. Take any $g \in G$ and $a \in M \cup \bar{M}$, either $(g, a) \in I \cup \bar{I}$ or $(g, a) \notin I \cup \bar{I}$. The former means either $(g, a) \in I$ or $(g, a) \in \bar{I}$. However, the latter implies both $(g, a) \notin I$ and $(g, a) \notin \bar{I}$.

Definition 3 Take $(F|\bar{F}) = (G, M \cup \bar{M}, I \cup \bar{I})$. For $X \subseteq G$ and $B \subseteq M \cup \bar{M}$, the definitions of $f : \mathcal{P}(G) \rightarrow \mathcal{P}(M \cup \bar{M})$ and $g : \mathcal{P}(M \cup \bar{M}) \rightarrow \mathcal{P}(G)$ are shown:

$$f(X) = \{a \in M \cup \bar{M} | \forall g \in X, (g, a) \in I \cup \bar{I}\}, \quad (18)$$

$$g(B) = \{g \in G | \forall a \in B, (g, a) \in I \cup \bar{I}\}. \quad (19)$$

Then (X, B) forms a formal concept of $(F|\bar{F})$ whenever $f(X) = B$ and $g(B) = X$.

Take $L(G, M \cup \bar{M}, I \cup \bar{I})$ be the collection of all concepts of $(F|\bar{F})$. Denote by $L_G(G, M \cup \bar{M}, I \cup \bar{I})$ and $L_{M \cup \bar{M}}(G, M \cup \bar{M}, I \cup \bar{I})$ the set of all extensions and the collection of all intensions of $L(G, M \cup \bar{M}, I \cup \bar{I})$, respectively. To trim some proofs, we introduce the following definition derived from Reference [3].

Definition 4 Let $L(G, M_1, I_1)$ and $L(G, M_2, I_2)$ be concept lattices. For $(X, B) \in L(G, M_2, I_2)$, if there is $(Y, C) \in L(G, M_1, I_1)$ meeting $X = Y$, we record as $L(G, M_1, I_1) \leq L(G, M_2, I_2)$ and call the former is finer than the latter. Moreover, if $L(G, M_1, I_1) \leq L(G, M_2, I_2)$ and $L(G, M_2, I_2) \leq L(G, M_1, I_1)$, then $L(G, M_1, I_1)$ and $L(G, M_2, I_2)$ are isomorphism.

It is conceivable that two concept lattices are isomorphism if all of their extents are corresponding equality. Similarly, $L(G, M_1, I_1)$ and $L(G, M_2, I_2)$ are also isomorphism with respect to intents if $L_{M_1}(G, M_1, I_1) = L_{M_2}(G, M_2, I_2)$. Then the following results are conceivable.

Theorem 6 Take the mixed formal context $(F|\bar{F}) = (G, M \cup \bar{M}, I \cup \bar{I})$ w.r.t. F , we have

1. $OML_M(G, M, I) = L_{M \cup \bar{M}}(G, M \cup \bar{M}, I \cup \bar{I})$;
2. $OML(G, M, I)$ and $L(G, M \cup \bar{M}, I \cup \bar{I})$ are isomorphism w.r.t. intentions.

Proof (1) For $(X, B) \in OML(G, M, I)$, we have $B = X^\uparrow = f(X)$, i.e., $OML_M(G, M, I) \subseteq L_{M \cup \bar{M}}(G, M \cup \bar{M}, I \cup \bar{I})$. Nevertheless, it also easily verifies that $B \in OML_M(G, M, I)$ for all $(X, B) \in L(G, M \cup \bar{M}, I \cup \bar{I})$. Thus, $OML_M(G, M, I) = L_{M \cup \bar{M}}(G, M \cup \bar{M}, I \cup \bar{I})$.

(2) It follows from item (1).

Thus, the hierarchical structures of $OML(G, M, I)$ and $L(G, M \cup \bar{M}, I \cup \bar{I})$ are coincided with each other.

Theorem 7 $(X, (Pos(B), Neg(B))) \in OEL(G, M, I)$ if $(X, B) \in OML(G, M, I)$. Conversely, $(X, (B, C)) \in OEL(G, M, I)$ when $(X, B \cup \bar{C}) \in OML(G, M, I)$.

Proof $X^\leq = (Pos(B), Neg(B))$ if $(X, B) \in OML(G, M, I)$. Another dimension, $(Pos(B), Neg(B))^\geq = Pos(B)^* \cap Neg(B)^* = B^\downarrow = X$. Thus, $(X, (Pos(B), Neg(B))) \in OEL(G, M, I)$. Conversely, suppose that $(X, (B, C)) \in OEL(G, M, I)$, then $X^\uparrow = X^* \cup \bar{X}^* = B \cup \bar{C}$. Meanwhile, $(B \cup \bar{C})^\downarrow = B^\downarrow \cap \bar{C}^\downarrow = B^* \cap C^* = (B, C)^\geq = X$. Therefore, $(X, B \cup \bar{C}) \in OML(G, M, I)$.

Thus, it is easy to know that

1. $OEL_M^+(G, M, I) = \{Pos(B) | B \in OML_M(G, M, I)\}$,
2. $OEL_M^-(G, M, I) = \{Neg(B) | B \in OML_M(G, M, I)\}$.

4 Mixed decision rules acquisition

Definition 5 [24] A decision formal context is recorded as $S = (G, M, I, N, J)$, where (G, M, I) and (G, N, J) are formal contexts with $M \cap N = \emptyset$. And M and N here are the conditional attributes and decision attributes, respectively.

Let $(S|\bar{S}) = (G, M \cup \bar{M}, I \cup \bar{I}, N \cup \bar{N}, J \cup \bar{J})$ be the corresponding mixed decision formal context of S with respect to G . Similarly, take $OML(G, N, J)$ the object-induced mixed formal concept lattice of (G, N, J) . Other symbols are similar to the previous ones, which will not be repeated here. Simultaneously, in order to differentiate the mixed operators between (G, M, I) and (G, N, J) , denote $(\uparrow_M, \downarrow_M)$ and $(\uparrow_N, \downarrow_N)$ as the mixed maps on (G, M, I) and (G, N, J) , respectively.

4.1 Mixed decision rules in the O-consistent decision formal context

Definition 6 Take S be a decision formal context. For $(Y, D) \in OML(G, N, J)$, if there is $(X, B) \in OML(G, M, I)$ such that $X = Y$, then S is called object-induced consistent, O-consistent for short.

Definition 7 For any $(X, B) \in OML(G, M, I)$ and $(Y, D) \in OML(G, N, J)$ from S with none of X, Y, B and D is empty. Then $B \rightarrow D$ is said to be a mixed decision rule derived from S if $X \subseteq Y$. The set of all the mixed decision rules generated from S is denoted by $\mathcal{R}(S)$.

Thus, for $B \rightarrow D \in \mathcal{R}(S)$, any object that has all the mixed condition attributes in B has also all the mixed decision attributes in D . That is, $B \rightarrow D$ implies “If $\wedge B$ then $\wedge D$ ”.

Example 3 Table 5 depicts the corresponding mixed formal context of a decision formal context S . Fig. 3 and Fig. 4 are the Hasse diagram of $OML(G, M, I)$ and $OML(G, N, J)$, respectively. Then the mixed decision rules are accordingly received and shown in Table 6.

Classically, decision rules acquired from formal context are often described as “if B , then D ”. Even if from a negative formal context, the decision rules are “if not B , then not D ”. The common characteristic is that positive attributes and negative attributes separate from each other. In other terms, the premise of negative attributes does not have conclusion with positive decision attributes. In mixed decision rules, as illustrated by Example 3, both the premises and the conclusions may be composed by positive and negative attributes. Similarly, there also exists redundancy in mixed decision rules.

Definition 8 For $B_1 \rightarrow D_1, B_2 \rightarrow D_2 \in \mathcal{R}(S)$, if $B_1 \subseteq B_2$ and $D_2 \subseteq D_1$, then $B_2 \rightarrow D_2$ can be yielded by $B_1 \rightarrow D_1$, denoted by $B_1 \rightarrow D_1 \Rightarrow B_2 \rightarrow D_2$. Moreover, $B \rightarrow D \in \mathcal{R}(S)$ is called redundant in $\mathcal{R}(S)$ if there exists $B_0 \rightarrow D_0 \in \mathcal{R}(S) - \{B \rightarrow D\}$ such that

Table 5: A mixed decision formal context $(S|\bar{S}) = (G, M \cup \bar{M}, I \cup \bar{I}, N \cup \bar{N}, J \cup \bar{J})$

$I \cup \bar{I}$	m_1	m_2	m_3	m_4	m_5	m_6	$\bar{m}_1$	$\bar{m}_2$	$\bar{m}_3$	$\bar{m}_4$	$\bar{m}_5$	$\bar{m}_6$	n_1	n_2	n_3	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$
g_1	0	0	1	0	0	1	1	1	0	1	1	0	0	0	1	1	1	0
g_2	0	0	0	1	1	1	1	1	1	0	0	0	1	0	1	0	1	0
g_3	1	1	1	0	1	1	0	0	0	1	0	0	0	1	1	1	0	0
g_4	0	0	0	1	1	0	1	1	1	0	0	1	1	0	0	0	1	1

Fig. 3: The $OML(G, M, I)$ of Table 5 **Fig. 4:** The $OML(G, N, J)$ of Table 5

$B_0 \rightarrow D_0 \Rightarrow B \rightarrow D$; or else, $B \rightarrow D$ is known as non-redundant in $\mathcal{R}(S)$. We denoted by $\mathcal{R}^*(S)$ the collection of all the non-redundant mixed decision rules in $\mathcal{R}(S)$.

Example 4 From Example 2, we search for all the non-redundant mixed decision rules in $\mathcal{R}(S)$ and show them in Table 7.

Theorem 8 Let $(X, B) \in OML(G, M, I)$ and $(Y, D) \in OML(G, N, J)$ with $X \neq \emptyset$ and $B \neq \emptyset$ from a decision formal context S . If $X = Y$, then $B \rightarrow D$ is a non-redundant mixed decision rule in $\mathcal{R}(S)$.

Proof If there are $(X_0, B_0) \in OML(G, M, I)$, $(Y_0, D_0) \in OML(G, N, J)$ with $X_0 \neq \emptyset$ and $B_0 \neq \emptyset$ such that $B_0 \rightarrow D_0$ and it implies $B \rightarrow D$. Then $B_0 \subseteq B$ and $D \subseteq D_0$. That is to say, $X \subseteq X_0 \subseteq Y_0 \subseteq Y$. Since $X = Y$, then $X = X_0 = Y_0 = Y$, which means $B = B_0$ and $D = D_0$. Thereby, $B \rightarrow D$ is a non-redundant mixed decision rule in $\mathcal{R}(S)$.

Furthermore, if the considered decision formal context S is O-consistent, we have the following results.

Corollary 1 For an O-consistent decision formal context $S = (G, M, I, N, J)$, $(X, B) \in OML(G, M, I)$ and

Table 6: Mixed decision rules $\mathcal{R}(S)$

$m_6 \rightarrow n_3$	$m_3\bar{m}_4m_6 \rightarrow n_3$	$\bar{m}_1\bar{m}_2m_3\bar{m}_4\bar{m}_5m_6 \rightarrow \bar{n}_1n_3$
$\bar{m}_1\bar{m}_2 \rightarrow \bar{n}_2$	$\bar{m}_1\bar{m}_2m_6 \rightarrow n_3$	$\bar{m}_1\bar{m}_2\bar{m}_3m_4\bar{m}_5\bar{m}_6 \rightarrow \bar{n}_1n_3$
$m_3\bar{m}_4m_6 \rightarrow \bar{n}_1n_3$	$\bar{m}_1\bar{m}_2m_3\bar{m}_4\bar{m}_5m_6 \rightarrow n_3$	$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5m_6 \rightarrow \bar{n}_2n_3$
$\bar{m}_1\bar{m}_2m_6 \rightarrow \bar{n}_2n_3$	$m_1m_2m_3\bar{m}_4m_5m_6 \rightarrow n_3$	$m_1m_2m_3\bar{m}_4m_5m_6 \rightarrow \bar{n}_2n_3$
$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5 \rightarrow n_1\bar{n}_2$	$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5m_6 \rightarrow n_3$	$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5m_6 \rightarrow n_1\bar{n}_2$
$\bar{m}_1\bar{m}_2m_3\bar{m}_4\bar{m}_5m_6 \rightarrow \bar{n}_1\bar{n}_2n_3$	$\bar{m}_1\bar{m}_2m_6 \rightarrow \bar{n}_2$	$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5\bar{m}_6 \rightarrow n_1\bar{n}_2$
$m_1m_2m_3\bar{m}_4m_5m_6 \rightarrow \bar{n}_1n_2n_3$	$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5 \rightarrow \bar{n}_2$	
$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5m_6 \rightarrow n_1\bar{n}_2n_3$	$\bar{m}_1\bar{m}_2m_3\bar{m}_4\bar{m}_5m_6 \rightarrow \bar{n}_2$	
$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5\bar{m}_6 \rightarrow n_1\bar{n}_2\bar{n}_3$	$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5m_6 \rightarrow \bar{n}_2$	
	$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5\bar{m}_6 \rightarrow \bar{n}_2$	

Table 7: Non-redundant mixed decision rules in $\mathcal{R}(S)$

$m_6 \rightarrow n_3$
$\bar{m}_1\bar{m}_2 \rightarrow \bar{n}_2$
$m_3\bar{m}_4m_6 \rightarrow \bar{n}_1n_3$
$\bar{m}_1\bar{m}_2m_6 \rightarrow \bar{n}_2n_3$
$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5 \rightarrow n_1\bar{n}_2$
$\bar{m}_1\bar{m}_2m_3\bar{m}_4\bar{m}_5m_6 \rightarrow \bar{n}_1\bar{n}_2n_3$
$\bar{m}_1\bar{m}_2\bar{m}_3m_4\bar{m}_5\bar{m}_6 \rightarrow \bar{n}_1n_2n_3$
$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5m_6 \rightarrow n_1\bar{n}_2n_3$
$\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5\bar{m}_6 \rightarrow n_1\bar{n}_2\bar{n}_3$

$(Y, D) \in OML(G, N, J)$ with $X \neq \emptyset$ and $B \neq \emptyset$, then $B \rightarrow D$ is non-redundant iff $B^{\downarrow_M} = D^{\downarrow_N}$.

Proof We only proof the sufficiency. Suppose $B \rightarrow D$ is non-redundant in $\mathcal{R}(S)$ and $X \subset Y$. By O-consistent, there exists $(X_0, B_0) \in OML(G, M, I)$ such that $X_0 = Y$, then $B_0 \rightarrow D \in \mathcal{R}(S)$. Since $X \subset X_0$, we have $B_0 \subset B$. Thus, $B_0 \rightarrow D$ implies $B \rightarrow D$, which is contradicted to $B \rightarrow D$ is non-redundant in $\mathcal{R}(S)$.

With O-consistent, $g^{\uparrow_N \downarrow_N} = g^{\uparrow_M \downarrow_M}$ for any $g \in G$. Then $g^{\uparrow_M} \rightarrow g^{\uparrow_N} \in \mathcal{R}^*(S)$ for $g \in G$. Thereby, the result is available as follows:

Theorem 9 With respect to an O-consistent decision formal context S , mixed decision rule $B \rightarrow D \in \mathcal{R}(S)$, B and D are non-empty, is non-redundant iff $B \rightarrow D = \bigcap_{i \in T} g_i^{\uparrow_M} \rightarrow \bigcap_{i \in T} g_i^{\uparrow_N}$ with $(\bigcap_{i \in T} g_i^{\uparrow_M})^{\downarrow_M} = (\bigcap_{i \in T} g_i^{\uparrow_N})^{\downarrow_N}$, where T is an index set.

Proof Necessity. If $B \rightarrow D \in \mathcal{R}^*(S)$, then $B^{\downarrow_M} = D^{\downarrow_N}$. In this case, $B \rightarrow D = \bigcap_{g \in D^{\downarrow_N}} g^{\uparrow_M} \rightarrow \bigcap_{g \in D^{\downarrow_N}} g^{\uparrow_N}$. Therefore, an index set is existed with $B \rightarrow D = \bigcap_{i \in T} g_i^{\uparrow_M} \rightarrow \bigcap_{i \in T} g_i^{\uparrow_N}$.

Sufficiency. If $B \rightarrow D = \bigcap_{i \in T} g_i^{\uparrow_M} \rightarrow \bigcap_{i \in T} g_i^{\uparrow_N}$, suppose $Y = \{g_i | i \in T\}$, then $B = Y^{\uparrow_M}$ and $D = Y^{\uparrow_N}$. Since $(\bigcap_{i \in T} g_i^{\uparrow_M})^{\downarrow_M} = (\bigcap_{i \in T} g_i^{\uparrow_N})^{\downarrow_N}$, by Theorem 8, $B \rightarrow D$ is non-redundant.

It is generally known that $\bigcap_{i \in T} g_i^{\uparrow_M} \rightarrow \bigcap_{i \in T} g_i^{\uparrow_N} = \bigcap_{i \in T} (g_i^{\uparrow_M} \rightarrow g_i^{\uparrow_N})$. Thus, for an O-consistent decision formal context, each non-redundant mixed decision rule can be represented as a meet of several particular non-redundant mixed decision rules.

Definition 9 With respect to an O-consistent decision formal context S , we call $g^{\uparrow_M} \rightarrow g^{\uparrow_N}$ the core mixed decision rule for any $g \in G$. All the core mixed decision rules in S is recorded as $\mathcal{C}(S)$. Furthermore, a mixed decision rule $B \rightarrow D$ satisfying $B \neq \emptyset$, $D \neq \emptyset$ and $B \rightarrow D = \bigcap_{i \in T} (g_i^{\uparrow_M} \rightarrow g_i^{\uparrow_N})$ with $|T| \geq 2$ is called a relatively necessary mixed decision rule. And denote $\mathcal{N}(S)$ as the collection of all such mixed decision rules in S . Denote $w\mathcal{R}(S)$ as the join of $\mathcal{C}(S)$ and $\mathcal{N}(S)$.

For instance, from Table 6, the core mixed decision rules are

1. $\bar{m}_1\bar{m}_2m_3\bar{m}_4\bar{m}_5m_6 \rightarrow \bar{n}_1\bar{n}_2n_3$;
2. $m_1m_2m_3\bar{m}_4m_5m_6 \rightarrow \bar{n}_1n_2n_3$;
3. $\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5m_6 \rightarrow n_1\bar{n}_2n_3$;
4. $\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5\bar{m}_6 \rightarrow n_1\bar{n}_2\bar{n}_3$.

And the relatively necessary mixed decision rules are

1. $m_6 \rightarrow n_3$;
2. $m_5m_6 \rightarrow n_3$;
3. $\bar{m}_1\bar{m}_2 \rightarrow \bar{n}_2$;
4. $m_3\bar{m}_4m_6 \rightarrow \bar{n}_1n_3$;
5. $\bar{m}_1\bar{m}_2m_6 \rightarrow \bar{n}_2n_3$;
6. $\bar{m}_1\bar{m}_2\bar{m}_3m_4m_5 \rightarrow \bar{n}_1\bar{n}_2$.

Nevertheless, for $\mathcal{F} \subseteq \mathcal{C}(S)$, $\bigcap \mathcal{F}$ dose not a non-redundant mixed decision rule in S if the condition $(\bigcap_{i \in T} g_i^{\uparrow_M})^{\downarrow_M} = (\bigcap_{i \in T} g_i^{\uparrow_N})^{\downarrow_N}$ fails. Take $\mathcal{F}$ the collection of item (2) and item (3) of $\mathcal{C}(S)$, then $\bigcap \mathcal{F} = m_5m_6 \rightarrow n_3$. However, $(m_5m_6)^{\downarrow_M} \neq n_3^{\downarrow_N}$. Thus $m_5m_6 \rightarrow n_3 \notin \mathcal{R}^*(S)$.

As illustrated by above, the conclusion is available below.

Theorem 10 Let S be O-consistent. Then $\mathcal{R}^*(S) \subseteq w\mathcal{R}(S)$. If all rules in $\mathcal{N}(S)$ satisfying $(\bigcap_{i \in T} g_i^{\uparrow_M})^{\downarrow_M} = (\bigcap_{i \in T} g_i^{\uparrow_N})^{\downarrow_N}$, then $\mathcal{R}^*(S) = w\mathcal{R}(S)$.

Proof Straightforward.

We call $w\mathcal{R}(S)$ the **weak-basis** of $\mathcal{R}(S)$. $w\mathcal{R}(S)$ may be smaller than $\mathcal{R}(S)$ in a general way. In applications, it is unnecessary to obtain all the mixed decision rules. Subsequently, it is difficulty to search all the non-redundant mixed decision rules. By Theorem 10, we can

only explore the weak-basis from an O-consistent decision formal context S via $\mathcal{C}(S)$ and $\mathcal{N}(S)$ directly in order to reduce the computational complexity.

Theorem 11 *Let S be O-consistent with the weak-basis $w\mathcal{R}(S)$. For any $B \rightarrow D \in w\mathcal{R}(S)$, there does not exist $B_0 \rightarrow D \in w\mathcal{R}(S)$ such that $B_0 \subset B$ iff $B \rightarrow D \in \mathcal{R}^*(S)$.*

Proof Necessity. We only need to proof that there exists no $B \rightarrow D_0$ such that $D \subset D_0$. Actually, such $B \rightarrow D_0$ does not exist if $B \rightarrow D \in \mathcal{C}(S)$. Besides, $B \rightarrow D$ can be described as a meet of some core mixed decision rules when $B \rightarrow D \in \mathcal{N}(S)$. That is to say, there is an index set T in G such that $B \rightarrow D = \bigcap_{i \in T} (g_i^{\uparrow M} \rightarrow g_i^{\uparrow N})$. If such $B \rightarrow D_0$ with $D \subset D_0$ exists, then there is an index set T_0 in G such that $B \rightarrow D_0 = \bigcap_{i \in T_0} (g_i^{\uparrow M} \rightarrow g_i^{\uparrow N})$. Thus, $T_0 \subset T$ and $\bigcap_{i \in T} g_i^{\uparrow M} = \bigcap_{i \in T_0} g_i^{\uparrow M}$. In such case, for any $g_0 \in T - T_0$, we have $\bigcap_{i \in T_0} g_i^{\uparrow M} \subseteq g_0^{\uparrow M}$ while $\bigcap_{i \in T_0} g_i^{\uparrow N} \not\subseteq g_0^{\uparrow N}$. Subsequently, $g_0^{\uparrow M \downarrow M} \subseteq (\bigcap_{i \in T_0} g_i^{\uparrow M})^{\downarrow M} \subseteq (\bigcap_{i \in T_0} g_i^{\uparrow N})^{\downarrow N}$ with the definition of mixed decision rules. On the other hand, $g_0^{\uparrow M \downarrow M} = g_0^{\uparrow N \downarrow N}$ as S is an O-consistent. Consequently, $g_0^{\uparrow N \downarrow N} \subseteq (\bigcap_{i \in T_0} g_i^{\uparrow N})^{\downarrow N}$, which is a contradiction. Since there does not exist $B_0 \rightarrow D \in w\mathcal{R}(S)$ satisfying $B_0 \rightarrow B$, we can conclude that $B \rightarrow D \in \mathcal{R}^*(S)$.

Sufficiency. It is conceivable that there does not exist $B \rightarrow D_0 \in w\mathcal{R}(S)$ such that $D \subset D_0$ for any $B \rightarrow D \in w\mathcal{R}(S)$. Since $B \rightarrow D \in \mathcal{R}^*(S)$, then there exists no $B_0 \rightarrow D_0 \in w\mathcal{R}(S)$ such that $B_0 \subseteq B$ and $D \subseteq D_0$. Thereby, there does not exist $B_0 \rightarrow D \in w\mathcal{R}(S)$ satisfying $D \subset D_0$.

Corollary 2 *Let S be O-consistent. For any $B \rightarrow D \in \mathcal{N}(S)$, there exists no $B \rightarrow D_0 \in w\mathcal{R}(S)$ such that $D \subset D_0$.*

Proof It is immediate from Theorem 11.

Thus, Theorem 11 provides a suggestion for all the non-redundant mixed decision rules from the weak-basis from the weak-basis.

4.2 Mixed decision rules in the decision formal context

Generally, a decision formal context may not usually O-consistent. Thus, the decision formal context S discussed in the subsection is not required O-consistent. As previously stated, $B \rightarrow D \in \mathcal{R}(S)$ is non-redundant if $B^{\downarrow M} = D^{\downarrow N}$. In this subsection, we shall discuss the unequal situations.

It is worth pointing out $B \rightarrow X^{\uparrow N} \in \mathcal{R}(S)$ for $(X, B) \in OML(G, M, I)$ with none of X and B is empty.

Lemma 3 *For each $(X, B) \in OML(G, M, I)$ from S , there exists no $B_0 \rightarrow D_0 \in \mathcal{R}(S)$ such that $B_0 \subseteq B$ and $X^{\uparrow N} \subset D_0$.*

Proof Suppose that such $B_0 \rightarrow D_0 \in \mathcal{R}(S)$ is existed. Then $B_0^{\downarrow M} \subseteq D_0^{\downarrow N} \subset X^{\uparrow N \downarrow N}$ as $B_0 \rightarrow D_0 \in \mathcal{R}(S)$ and $X^{\uparrow N} \subset D_0$. Nevertheless, by $B_0 \subseteq B$, we have $X \subseteq B_0^{\downarrow M}$, which implies $X^{\uparrow N \downarrow N} \subseteq B_0^{\downarrow M \uparrow N \downarrow N}$, a contradiction.

It is conceivable that if there are not two identical rows in (G, M, I) , that is, $g_i^{\uparrow M} = g_j^{\uparrow M}$ means $g_i = g_j$, then $g^{\uparrow M} \rightarrow g^{\uparrow N} \in \mathcal{R}(S)$ and it is non-redundant clearly. More generally, if $g^{\uparrow M \downarrow M}$ is not a single point set, we can see that $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ is a mixed decision rule in S immediately following from $g^{\uparrow M \downarrow M} \subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$.

Corollary 3 *There exists no $(Y_0, D_0) \in OML(G, M, I)$ satisfying $g^{\uparrow M \downarrow M} \subseteq Y_0$ and $g^{\uparrow M \downarrow M \uparrow N} \subset D_0$ w.r.t. S .*

That is to say, $g^{\uparrow M} \rightarrow D_0 \notin \mathcal{R}(S)$. Furthermore, we shall have the following results.

Theorem 12 *For $(X, B) \in OML(G, M, I)$ from S with all of X , B and $X^{\uparrow N}$ are non-empty, then $B \rightarrow X^{\uparrow N} \in \mathcal{R}^*(S)$ iff there exists no $(X_0, B_0) \in OML(G, M, I)$ with both X_0 and B_0 are non-empty satisfying $B_0 \rightarrow X^{\uparrow N} \in \mathcal{R}(S)$ and $B_0 \subset B$.*

Proof Necessity. If there is $(X_0, B_0) \in OML(G, M, I)$ with both X_0 and B_0 are non-empty satisfying $B_0 \rightarrow X^{\uparrow N} \in \mathcal{R}(S)$ and $B_0 \subset B$. Then $B \rightarrow X^{\uparrow N}$ is redundant. This is in contradiction with $B \rightarrow X^{\uparrow N} \in \mathcal{R}^*(S)$.

Sufficiency. Assume that $B \rightarrow X^{\uparrow N}$ is a redundant mixed decision rule in $\mathcal{R}(S)$. By Lemma 3, we only need to show there exists $B_0 \rightarrow D_0 \in \mathcal{R}(S)$ such that $B_0 \subset B$ and $X^{\uparrow N} \subseteq D_0$. However, $X^{\uparrow N} \subset D_0$ does not hold. Thereby, we have $X^{\uparrow N} = D_0$. This implies $B_0 \rightarrow X^{\uparrow N}$ and $B_0 \subset B$, a contradiction.

Corollary 4 *$g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ for $g \in G$ is redundant w.r.t. S iff there is $(X_0, B_0) \in OML(G, M, I)$ with both X_0 and B_0 are non-empty satisfying the following two conditions:*

1. $B_0 \subset g^{\uparrow M}$;
2. there exists $g_1 \in g^{\uparrow M \downarrow M}$ satisfying $g_1^{\uparrow N} = g_0^{\uparrow N}$ for any $g_0 \in X_0 - g^{\uparrow M \downarrow M}$.

Proof Necessity. If $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ is redundant, then there is $(X_0, B_0) \in OML(G, M, I)$ with both X_0 and B_0 are non-empty such that $B_0 \subset g^{\uparrow M}$ and $B_0 \rightarrow g^{\uparrow M \downarrow M \uparrow N}$. Then $X_0 \subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$, e.i., $g^{\uparrow M \downarrow M \uparrow N} \subseteq X_0^{\uparrow N}$. Meanwhile, $g^{\uparrow M \downarrow M} \subset X_0$ as $B_0 \subset g^{\uparrow M}$. This yields $X_0^{\uparrow N} \subseteq g^{\uparrow M \downarrow M \uparrow N}$. Thus, $X_0^{\uparrow N} = g^{\uparrow M \downarrow M \uparrow N}$. Then we have $g^{\uparrow M \downarrow M \uparrow N} \subseteq g_0^{\uparrow N}$ for each $g_0 \in X_0 - g^{\uparrow M \downarrow M}$.

Namely, $g_0^{\uparrow N \downarrow N} \subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$, which implies there exists $g_1 \in g^{\uparrow M \downarrow M}$ such that $g_0 \in g_1^{\uparrow N \downarrow N}$ by Lemma 2. That is to say, $g_0^{\uparrow N} = g_1^{\uparrow N}$.

Sufficiency. By item (1), $g^{\uparrow M \downarrow M} \subset X_0$. Furthermore, with item (2), it is easy to conclude that $g^{\uparrow M \downarrow M \uparrow N} \subseteq g_0^{\uparrow N}$ for $g_0 \in X_0 - g^{\uparrow M \downarrow M}$. This implies $X_0^{\uparrow N} = g^{\uparrow M \downarrow M \uparrow N}$. Then $B_0 \rightarrow g^{\uparrow M \downarrow M \uparrow N} \in \mathcal{R}(S)$ yields $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ is redundant.

Corollary 5 $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N} \in \mathcal{R}^*(S)$ for $g \in G$ w.r.t. S if and only if it is incompatible with $(X_0, B_0) \in OML(G, M, I)$ with $X_0 \neq \emptyset$ and $B_0 \neq \emptyset$ meeting $X_0^{\uparrow N} = g^{\uparrow M \downarrow M \uparrow N}$ and $B_0 \subset g^{\uparrow M}$.

Proof It easily follows from the above Corollary.

Theorem 13 $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N} \in \mathcal{R}^*(S)$ for $g \in G$ if and only if $(g_0^{\uparrow M} \cap g^{\uparrow M})^{\downarrow M} \not\subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$ for $g_0 \in g^{\uparrow M \downarrow M \uparrow N \downarrow N} - g^{\uparrow M \downarrow M}$.

Proof Necessity. Suppose $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N} \in \mathcal{R}^*(S)$. It is conceivable that $g_0^{\uparrow M} \neq g^{\uparrow M}$ for $g_0 \in g^{\uparrow M \downarrow M \uparrow N \downarrow N} - g^{\uparrow M \downarrow M}$. That is, $\{g_0, g\}^{\uparrow M} \subset g^{\uparrow M}$. In other hand, if $(g_0^{\uparrow M} \cap g^{\uparrow M})^{\downarrow M} \subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$, then $\{g_0, g\}^{\uparrow M \downarrow M} \subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$ implying $\{g_0, g\}^{\uparrow M} \neq \emptyset$ and $\{g_0, g\}^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N} \in \mathcal{R}(S)$. Thereby, $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ is redundant in $\mathcal{R}(S)$, a contradiction.

Sufficiency. If $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ is redundant, by Corollary 4, there is $(X_0, B_0) \in OML(G, M, I)$ with $X_0 \neq \emptyset$, $B_0 \neq \emptyset$ and $B_0 \subset g^{\uparrow M}$ meeting there exists $g_1 \in g^{\uparrow M \downarrow M}$ satisfying $g_0^{\uparrow N} = g_1^{\uparrow N}$ for any $g_0 \in X_0 - g^{\uparrow M \downarrow M}$. That is to say $g_0 \in g^{\uparrow M \downarrow M \uparrow N \downarrow N} - g^{\uparrow M \downarrow M}$, then $(g_0^{\uparrow M} \cap g^{\uparrow M})^{\downarrow M} = \{g_0, g\}^{\uparrow M \downarrow M} \subseteq X_0^{\uparrow M \downarrow M} = X_0 \subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$, which contradicts to $(g_0^{\uparrow M} \cap g^{\uparrow M})^{\downarrow M} \not\subseteq g^{\uparrow M \downarrow M \uparrow N \downarrow N}$ for every $g_0 \in g^{\uparrow M \downarrow M \uparrow N \downarrow N} - g^{\uparrow M \downarrow M}$.

Corollary 6 For $(X, B) \in OML(G, M, I)$ with none of X and B is empty, $B \rightarrow X^{\uparrow N}$ is non-redundant w.r.t. S iff $(g^{\uparrow M} \cap B)^{\downarrow M} \not\subseteq X^{\uparrow N \downarrow N}$ for any $g \in X^{\uparrow N \downarrow N} - X$.

Corollary 7 For $(X, B) \in OML(G, M, I)$ with none of X and B is empty, if $B \rightarrow X^{\uparrow N} = \bigcap_{g \in X^{\uparrow N \downarrow N}} g^{\uparrow M} \rightarrow X^{\uparrow N}$, then $B \rightarrow X^{\uparrow N}$ is redundant.

For $B \rightarrow D \in \mathcal{R}(S)$ with $(X, B) \in OML(G, M, I)$, if $D \subseteq X^{\uparrow N}$, then $B \rightarrow D$ is redundant in $\mathcal{R}(S)$. Thus, $\mathcal{R}^*(S)$ is available from $\{B \rightarrow X^{\uparrow N} | (X, B) \in OML(G, M, I)\}$. To trim some proofs, we shall present the following Lemma.

Lemma 4 For any $X \in OML_G(G, M, I)$ from S , if $X \neq \emptyset$, the following results are available:

1. $X = \bigcup_{g \in X} g^{\uparrow M \downarrow M}$;
2. $\bigcup_{g \in X} g^{\uparrow M \downarrow M \uparrow N \downarrow N} \subseteq X^{\uparrow N \downarrow N}$.

Proof For item (1), since $X = (\bigcup_{g \in X} g^{\uparrow M \downarrow M})^{\uparrow M \downarrow M}$, $\bigcup_{g \in X} g^{\uparrow M \downarrow M} \subseteq X$ is available. On the other hand, $g \in g^{\uparrow M \downarrow M}$, then $X \subseteq \bigcup_{g \in X} g^{\uparrow M \downarrow M}$.

Item (2). We have $X^{\uparrow N \downarrow N} = (\bigcup_{g \in X} g^{\uparrow M \downarrow M})^{\uparrow N \downarrow N}$ from item (1). It is available that $g^{\uparrow M \downarrow M \uparrow N \downarrow N} \subseteq X^{\uparrow N \downarrow N}$ since $g^{\uparrow M \downarrow M} \subseteq X$ for $g \in X$. Thus, $\bigcup_{g \in X} g^{\uparrow M \downarrow M \uparrow N \downarrow N} \subseteq X^{\uparrow N \downarrow N}$.

Consequently, each mixed decision rule $B \rightarrow X^{\uparrow N}$ is the mergence of the rules $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ ($g \in X$), that is,

$$B \rightarrow X^{\uparrow N} = \bigcap_{g \in X} (g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}). \quad (20)$$

Then the following theorem is available.

Theorem 14 For any $(X, B) \in OML(G, M, I)$ from S with none of X and B is empty, $g_0^{\uparrow M} \rightarrow g_0^{\uparrow M \downarrow M \uparrow N}$ is non-redundant for any $g_0 \in X$ if $B \rightarrow X^{\uparrow N}$ is non-redundant.

Proof Hypothetically, suppose an object $g_0 \in X$ such that $g_0^{\uparrow M} \rightarrow g_0^{\uparrow M \downarrow M \uparrow N}$ is redundant is available. By Theorem 13, there exists $g_1 \in g_0^{\uparrow M \downarrow M \uparrow N \downarrow N} - g_0^{\uparrow M \downarrow M}$ such that $\{g_0, g_1\}^{\uparrow M \downarrow M} \subseteq g_0^{\uparrow N \downarrow N}$. Since $B \rightarrow X^{\uparrow N}$ is non-redundant, then $(X \cup \{g\})^{\uparrow M \downarrow M} \not\subseteq X^{\uparrow N \downarrow N}$ is available for $g \in X^{\uparrow N \downarrow N} - X$. In other terms, each $g_1 \in \bigcup_{g \in X} g^{\uparrow M \downarrow M \uparrow N \downarrow N} - X$ satisfies $(X \cup \{g\})^{\uparrow M \downarrow M} \not\subseteq X^{\uparrow N \downarrow N}$. Subsequently, such $g_1, g_2 \in X$ are available for $g_1 \in g_2^{\uparrow M \downarrow M \uparrow N \downarrow N} - g_2^{\uparrow M \downarrow M}$ satisfying $\{g_1, g_2\}^{\uparrow M \downarrow M} \not\subseteq X^{\uparrow N \downarrow N}$. In this case, assume that $g_3 \in \{g_1, g_2\}^{\uparrow M \downarrow M}$ while $g_3 \notin X^{\uparrow N \downarrow N}$. Then $g_3 \notin \bigcup_{g \in X} g^{\uparrow M \downarrow M \uparrow N \downarrow N}$ so that $g_3 \notin g_2^{\uparrow M \downarrow M \uparrow N \downarrow N}$, which means $\{g_1, g_2\}^{\uparrow M \downarrow M} \not\subseteq g_2^{\uparrow M \downarrow M \uparrow N \downarrow N}$, a contradiction.

Similar to Definition 9, we also present the definition of core mixed decision rules and relatively necessary mixed decision rules as follows.

Definition 10 For each $g \in G$ from S , we call $g^{\uparrow M} \rightarrow g^{\uparrow M \downarrow M \uparrow N}$ the core mixed decision rule in $\mathcal{R}(S)$. Denote $\mathcal{C}(S)$ the set of all the core mixed decision rules in $\mathcal{R}(S)$. Furthermore, for $(X, B) \in OML(G, M, I)$ with $X \neq \emptyset$ and $B \neq \emptyset$, if there exists an index set T in $\mathcal{C}(S)$ such that $B \rightarrow X^{\uparrow N} = \bigcap_{i \in T} (g_i^{\uparrow M} \rightarrow g_i^{\uparrow M \downarrow M \uparrow N})$, then $B \rightarrow X^{\uparrow N}$ is said to be the relatively necessary mixed decision rule in $\mathcal{R}(S)$. And the collection of all relatively necessary mixed decision rules is denoted by $\mathcal{N}(S)$. In addition, denote $w\mathcal{R}(S)$ the join of $\mathcal{C}(S)$ and $\mathcal{N}(S)$.

Thus, any non-redundant mixed decision rule is available to decompose into the meet of several core mixed decision rules. As formulated by above, we easily have the following conclusion.

Theorem 15 $\mathcal{R}^*(S) \subseteq w\mathcal{R}(S)$ and if all the relatively necessary mixed decision rules are non-redundant, then equality holds.

Proof Straightforward.

Here, we also call $w\mathcal{R}(S)$ the **weak-basis** of $\mathcal{R}(S)$. Generally, $w\mathcal{R}(S)$ is much less than $\mathcal{R}(S)$. To search for $\mathcal{R}^*(S)$, it is available to start from weak-basis $w\mathcal{R}(S)$. In applications, weak-basis is easier available than all the non-redundant mixed decision rules. Therefore, one can search for the weak-basis instead of the basis to improve efficiency.

5 The relationship between mixed decision rules and object-induced three-way rules

In Section 3, we have introduced the connection between MCA and 3WCA. Here, we shall discuss the link between the decision rules of them.

Definition 11 [25] With a decision formal context S , if $(X, (A, B)) \in OEL(G, M, I)$ such that $X = Y$ is available for $(Y, (C, D)) \in OEL(G, N, J)$, then we denote by $OEL(G, M, I) \leq OEL(G, N, J)$ and call the former is finer than the latter. In such case, S is called object-induced three-way consistent (OE-consistent conveniently).

According to Definition 6, it exhibits that O-consistent is equivalent to OE-consistent. Then object-induced three-way rules is defined under the situation of OE-consistent.

Definition 12 [25] If $(X, (A, B)) \in OEL(G, M, I)$ and $(Y, (C, D)) \in OEL(G, N, J)$ ($Y \neq \emptyset, G$) from an OE-consistent decision formal context S meeting $X \subseteq Y$, then $A \rightarrow C$ ($C \neq \emptyset$) and $not B \rightarrow not D$ ($D \neq \emptyset$) are available, which are called positive and negative object-induced three-way decision rules, respectively (OE-P and OE-N decision rules for short). Furthermore, all the OE-P decision rules and all the OE-N decision rules are recorded as $\mathcal{OE} - \mathcal{PR}$ and $\mathcal{OE} - \mathcal{NR}$, respectively. Any rule in $\mathcal{OE} - \mathcal{PR}$ or $\mathcal{OE} - \mathcal{NR}$ is said to be object-induced three-way rule (OE decision rule for short), and all the OE decision rules is denoted by $\mathcal{OER}$.

Likewise, if OE-P decision rules $A \rightarrow C$ and $A_0 \rightarrow C_0$ satisfy $A \subseteq A_0$ and $C_0 \subseteq C$, then $A \rightarrow C$ yields $A_0 \rightarrow C_0$, which implies $A_0 \rightarrow C_0$ is redundant. And the definition of redundant OE-N decision rules from [25] is presented in the following.

Definition 13 [25] Let S be OE-consistent. If $not B \rightarrow not D$ and $not B_0 \rightarrow not D_0$ in $\mathcal{OE} - \mathcal{NR}$ such that $B \subseteq B_0$ and $D_0 \subseteq D$, then $not B \rightarrow not D$ yields $not B_0 \rightarrow not D_0$, and the latter is redundant.

Table 8: A decision formal context $S = (G, M, I, N, J)$

I	m_1	m_2	m_3	m_4	m_5	m_6	n_1	n_2	n_3
g_1	0	0	1	0	0	1	0	0	1
g_2	0	0	0	1	1	1	1	0	1
g_3	1	1	1	0	1	1	0	1	1
g_4	0	0	0	1	1	0	1	0	0

Example 5 Table 8 is derived from Table 5. From Table 8, it is easily checked that S is OE-consistent. Considering Definition 12, all OE decision rules $\mathcal{OER}$ from S and all the non-redundant rules are depicted in Table 9 and Table 10, respectively.

Table 9: OE decision rules in S

$m_6 \rightarrow n_3$	$not m_1 m_2 \rightarrow not n_2$
$m_3 m_6 \rightarrow n_3$	$not m_1 m_2 m_3 \rightarrow not n_2$
$m_5 m_6 \rightarrow n_3$	$not m_1 m_2 m_4 m_5 \rightarrow not n_2$
$m_4 m_5 m_6 \rightarrow n_3$	$not m_1 m_2 m_3 m_6 \rightarrow not n_2$
$m_1 m_2 m_3 m_5 m_6 \rightarrow n_3$	$not m_4 \rightarrow not n_1$
$m_4 m_5 \rightarrow n_1$	$not m_1 m_2 m_4 m_5 \rightarrow not n_1$
$m_4 m_5 m_6 \rightarrow n_1$	$not m_1 m_2 m_4 m_5 \rightarrow not n_1 n_2$
$m_1 m_2 m_3 m_5 m_6 \rightarrow n_2 n_3$	$not m_1 m_2 m_3 m_6 \rightarrow not n_2 n_3$
$m_4 m_5 m_6 \rightarrow n_1 n_3$	

Table 10: Non-redundant rules in $\mathcal{OER}$

$m_6 \rightarrow n_3$	$not m_1 m_2 \rightarrow not n_2$
$m_4 m_5 \rightarrow n_1$	$not m_4 \rightarrow not n_1$
$m_4 m_5 m_6 \rightarrow n_1 n_3$	$not m_1 m_2 m_4 m_5 \rightarrow not n_1 n_2$
$m_1 m_2 m_3 m_5 m_6 \rightarrow n_2 n_3$	$not m_1 m_2 m_3 m_6 \rightarrow not n_2 n_3$

Theorem 16 Let S is O-consistent. $Pos(B) \rightarrow Pos(D) \in \mathcal{OE} - \mathcal{PR}$ if $Pos(B), Pos(D) \neq \emptyset$, and $not Neg(B) \rightarrow not Neg(D) \in \mathcal{OE} - \mathcal{NR}$ if $Neg(B), Neg(D) \neq \emptyset$ for $B \rightarrow D \in \mathcal{R}(S)$.

Proof Assume $(X, B) \in OML(G, M, I)$ and $(Y, D) \in OML(G, N, J)$. Then $(X, (Pos(B), Neg(B))) \in OEL(G, M, I)$ and $(Y, (Pos(D), Neg(D))) \in OEL(G, N, J)$. If $B \rightarrow D \in \mathcal{R}(S)$, by definition, we get $X \subseteq Y$. With the fact that O-consistent is equivalent to OE-consistent, it means $Pos(B) \rightarrow Pos(D)$ is an OE-P decision rule if $Pos(B), Pos(D) \neq \emptyset$ and $not Neg(B) \rightarrow not Neg(D)$ is an OE-N decision rule if $Neg(B), Neg(D) \neq \emptyset$, respectively, according to Definition 12.

Theorem 17 Let $S = (G, M, I, N, J)$ be OE-consistent. Take $(X, (A, B)) \in OEL(G, M, I)$ and $(Y, (C, D)) \in OEL(G, N, J)$ with $X, Y \neq \emptyset, G$. If $X \subseteq Y$, then $A \cup \overline{B} \rightarrow C \cup \overline{D} \in \mathcal{R}(S)$.

Proof It follows from that $(X, A \cup \overline{B}) \in OML(G, M, I)$, $(Y, C \cup \overline{D}) \in OML(G, N, J)$ and $X \subseteq Y$.

Corollary 8 Let S be OE-consistent. Take $(X, (A, B)) \in OEL(G, M, I)$ and $(Y, (C, D)) \in OEL(G, N, J)$ with $X, Y \neq \emptyset, G$. If $A \rightarrow C \in \mathcal{OE} - \mathcal{PR}$ and not $B \rightarrow$ not $D \in \mathcal{OE} - \mathcal{NR}$, then $A \cup \overline{B} \rightarrow C \cup \overline{D} \in \mathcal{R}(S)$.

The above results show that $\mathcal{OE} - \mathcal{PR}$ and $\mathcal{OE} - \mathcal{NR}$ are available by decomposing all mixed decision rules in $\mathcal{R}(S)$ under O-consistent decision formal context. Nevertheless, not all the mixed decision rules are the merge of the OE-P decision rules and OE-N decision rules.

Example 6 Comparing with Table 6 and Table 9, it easy to see that each OE decision rule may be derived from a mixed decision rule. Nevertheless, $m_3\overline{m}_4m_6 \rightarrow n_3$ fails to be a merge of an OE-P decision rule and an OE-N decision rule.

There are not certain relationships between non-redundant OE decision rules and non-redundant mixed decision rules, as well as redundant OE decision rules and redundant mixed decision rules. For instance, the mixed decision rule $m_3\overline{m}_4m_6 \rightarrow n_3$ in Table 6 is non-redundant while the OE-P decision rule $m_3m_6 \rightarrow n_3$ derived from it is redundant in $\mathcal{OER}$ in Table 9. Moreover, a mixed decision rule $\overline{m}_1\overline{m}_2m_6 \rightarrow n_3$ in Table 6 is redundant. However, $m_6 \rightarrow n_3$ induced from it is non-redundant.

In addition, OE decision rules in $\mathcal{OER}$ fail to reflect the relationship between positive condition attributes and negative decision attributes as well as negative condition attributes and positive decision rules. In fact, mixed decision rules overcome such weak point. It shows, $\mathcal{R}(S)$ are more complete than $\mathcal{OER}$.

6 Discussion and conclusion

In most decision making analysis, decision rules are typically built using positive information only. Nevertheless, driven by demands of practical situations, negative information also needs to be exactly represented and managed. The main results of this work is classifying mixed decision rules into diverse parts and introducing the weak-basis of mixed decision rules. We have firstly discussed two types of mixed concept lattices and explored the connections between mixed concept lattice and classical concept lattice, mixed concept lattice and three-way lattice in depth. Subsequently, non-redundant mixed decision rules have been devoted to and the weak-basis has been investigated to approximate the basis of mixed decision rules. In the last part, we have studied the relationship between mixed decision and three-way rules. And some theoretical examples have been given to show the main results of our work.

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