Complex Dynamics of An Epidemic Model With Saturated Media Coverage and Recovery

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Abstract During the outbreak of emerging infectious diseases, media coverage and medical resource play important roles in affecting the disease transmission. To investigate the effects of the saturation of media coverage and limited medical resources, we proposed a mathematical model with extra compartment of media coverage and two nonlinear functions. We theoretically obtained that saturated recovery significantly contributes the occurrence of backward bifurcation and rich dynamics. Then it is reasonable to only considering nonlinear recovery, we theoretically showed that backward bifurcation can occur and multiple equilibria may coexist under certain conditions in this case. And numerical simulations reveals the rich dynamic behaviors, including forward-backward bifurcation, Hopf bifurcation, Saddle-Node bifurcation, Homoclinic bifurcation and unstable limit cycle. Comparing the system with linear recovery, where the threshold dynamic are almost completely characterized by a threshold condition called the basic reproduction number, we concluded that only saturated media impact hardly induces the complicated dynamics, while the nonlinear recovery function, associated with limitation of medical resources, may induce the coexistence of the disease-free equilibrium (DFE) and a endemic state or multiple endemic states, which means that the limitation of medical resources causes much difficulties in eliminating the infectious diseases.

Keywords Mathematical model · Basic reproduction number · Media impact · Nonlinear recovery function · Saddle-Node bifurcation · Hopf bifurcation

1 Introduction

The emerging or re-emergent infectious diseases, including SARS(2003), HINI(2009) and COVID-19, have been threatening the public health and caused worrying concern amongst

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the public and health authorities. In the initial stage of the epidemic, massive news coverage and fast information flow significantly generate profound psychological/behavioural impacts on the public, which may influence disease spread and then the implementation of public interventions. There are a number of mathematical models considering the media impact on disease. Some included a factor in the incidence rate to lower the infection [7, 10, 17, 20, 22], some proposed the on-off media functions [20, 22, 23], where the media impact is triggered by the disease infection, others modelled the level of media coverage as an independent compartment [13, 15, 17, 25, 26, 27], and Song et al. [15] further extended these models by considering the delayed media function and examined local/global bifurcation caused by time delays.

A common assumption in above studies is that either recovery or the growth of media items linearly depends on the number of infected individuals. However, this is not how the thing looks like. Mostly, the limitation of medical resources is the most serious threat during the early stage of emerging infectious diseases, and occurs in almost all countries. For example, with sharp the increase of COVID-19 cases in mainland China in early 2020 [14, 21] the shortage of testing kits, hospital beds is the most serious problem, which implies recovery is a nonlinear function of the number of infected individuals. It is also known that, facing the emerging or re-emerging infectious diseases, news items surge in a certain period but have saturation due to the fatigue of reporting, and consequently the saturated growth of news items should be considered especially during whole epidemic outbreak. Then, it is more reasonable to include the saturation of media coverage and medical resource in a mathematical modeling framework and analyze the impact of the saturations, which fall within the scope of this study.

The paper is organized as follows. In section 2, we proposed a mathematical model with double nonlinear functions to investigate the impacts of saturated media coverage and limited medical resources on disease transmission and investigated that nonlinear recovery function significantly contributes the occurrence of backward bifurcation. In section 3, we consider a special case, linear media response, taking account of high nonlinearity of the system. Theoretical analysis shows that backward bifurcation occurs under certain parameters, and multiple equilibria can coexist in such system and numerical simulations reveal some rich dynamic behaviors, including forward-backward bifurcation, Hopf bifurcation, Saddle-Node bifurcation, Homoclinic bifurcation and unstable limit cycle. In section 4, we further consider another special case, the linear recovery term, to clear the contribution of saturated media on complex dynamics, in which the threshold dynamics are investigated and the global stability of the endemic state under certain conditions is proved. In section 5, a brief discussion is given.

2 Model Description

We divide the population into susceptible individuals (S), exposed but not yet infectious individuals (E) and infected individuals (I). We further assume that the susceptible individuals are infected by infectious individuals with a rate of β , and become exposed. The exposed individuals become infectious with a rate of σ , and then recover with a saturated function $\frac{\gamma I}{1+h_1 I}$, where γ is the recovery rate of the infective individuals and h_1 is the parameter that measures the effect of medical resource limitation [3, 28].

In order to evaluate the effect of media coverage and fast information flow on disease transmission, we include the average number of daily news items related to the outbreak as an independent variable, denoted as $M(t)$, the changing rate of which depends on the

number of newly observed infectious individuals (σE) with a saturated function $\frac{\delta}{1+h_2\sigma E}$, where δ represents the reporting rate and h_2 is the parameter that measures the saturation effect of media reports. We further assume that media reports wane at the rate of μ . To respond the effect of media coverage on reducing the effective contact of susceptible with infectious individuals, we further include an exponential decreasing factor [11, 15], which is represented by $e^{-\alpha M(t)}$, to describe the reduction factor in the incidence. Thus the model equations are

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta e^{-\alpha M} SI - dS + \frac{\gamma I}{1+h_1 I} \\ \frac{dE}{dt} = \beta e^{-\alpha M} SI - (d + \sigma)E \\ \frac{dI}{dt} = \sigma E - dI - \frac{\gamma I}{1+h_1 I} \\ \frac{dM}{dt} = \frac{\delta}{1+h_2\sigma E} \sigma E - \mu M \end{cases} \quad (2.1)$$

where Λ is the rate of flowing into the population and d is the natural death rate. All parameters are non-negative constants.

We can testify that the feasible region of model (2.1) is

$$\Gamma = \{(S, E, I, M) \in R_+^4 : 0 < S + E + I \leq \frac{\Lambda}{d}, 0 < M < \frac{\Lambda \delta \sigma}{d\mu}\}.$$

The disease-free equilibrium $E_0 = (\frac{\Lambda}{d}, 0, 0, 0)$ is always feasible. By calculating the spectral radius of the next generation for the model (2.1), we get the basic reproduction number of this model is

$$R_0 = \beta \frac{\Lambda \sigma}{d(d + \sigma)(d + \gamma)},$$

where R_0 is the expected number of secondary cases produced, in a completely susceptible population, by a typical infected individual during his/her infectious period [5]. The local stability of disease-free equilibrium also can be proved.

Theorem 1 *The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. The bifurcation at $R_0 = 1$ is backward when $h_1 > \tilde{h}_1$; the bifurcation at $R_0 = 1$ is forward when $h_1 < \tilde{h}_1$, that is to say when $R_0 = 1$, E_0 is locally asymptotically stable if $h_1 < \tilde{h}_1$ and unstable if $h_1 > \tilde{h}_1$, where*

$$\tilde{h}_1 = \frac{(d + \gamma)(d + \gamma + \sigma)}{\delta \sigma} \frac{d}{\Lambda} + \frac{\sigma \alpha (d + \gamma)^2}{\mu \gamma}. \quad (2.2)$$

Proof The Jacobian matrix concerned with the linearization of system (2.1) at E_0 is

$$J_{E_0} = \begin{pmatrix} -d & 0 & -\beta \frac{\Lambda}{d} + \gamma & 0 \\ 0 & -(d + \sigma) & \beta \frac{\Lambda}{d} & 0 \\ 0 & \sigma & -(d + \gamma) & 0 \\ 0 & \delta \sigma & 0 & -\mu \end{pmatrix}.$$

So, the corresponding characteristic equation is

$$(\lambda + d)(\lambda + \mu)[\lambda^2 + (2d + \gamma + \sigma)\lambda + (d + \gamma)(d + \sigma)(1 - R_0)] = 0. \quad (2.3)$$

This implies that the disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

When $R_0 = 1$, zero is a simple eigenvalue of J_{E_0} and all other eigenvalues of J_{E_0} have negative real parts. Hence E_0 is a nonhyperbolic equilibrium, and the linearization cannot

determine its stability. Here, we use center manifold [1, 16] to analyze its stability. Let $X = (S, E, I, M)^T = (x_1, x_2, x_3, x_4)^T$, where $(\cdot)^T$ is the transpose of the matrix, so model (2.1) can be denoted as $\dot{X} = F = (f_1, f_2, f_3, f_4)^T$. We denoted the left and right eigenvalues of J_{E_0} as v and w respectively, where $v = (v_1, v_2, v_3, v_4)^T$, $w = (w_1, w_2, w_3, w_4)^T$. Algebraic calculations show that

$$v = (0, \sigma, d + \sigma, 0)n,$$

$$w = (-u(d + \gamma + \sigma), \mu(d + \gamma), \mu\delta, \delta\sigma(d + \gamma))m.$$

where $m > 0$. One can further choose m and n such that $mn > 0$ and $vw = 1$.

Let $\hat{\gamma} = \beta \frac{\Lambda}{d} \frac{\sigma}{d + \sigma} - d$ and $\phi = \hat{\gamma} - \gamma$, then $R_0 < 1$ if and only if $\phi < 0$. According to Theorem 4.1 in [2], the local dynamic behavior near $R_0 = 1$ is determined by the following constants a and b :

$$a = \sum_{i,j,k=1}^4 v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E_{0f}, 0), \quad b = \sum_{i,k=1}^4 v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(E_{0f}, 0).$$

In terms of a and b , we just need to find the nonzero partial derivative of f_2 and f_3 at E_0 when $R_0 = 1$ and $\phi = 0$. Algebraic calculations show that

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_3}(E_{0f}, 0) = \beta, \quad \frac{\partial^2 f_2}{\partial x_3 \partial x_1}(E_{0f}, 0) = \beta, \quad \frac{\partial^2 f_2}{\partial x_3 \partial x_4}(E_{0f}, 0) = -\alpha\beta \frac{\Lambda}{d},$$

$$\frac{\partial^2 f_2}{\partial x_4 \partial x_3}(E_{0f}, 0) = -\alpha\beta \frac{\Lambda}{d}, \quad \frac{\partial^2 f_3}{\partial x_3^2}(E_{0f}, 0) = 2\gamma h_1, \quad \frac{\partial^2 f_3}{\partial \phi \partial x_3}(E_{0f}, 0) = 1.$$

All other partial derivative of f_2 and f_3 are zero. So, we can get that

$$b = (d + \sigma)\mu\delta nm > 0;$$

$$a = 2nm^2\mu\{-\sigma\delta[\mu(d + \gamma + \sigma) + \alpha \frac{\Lambda}{d}\delta\sigma(d + \gamma)] + (d + \sigma)\mu\delta^2\gamma h_1\}.$$

Further, when $R_0 = 1$, $\beta = \frac{(d + \sigma)(d + \gamma)}{\sigma} \frac{d}{\Lambda}$ must hold, then we can get $a < 0$ if and only if

$$h_1 < \frac{(d + \gamma)(d + \gamma + \sigma)}{\delta\sigma} \frac{d}{\Lambda} + \sigma\alpha \frac{(d + \gamma)^2}{\mu\gamma} \doteq \tilde{h}_1 \quad (2.4)$$

So we get the bifurcation at $R_0 = 1$ is backward when $h_1 > \tilde{h}_1$; the bifurcation at $R_0 = 1$ is forward when $h_1 < \tilde{h}_1$, that is to say when $R_0 = 1$, E_0 is locally asymptotically stable if $h_1 < \tilde{h}_1$ and unstable if $h_1 > \tilde{h}_1$. In particular, when $h_1 = 0$, $h_1 < \tilde{h}_1$ holds true, so the bifurcation at $R_0 = 1$ is forward and backward bifurcation can not occur in such scenario. That completes the proof.

It is worthy noticing that the system (2.1) may occur backward bifurcation at $R_0 = 1$, which is mainly associated with parameter h_1 , instead of parameter h_2 . This means that nonlinear recovery function significantly contributes the occurrence of backward bifurcation. We can also easily prove the persistence of the system (2.1) for $R_0 > 1$ by using the standard methods [19], shown in the following theorem (see Appendix A.1 for details).

Theorem 2 *If $R_0 > 1$, there exists a positive constant η_0 such that for the system (2.1)*

$$\liminf_{t \rightarrow \infty} (S(t), E(t), I(t), M(t)) \geq (\eta_0, \eta_0, \eta_0, \eta_0).$$

Note that system (2.1) has high nonlinearity, which is complicated to analyze the dynamical behaviors of this system and the occurring of backward bifurcation for system (2.1) at $R_0 = 1$ is mainly associated with parameter h_1 , instead of parameter h_2 , we initially consider a special case: linear media response (i.e., $h_2 = 0$), and conclude the dynamics of the proposed system theoretically and numerically.

3 Dynamical behaviors of the system (2.1) with $h_2 = 0$

When only considering the saturated recovery (i.e., $h_2 = 0$), the corresponding model equations are

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta e^{-\alpha M} SI - dS + \frac{\gamma}{1+h_1 I} I \\ \frac{dE}{dt} = \beta e^{-\alpha M} SI - (d + \sigma) E \\ \frac{dI}{dt} = \sigma E - dI - \frac{\gamma}{1+h_1 I} I \\ \frac{dM}{dt} = \delta \sigma E - \mu M \end{cases} \quad (3.5)$$

The disease-free equilibrium of system (3.5) is the same as that for system (2.1), and in the following we focus on the stability of the DFE, the feasibility of the endemic states and possible bifurcations around them.

3.1 Existence of the endemic equilibrium

As for system (3.5), the possible endemic equilibrium $E^{**} = (S^*, E^*, I^*, M^*)$ satisfies

$$\begin{aligned} E^* &= \frac{1}{\sigma} \left(d + \frac{\gamma}{1+h_1 I^*} \right) I^*, \quad M^* = \frac{\delta}{\mu} \left(d + \frac{\gamma}{1+h_1 I^*} \right) I^*, \\ S^* &= \frac{(d + \sigma) \left(d + \frac{\gamma}{1+h_1 I^*} \right)}{\sigma \beta} e^{\alpha \frac{\delta}{\mu} \left(d + \frac{\gamma}{1+h_1 I^*} \right) I^*}, \\ \Lambda - \left[d + \frac{d}{\sigma} \left(d + \frac{\gamma}{1+h_1 I^*} \right) \right] I^* - d \frac{(d + \sigma) \left(d + \frac{\gamma}{1+h_1 I^*} \right)}{\sigma \beta} e^{\alpha \frac{\delta}{\mu} \left(d + \frac{\gamma}{1+h_1 I^*} \right) I^*} &= 0. \end{aligned} \quad (3.6)$$

Denote $A(I) = d + \frac{\gamma}{1+h_1 I}$, and

$$f_1(I) = \frac{\Lambda - \left[d + \frac{d}{\sigma} A(I) \right] I}{A(I)}, \quad f_2(I) = d \frac{d + \sigma}{\sigma \beta} e^{\alpha \frac{\delta}{\mu} A(I) I}.$$

Then the existence of the solution of the last equation in (3.6) can be translated into the existence of the intersection of the functions $f_1(I)$ and $f_2(I)$. Algebraic calculation yields

$$f_1'(I) = -\frac{d}{\sigma A^2(I)(1+h_1 I)^2} B(I), \quad f_2'(I) = \alpha \frac{\delta}{\mu} \left[d + \frac{\gamma}{(1+h_1 I)^2} \right] f_2(I) > 0,$$

for $\forall I > 0$, where

$$B(I) = d(d + \sigma)h_1^2 I^2 + 2h_1(d + \sigma)(d + \gamma)I + (d + \gamma)(d + \gamma + \sigma) - \sigma h_1 \gamma \frac{\Lambda}{d}.$$

Further, we can testify that

$$f_1(0) - f_2(0) = d \frac{d + \sigma}{\sigma \beta} (R_0 - 1) > 0 \quad (3.7)$$

if and only if $R_0 > 1$ and

$$\begin{aligned}
 f_1(\hat{I}) - f_2(\hat{I}) &= \frac{1}{A(\hat{I})} \left[\Lambda - \left(d + \frac{d}{\sigma} A(\hat{I}) \right) \hat{I} - d \frac{(d + \sigma) A(\hat{I})}{\sigma \beta} e^{\alpha \frac{\delta}{\mu} A(\hat{I}) \hat{I}} \right] \\
 &\leq \frac{1}{A(\hat{I})} \left[\Lambda - \left(d + \frac{d}{\sigma} d \right) \hat{I} - d \frac{(d + \sigma) A(\hat{I})}{\sigma \beta} e^{\alpha \frac{\delta}{\mu} A(\hat{I}) \hat{I}} \right] \\
 &= -d \frac{(d + \sigma) A(\hat{I})}{\sigma \beta} e^{\alpha \frac{\delta}{\mu} A(\hat{I}) \hat{I}} \\
 &< 0,
 \end{aligned} \tag{3.8}$$

where $\hat{I} = \frac{\Lambda}{d} \frac{\sigma}{d + \sigma}$.

When $(d + \gamma)(d + \gamma + \sigma) > \sigma h_1 \gamma \frac{\Lambda}{d}$, that is to say

$$h_1 < \frac{(d + \gamma)(d + \gamma + \sigma)}{\sigma \gamma} \frac{d}{\Lambda} \doteq \hat{h}_1, \tag{3.9}$$

we have $B(I) > 0$ for $\forall I > 0$, so $f'_1(I) < 0$ must holds for $\forall I > 0$ in this case, which implies that function $f_1(I)$ decreases monotonically on $[0, \infty)$ for $h_1 < \hat{h}_1$. Considering the zero point theorem and combining the monotonicity of $f_2(I)$ for $\forall I > 0$, we conclude that function $f_1(I)$ and $f_2(I)$ have a unique intersection on the interval $(0, \frac{\Lambda}{d})$ for $h_1 < \hat{h}_1$ and $R_0 > 1$; function $f_1(I)$ and $f_1(I)$ have no any intersection on the interval $(0, \frac{\Lambda}{d})$ for $h_1 < \hat{h}_1$ and $R_0 < 1$.

When $(d + \gamma)(d + \gamma + \sigma) < \sigma h_1 \gamma \frac{\Lambda}{d}$, that is to say $h_1 > \hat{h}_1$. Denote $f(I) = f_1(I) - f_2(I)$, we get $f(\hat{I}) < 0$ and $f'(I) < 0$ for $I > \hat{I}$, then $f(I) < 0$ must hold true for $I \geq \hat{I}$, so we just need to consider if functions $f_1(I)$ and $f_2(I)$ have an intersection on $[0, \hat{I}]$. For $R_0 > 1$, $f(0) > 0$ must hold, which implies that the system (3.5) at least has one endemic equilibrium for $R_0 > 1$ on the basis of the zero point theorem. For $R_0 < 1$, $f(0) < 0$ must hold. Denote $f_{\max} = \max_{0 \leq I \leq \hat{I}} \{f(I)\}$. When $f_{\max} < 0$, functions $f_1(I)$ and $f_2(I)$ have no intersection on $[0, \frac{\Lambda}{d}]$.

For $f_{\max} > 0$, there must be a $\tilde{I} \in (0, \frac{\Lambda}{d} \frac{\sigma}{d + \sigma})$, such that $f(\tilde{I}) > 0$, so $f(0)f(\tilde{I}) < 0$ and $f(\tilde{I})f(1) < 0$. According to the zero point theorem, $f(I)$ has at least two zero point, one is in the interval of $(0, \tilde{I})$ and the other in $(\tilde{I}, \frac{\Lambda}{d} \frac{\sigma}{d + \sigma})$, i.e., the system has at least two positive equilibriums.

According to the above analysis, we obtain the following conclusion.

Theorem 3 If $h_1 < \hat{h}_1$, the system (3.5) has a unique endemic equilibrium for $R_0 > 1$, denoted by $E_1 = (S^*, E^*, I^*, M^*)$, which satisfies (3.6) and $0 < I^* < \hat{I}$; but it has no endemic equilibrium for $R_0 < 1$. If $h_1 > \hat{h}_1$, the system (3.5) exists at least one endemic equilibrium for $R_0 > 1$; the system (3.5) has no endemic equilibrium for $R_0 < 1$ and $f_{\max} < 0$, but it at least exists two endemic equilibrium for $R_0 < 1$ and $f_{\max} > 0$.

Theorem 4 When $h_1 < \hat{h}_1$ and $R_0 > 1$, the unique endemic equilibrium of system (3.5) is locally asymptotically stable.

Proof The Jacobian matrix concerned the linearization of system (3.5) at E_1 is

$$J_{E_1} = \begin{pmatrix} -\beta e^{-\alpha M^*} I^* - d & 0 & -\beta e^{-\alpha M^*} S^* + \frac{\gamma}{(1 + h_1 I^*)^2} & \alpha \beta e^{-\alpha M^*} S^* I^* \\ \beta e^{-\alpha M^*} I^* & -(d + \sigma) & \beta e^{-\alpha M^*} S^* & -\alpha \beta e^{-\alpha M^*} S^* I^* \\ 0 & \sigma & -(d + \frac{\gamma}{(1 + h_1 I^*)^2}) & 0 \\ 0 & \delta \sigma & 0 & -\mu \end{pmatrix}.$$

Let $r(I^*) = \frac{\gamma}{1+h_1 I^*}$, we can get

$$\begin{aligned}\beta e^{-\alpha M^*} (S^* - I^*) &= \beta e^{-\alpha M^*} \left[\frac{(d+\sigma)(d+r(I^*))}{\sigma\beta} e^{\alpha M^*} - I^* \right] \\ &= \frac{(d+\sigma)[d+r(I^*)]}{\sigma} - \beta I^* e^{-\alpha M^*}\end{aligned}$$

and

$$\begin{aligned}\alpha\beta e^{-\alpha M^*} S^* I^* &= \alpha\beta e^{-\alpha M^*} \frac{(d+\sigma)[d+r(I^*)]}{\sigma\beta} e^{\alpha M^*} I^* \\ &= \alpha \frac{(d+\sigma)[d+r(I^*)]}{\sigma} I^*\end{aligned}$$

Then the characteristic equation yields

$$\begin{aligned}|\lambda E - J_{E_1}| &= (\lambda + d) \begin{vmatrix} \lambda + \beta I^* e^{-\alpha M^*} + (d+\sigma) & -\beta e^{-\alpha M^*} (S^* - I^*) & \alpha\beta e^{-\alpha M^*} S^* I^* \\ -\sigma & \lambda + (d + \frac{\gamma}{(1+h_1 I^*)^2}) & 0 \\ -\delta\sigma & 0 & \lambda + \mu \end{vmatrix} \\ &\doteq (\lambda + d)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) = 0\end{aligned}\tag{3.10}$$

with

$$\begin{aligned}a_1 &= \mu + 2d + \frac{\gamma}{(1+h_1 I^*)^2} + \sigma + g(I^*), \\ a_2 &= g(I^*)[d + \frac{\gamma}{(1+h_1 I^*)^2} + \mu + \sigma] + \mu[2d + \sigma + \frac{\gamma}{(1+h_1 I^*)^2}] \\ &\quad + \alpha\delta(d+\sigma)[d+r(I^*)]I^* - (d+\sigma)\gamma\frac{h_1 I^*}{(1+h_1 I^*)^2}, \\ a_3 &= \mu[d + \frac{\gamma}{(1+h_1 I^*)^2} + \sigma]g(I^*) - \mu(d+\sigma)\gamma\frac{h_1 I^*}{(1+h_1 I^*)^2} \\ &\quad + \alpha\delta[d+r(I^*)](d+\sigma)I^*[d + \frac{\gamma}{(1+h_1 I^*)^2}],\end{aligned}\tag{3.11}$$

where $g(I^*) := \beta I^* e^{-\alpha M^*} = \frac{[d+r(I^*)](d+\sigma)I^*}{\sigma\frac{\Lambda}{d} - [d+r(I^*)+\sigma]I^*}$.

Further, we have

$$\begin{aligned}a_1 a_2 - a_3 &= \mu^2[g(I^*) + 2d + \sigma + D] + \mu[2d + D + \sigma + g(I^*)]^2 \\ &\quad + [2d + D + \sigma + g(I^*)](d + D + \sigma)g(I^*) \\ &\quad + \alpha\delta(d+\sigma)(d+r)I^*[d + \sigma + \mu + g(I^*)] \\ &\quad - [2d + D + \sigma + g(I^*)](d+\sigma)Dh_1 I^*\end{aligned},\tag{3.12}$$

where $D = \frac{\gamma}{(1+h_1 I^*)^2}$.

We can get the system (3.5) has a unique endemic equilibrium for $R_0 > 1$ in the above cases, on the basis of Theorem 3. And $a_0 > 0$ must be true.

When $h_1 < \hat{h}_1$, the inequality

$$[2(d+\sigma)(d+\gamma) + (d+\sigma)dh_1 I]h_1 I > h_1 \gamma \sigma \frac{\Lambda}{d} - (d+\sigma+\gamma)(d+\gamma)\tag{3.13}$$

holds true since the right side of inequality (3.13) is less than 0 if $h_1 < \hat{h}_1$ and the left side of inequality (3.13) must be greater than 0. And the inequality (3.13) is equivalent to the inequality

$$[(d + \sigma)(d + r) + r(d + r + \sigma)]h_1 I^* > h_1 r \sigma \frac{\Lambda}{d} - (d + \sigma + r)(d + r), \quad (3.14)$$

where $r = \frac{\gamma}{1 + h_1 I^*}$. The inequality (3.14) is further equivalent to the following inequality

$$[d + \frac{\gamma}{(1 + h_1 I^*)^2} + \sigma]g(I^*) > (d + \sigma)\gamma \frac{h_1 I^*}{(1 + h_1 I^*)^2}, \quad (3.15)$$

which is naturally satisfied. Then in this case, $a_i > 0$ ($i = 2, 3$) and $a_1 a_2 - a_3 > 0$ must hold true from the expressions of a_i ($i = 2, 3$) and $a_1 a_2 - a_3$.

So if $h_1 < \hat{h}_1$, $a_i > 0$ ($i = 1, 2, 3$) and $a_1 a_2 - a_3 > 0$ hold true. According to the Hurwitz criterion, the roots of the polynomial $\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0$ all have negative real parts, which means that the roots of the characteristic equation (3.10) all have negative real parts. So this unique endemic equilibrium is locally asymptotically stable for $h_1 < \hat{h}_1$. That completes the proof.

Theorem 5 *The disease-free equilibrium E_0 of system (3.5) is globally asymptotically stable for $h_1 < \hat{h}_1$ and $R_0 < \frac{d + \gamma h_0}{d + \gamma}$ with $h_0 = \frac{\sigma \gamma}{\sigma \gamma + (d + \gamma)(d + \gamma + \sigma)}$.*

Proof Let's consider a Lyapunov function

$$V = \sigma E + (d + \sigma)I.$$

For $h_1 < \hat{h}_1$, we can obtain

$$\begin{aligned} \frac{dV}{dt} &= \sigma \beta e^{-\alpha M} S I - (d + \sigma)(d + \frac{\gamma}{1 + h_1 I})I \\ &\leq \sigma \beta \frac{\Lambda}{d} I - (d + \sigma)(d + \frac{\gamma}{1 + \hat{h}_1 \frac{\Lambda}{d}})I \\ &= (d + \sigma)(d + \gamma)I(R_0 - \frac{d + \gamma h_0}{d + \gamma}), \end{aligned}$$

So, if $R_0 < \frac{d + \gamma h_0}{d + \gamma}$, we have $\frac{dV}{dt} \leq 0$, and $\frac{dV}{dt} = 0$ holds if and only if $I = 0$. Applying LaSalle's invariance principle, we get that $\lim_{t \rightarrow \infty} I(t) = 0$. Then applying comparison principle, we can testify $\lim_{t \rightarrow \infty} E(t) = \lim_{t \rightarrow \infty} M(t) = 0$ and $\lim_{t \rightarrow \infty} S(t) = \frac{\Lambda}{d}$. So the disease-free equilibrium E_0 is globally asymptotically stable if $h_1 < \hat{h}_1$ and $R_0 < \frac{d + \gamma h_0}{d + \gamma}$. That completes the proof.

Note that system (3.5) has a unique positive equilibrium, which is locally stable, when $R_0 > 1$ and $h_1 < \hat{h}_1$, we can further develop the global result of this system under certain parameter restrictions by using compound matrices and geometric ideas [8, 9, 12, 18].

Theorem 6 *When $h_1 < \hat{h}_1$ and $R_0 > 1$, the unique endemic equilibrium E_1 of the system (3.5) is globally asymptotically stable if $\max\{\frac{\delta \sigma}{d + \mu - \gamma}, \frac{\delta \sigma}{d + \mu - 2\sigma}\} < \frac{d}{\alpha \Lambda}$, $d + \mu + \frac{\gamma}{(1 + h_1 \frac{\Lambda}{d})^2} > 2\sigma$ and $\gamma < \min\{2d + \sigma, d + \mu\}$.*

Proof The Jacobian matrix of system (3.5) reads

$$J = \begin{pmatrix} -\beta e^{-\alpha M} I - d & 0 & -\beta e^{-\alpha M} S + \frac{\gamma}{(1+h_1 I)^2} & \alpha \beta e^{-\alpha M} S I \\ \beta e^{-\alpha M} I & -(d + \sigma) & \beta e^{-\alpha M} S & -\alpha \beta e^{-\alpha M} S I \\ 0 & \sigma & -(d + \frac{\gamma}{(1+h_1 I)^2}) & 0 \\ 0 & \delta \sigma & 0 & -\mu \end{pmatrix}.$$

The third additive compound matrix of J is

$$J^{[3]} = \begin{pmatrix} a_{11} & 0 & \alpha \beta e^{-\alpha M} S I & \alpha \beta e^{-\alpha M} S I \\ 0 & a_{22} & \beta e^{-\alpha M} S & \beta e^{-\alpha M} S - \frac{\gamma}{(1+h_1 I)^2} \\ -\delta \sigma & \sigma & a_{33} & 0 \\ 0 & 0 & \beta e^{-\alpha M} I & a_{44} \end{pmatrix},$$

and the corresponding linear compound system reads

$$(X, Y, Z, W)^T = J^{[3]}(X, Y, Z, W)^T \quad (3.16)$$

where

$$a_{11} = -\beta e^{-\alpha M} I - d - (d + \sigma) - (d + \frac{\gamma}{(1+h_1 I)^2}), \quad a_{22} = -\beta e^{-\alpha M} I - d - (d + \sigma) - \mu, \\ a_{33} = -\beta e^{-\alpha M} I - d - (d + \frac{\gamma}{(1+h_1 I)^2}) - \mu, \quad a_{44} = -(d + \sigma) - (d + \frac{\gamma}{(1+h_1 I)^2}) - \mu.$$

Then we just need to testify the global stability of this linear compound system according to [8, 9]. To do this, we choose the Lyapunov functional as

$$V(S, E, I, R, X, Y, Z, W) = \begin{cases} \max\{t_1 \frac{|X|}{I}, \frac{|Y|+|W|}{E}, \frac{|Z|+|W|}{I}\} & \text{if } E \geq I, \\ \max\{t_2 \frac{|X|}{E}, \frac{|Y|+|W|}{E}, \frac{|Z|+|W|}{E}\} & \text{if } E < I, \end{cases}$$

where t_1, t_2 will be determined later. It is easy to show the system (3.5) admits a unique nonnegative bounded solution for all $t \geq 0$ and any non-negative initial value. Combing the Theorem 2, there exist two positive constants C_1, C_2 such that

$$C_1(|X| + |Y| + |Z| + |W|) \leq V \leq C_2(|X| + |Y| + |Z| + |W|).$$

Recalling that

$$\frac{\dot{E}}{E} = \frac{\beta e^{-\alpha M} S I}{E} - (d + \sigma), \quad \frac{\dot{I}}{I} = \frac{\sigma E}{I} - (d + \frac{\gamma}{1+h_1 I}).$$

Then, discussing the derivative of V along the trajectory of the linear compound system (3.16) for different conditions and denoting D_+ the right-hand (total) derivative with respect to t .

Case (1): $E \geq I$ and $V = t_1 \frac{|X|}{I}$, which implies $t_1 \frac{|X|}{I} > \{\frac{|Y|+|W|}{E}, \frac{|Z|+|W|}{I}\}$. Algebraic calculation yields

$$\begin{aligned} D_+ V &\leq t_1 \frac{[-\beta e^{-\alpha M} I - d - (d + \sigma) - (d + \frac{\gamma}{(1+h_1 I)^2})]|X| + \alpha \beta e^{-\alpha M} S I[|Z| + |W|]}{I} - \frac{I'}{I} t_1 \frac{|X|}{I} \\ &= t_1 \frac{|X|}{I} [-\beta e^{-\alpha M} I - d - (d + \sigma) - (d + \frac{\gamma}{(1+h_1 I)^2}) - \frac{I'}{I}] + t_1 \alpha \beta e^{-\alpha M} S I \frac{|Z| + |W|}{I} \\ &\leq [(t_1 \alpha S - 1) \beta e^{-\alpha M} I - d - (d + \sigma) - \sigma \frac{E}{I} + \frac{\gamma h_1 I}{(1+h_1 I)^2}] V \\ &\leq [(t_1 \alpha S - 1) \beta e^{-\alpha M} I - (2d + \sigma - \gamma) - \sigma \frac{E}{I}] V. \end{aligned}$$

Case (2): $E \geq I$ and $V = \frac{|Y|+|W|}{E}$, which implies $\frac{|Y|+|W|}{E} > \{t_1 \frac{|X|}{I}, \frac{|Z|+|W|}{I}\}$. Algebraic calculation yields

$$\begin{aligned}
 D_+V &\leq \frac{[-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu]|Y| + \beta e^{-\alpha M}(S+I)|Z| + [\beta e^{-\alpha M}S - (d + \sigma) - d - \mu]|W|}{E} \\
 &\quad - \frac{E'}{E} \frac{|Y| + |W|}{E} \\
 &= \frac{|Y| + |W|}{E} [-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu - \frac{E'}{E}] + I \frac{\beta e^{-\alpha M}(S+I)}{E} \frac{|Z| + |W|}{I} \\
 &\leq [-\beta e^{-\alpha M}I - d - \mu - \frac{\beta e^{-\alpha M}SI}{E} + \frac{\beta e^{-\alpha M}(SI + I^2)}{E}]V \\
 &= [-\beta e^{-\alpha M}I(1 - \frac{I}{E}) - d - \mu]V.
 \end{aligned}$$

Since $E \geq I$, $1 - \frac{I}{E} \geq 0$ must hold, which means $D_+V \leq -(d + \mu)V$.

Case (3): $E \geq I$ and $V = \frac{|Z|+|W|}{I}$, which implies $\frac{|Z|+|W|}{I} > \{\frac{|Y|+|W|}{E}, t_1 \frac{|X|}{I}\}$. Algebraic calculation yields

$$\begin{aligned}
 D_+V &\leq \frac{\delta\sigma|X| + \sigma|Y| + [-d - (d + \frac{\gamma}{(1+h_1I)^2}) - \mu]|Z| + [-(d + \sigma) - (d + \frac{\gamma}{(1+h_1I)^2}) - \mu]|W|}{I} - \frac{I'}{I} \frac{|Z| + |W|}{I} \\
 &= t_1 \frac{|X|}{I} \frac{\delta\sigma}{t_1} + \sigma \frac{E}{I} \frac{|Y| + |W|}{E} + \frac{|Z| + |W|}{I} [-d - (d + \frac{\gamma}{(1+h_1I)^2}) - \mu - \frac{I'}{I}] \\
 &\leq [\frac{\delta\sigma}{t_1} - d - \mu + \frac{\gamma h_1 I}{(1+h_1I)^2}]V \\
 &\leq [\frac{\delta\sigma}{t_1} - (d + \mu - \gamma)]V.
 \end{aligned}$$

Case (4): $E < I$ and $V = t_2 \frac{|X|}{E}$, which implies $t_2 \frac{|X|}{E} > \{\frac{|Y|+|W|}{E}, \frac{|Z|+|W|}{E}\}$. Algebraic calculation yields

$$\begin{aligned}
 D_+V &\leq t_2 \frac{[-\beta e^{-\alpha M}I - d - (d + \sigma) - (d + \frac{\gamma}{(1+h_1I)^2})]|X| + \alpha\beta e^{-\alpha M}SI[|Z| + |W|]}{E} - \frac{E'}{E} t_2 \frac{|X|}{E} \\
 &= t_2 \frac{|X|}{E} [-\beta e^{-\alpha M}I - d - (d + \sigma) - (d + \frac{\gamma}{(1+h_1I)^2}) - \frac{E'}{E}] + t_2 \alpha\beta e^{-\alpha M}SI \frac{|Z| + |W|}{E} \\
 &\leq [(t_2\alpha S - 1)\beta e^{-\alpha M}I - d - (d + \frac{\gamma}{(1+h_1I)^2}) - \frac{\beta e^{-\alpha M}SI}{E}]V.
 \end{aligned}$$

Case (5): $E < I$ and $V = \frac{|Y|+|W|}{E}$, which implies $\frac{|Y|+|W|}{E} > \{t_2 \frac{|X|}{E}, \frac{|Z|+|W|}{E}\}$. Algebraic calculation yields

$$\begin{aligned} D_+V &\leq \frac{[-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu]|Y| + \beta e^{-\alpha M}(S+I)|Z| + [\beta e^{-\alpha M}S - (d + \sigma) - d - \mu]|W|}{E} \\ &\quad - \frac{E'}{E} \frac{|Y|+|W|}{E} \\ &= \frac{|Y|+|W|}{E} [-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu - \frac{E'}{E}] + \beta e^{-\alpha M}(S+I) \frac{|Z|+|W|}{E} \\ &\leq [-\beta e^{-\alpha M}I - d - \mu - \frac{\beta e^{-\alpha M}SI}{E} + \beta e^{-\alpha M}(S+I)]V \\ &= [-\beta e^{-\alpha M}S(\frac{I}{E} - 1) - d - \mu]V. \end{aligned}$$

Since $E < I$, $\frac{I}{E} - 1 > 0$ must hold, which means $D_+V \leq -(d + \mu)V$

Case (6): $E < I$ and $V = \frac{|Z|+|W|}{E}$, which implies $\frac{|Z|+|W|}{E} > \{\frac{|Y|+|W|}{E}, t_2 \frac{|X|}{E}\}$. Algebraic calculation yields

$$\begin{aligned} D_+V &\leq \frac{\delta\sigma|X| + \sigma|Y| + [-d - (d + \frac{\gamma}{(1+h_1I)^2}) - \mu]|Z| + [-(d + \sigma) - (d + \frac{\gamma}{(1+h_1I)^2}) - \mu]|W|}{E} + \frac{E'}{E} \frac{|Z|+|W|}{E} \\ &= t_2 \frac{|X|}{E} \frac{\delta\sigma}{t_2} + \sigma \frac{|Y|}{E} + \frac{|Z|+|W|}{E} [-d - (d + \frac{\gamma}{(1+h_1I)^2}) - \mu - \frac{\beta e^{-\alpha M}SI}{E} + (d + \sigma)]V \\ &\leq [\frac{\delta\sigma}{t_2} + 2\sigma - (d + \frac{\gamma}{(1+h_1I)^2}) - \mu - \frac{\beta e^{-\alpha M}SI}{E}]V \\ &\leq [\frac{\delta\sigma}{t_2} + 2\sigma - (d + \frac{\gamma}{(1+\hat{h}_1\frac{\Lambda}{d})^2}) - \mu - \frac{\beta e^{-\alpha M}SI}{E}]V. \end{aligned}$$

Since $\max\{\frac{\delta\sigma}{d+\mu-\gamma}, \frac{\delta\sigma}{d+\frac{\gamma}{(1+\hat{h}_1\frac{\Lambda}{d})^2}+\mu-2\sigma}\} < \frac{1}{\alpha} \frac{d}{\Lambda}$, $d + \frac{\gamma}{(1+h_1\frac{\Lambda}{d})^2} + \mu > 2\sigma$ and $\gamma < \min\{2d + \sigma, d + \mu\}$, we can choose two positive constants t_1 and t_2 such that $\frac{\delta\sigma}{d+\mu-\gamma} < t_1 < \frac{1}{\alpha} \frac{d}{\Lambda}$ and $\frac{\delta\sigma}{d+\frac{\gamma}{(1+\hat{h}_1\frac{\Lambda}{d})^2}+\mu-2\sigma} < t_2 < \frac{1}{\alpha} \frac{d}{\Lambda}$ respectively, which means that $t_1\alpha S - 1 < 0$, $2d + \sigma - \gamma > 0$, $\frac{\delta\sigma}{t_1} - d - \mu + \gamma < 0$, $t_2\alpha S - 1 < 0$ and $\frac{\delta\sigma}{t_2} + 2\sigma - d - \frac{\gamma}{(1+\hat{h}_1\frac{\Lambda}{d})^2} - \mu < 0$. Therefore, there exists a positive constant κ such that $-\kappa > \max\{-(2d + \sigma - \gamma), -d, \frac{\delta\sigma}{t_1} - d - \mu + \gamma, \frac{\delta\sigma}{t_2} + 2\sigma - d - \frac{\gamma}{(1+\hat{h}_1\frac{\Lambda}{d})^2} - \mu\}$, in this case, $D_+V \leq -\kappa V$ must hold. By Corollary 3.2 in [9], E_1 of the system (3.5) is globally asymptotically stable. That completes the proof.

To further analyze the system (3.5) in details, we investigate the number of positive equilibria and their stability for $h_1 > \hat{h}_1$ on the basis of Theorem 3. Here we have only analyzed a few specific cases, and others are similar but complicated.

Let's initially consider the case when $h_1 > \hat{h}_1$ and $R_0 > 1$. Algebraic calculation yields the quadratic equation $B(I) = 0$ has a positive real root, denoted by I_1 , and a negative real root,

$$I_1 = \frac{-2h_1(d + \sigma)(d + \gamma) + \sqrt{\Delta}}{2d(d + \sigma)h_1^2},$$

where $\Delta = 4h_1^2(d + \sigma)^2(d + \gamma)^2 - 4d(d + \sigma)h_1^2[(d + \gamma)(d + \gamma + \sigma) - \frac{\Lambda}{d}\sigma h_1\gamma]$ and I_1 satisfies $I_1 < \frac{\Lambda}{d} \frac{\sigma}{d + \sigma}$. Then we have $f_1'(I) > 0$ for $0 < I < I_1$ and $f_1'(I) < 0$ for $I_1 < I < \frac{\Lambda}{d}$. Further, we have

$$f_1''(I) = -2h_1d\gamma \frac{d + \gamma + h_1\Lambda}{A^3(I)(1 + h_1I)^3} < 0,$$

we then obtain that the function $f_1(I)$ is a strictly convex function that increases on the interval $(0, I_1]$ and decreases on the interval $(I_1, \frac{\Lambda}{d})$. Similarly we can analyze the concavity of the function $f_2(I)$, which is given in Table A1 by considering eight cases (case A-H), see Appendix A.2 for details.

Combining the monotonicity and concavity of functions $f_1(I)$ and $f_2(I)$, we discuss the number of local equilibria in three scenarios according to the number of intersection of functions $f_1(I)$ and $f_2(I)$ in the interval of $(0, \frac{\Lambda}{d}]$ (Table 1).

Scenario 1): If cases B, E or F hold, that is to say, $f_2(I)$ is a strictly concave function of monotonically increasing. And the function $f_1(I)$ is a strictly convex function that increases on the interval $(0, I_1]$ and decreases on the interval $(I_1, \frac{\Lambda}{d})$. Combining (3.7) and (3.8), we can conclude the system (3.5) has a unique endemic equilibrium for $R_0 > 1$. When case A holds, $f_2(I)$ is a concave function of monotonically increasing and we can do the similar analysis.

Scenario 2): If case C holds, system (3.5) may have 1 to 3 endemic equilibria we use a table (i.e., Table A2) to illustrate for simplicity. For the sake of easy to representation, if there is one, two or three endemic equilibria, we denote them as $E_1 = (S^*, E^*, I^*, M^*)$; $E_{21} = (S_{21}^*, E_{21}^*, I_{21}^*, M_{21}^*)$ and $E_{22} = (S_{22}^*, E_{22}^*, I_{22}^*, M_{22}^*)$ ($I_{21}^* < I_{22}^*$); $E_{31} = (S_{31}^*, E_{31}^*, I_{31}^*, M_{31}^*)$, $E_{32} = (S_{32}^*, E_{32}^*, I_{32}^*, M_{32}^*)$ and $E_{33} = (S_{33}^*, E_{33}^*, I_{33}^*, M_{33}^*)$ ($I_{31}^* < I_{32}^* < I_{33}^*$) respectively. As for the case D, we can do the same analysis.

Scenario 3): If case G or H holds, we can get the system (3.5) has one or two or three endemic equilibria for $R_0 > 1$ similarly.

As for the case when $h_1 > \hat{h}_1$, $R_0 < 1$, we can use the same method to prove that system (3.5) may have 0 to 4 endemic equilibria (see details for Table 1), which means that the system (3.5) may occur backward bifurcation.

Due to computational difficulties, we will implement numerical simulations to analyze dynamical behaviors of (3.5) and possible bifurcations under various cases with the help of the MATCONT package [4]. We shall illustrate forward bifurcation, backward bifurcation, forward-backward bifurcations, Hopf bifurcation and Saddle-node bifurcations in the following.

Table 1 The number of endemic equilibria.

scenarios	Conditions	Concavity	Number of equilibria for $R_0 > 1$	Number of equilibria for $R_0 < 1$
scenario 1	Case B : (A.3) holds $Q > 0$ and $\tilde{I} > \frac{\Lambda}{d}$	$f_2''(I) > 0$ for $0 < I < \frac{\Lambda}{d}$	1	0 or 1 or 2
	Case E : (A.5) holds $Q > 0$ and $\frac{\Lambda}{d} < \tilde{I}_1$			
	Case F : (A.6) holds and $Q > 0$			
	Case A : (A.3) holds $Q > 0$ and $\tilde{I} < \frac{\Lambda}{d}$	$f_2''(I) > 0$ for $0 < I < \tilde{I}$ and $\tilde{I} < I < \frac{\Lambda}{d}$ $f_2''(I) = 0$ for $I = \tilde{I}$		
scenario 2	Case C : (A.5) holds $Q > 0$ and $\tilde{I}_2 < \frac{\Lambda}{d}$	$f_2''(I) > 0$ for $0 < I < \tilde{I}_1$ and $\tilde{I}_2 < I < \frac{\Lambda}{d}$ $f_2''(I) < 0$ for $\tilde{I}_1 < I < \tilde{I}_2$	1 or 2 or 3 (See Table A2 for details))	0 or 1 or 2 or 3 or 4
	Case D : (A.5) holds $Q > 0$ and $\tilde{I}_1 < \frac{\Lambda}{d} < \tilde{I}_2$	$f_2''(I) > 0$ for $0 < I < \tilde{I}_1$ $f_2''(I) < 0$ for $\tilde{I}_1 < I < \frac{\Lambda}{d}$	1 or 2 or 3	
scenario 3	Case G : $Q < 0$ and $\tilde{I} < \frac{\Lambda}{d}$	$f_2''(I) < 0$ for $0 < I < \tilde{I}$ $f_2''(I) > 0$ for $\tilde{I} < I < \frac{\Lambda}{d}$	1 or 2 or 3	0 or 1 or 2
	Case H : $Q < 0$ and $\tilde{I} > \frac{\Lambda}{d}$	$f_2''(I) < 0$ for $0 < I < \frac{\Lambda}{d}$		

¹ Case A, B, ... and H are given for discussing concavity of function $f_2(I)$, referring to Table A1.

A forward bifurcation is illustrated in Fig.1, and Fig.1(a) gives variation in equilibrium level of infections (I^*) with the basic reproduction number (associated with parameter Λ) and corresponding bifurcations, say, $R_0 = 5.0738$ ($\Lambda = 1.0249$) and $R_0 = 30.1368$ ($\Lambda = 6.0878$), the system (3.5) undergoes Hopf bifurcation. Fig.1(b-e) illustrate the time series of the system (3.5) with different parameters Λ which stabilize to either the disease-free equilibrium or endemic state or bifurcated periodic solution. The first Lyapunov coefficient at two Hopf points are -7.9051×10^7 and -2.2751×10^7 respectively, which means that the Hopf bifurcation is supercritical and the periodic orbits are born stably, that is to say the system exists a stable limit cycle when $5.0738 < R_0 < 30.1368$ (or $1.0249 < \Lambda < 6.0878$). And we can see that when $R_0 < 1$ ($0 < \Lambda < 0.2020$), the system (3.5) only exists the disease-free equilibrium, which is globally asymptotically stable, shown in Fig.1(b); when $1 < R_0 < 5.0738$ ($0.2020 < \Lambda < 1.0249$) and $R_0 > 30.1368$ ($\Lambda > 6.0878$), the system (3.5) exists an unstable disease-free equilibrium and a globally stable endemic equilibrium, shown in Fig.1(c) and (e); when $5.0738 < R_0 < 30.1368$ ($1.0249 < \Lambda < 6.0878$), the system (3.5) exists an unstable disease-free equilibrium and an unstable endemic equilibrium, but exists a bifurcated periodic solution around this unstable endemic equilibrium, shown in Fig.1(d).

Fig. 1 (a): Bifurcation diagram in (R_0, I) plane with R_0 (associated with parameter Λ), where $\sigma = 0.2$, $d = 10^{-3}$, $\beta = 10^{-3}$, $\gamma = 0.2$, $h_1 = 0.011$, $h_2 = 0$, $\mu = 0.01$, $\delta = 0.1$, $\alpha = 0.0430$, $\hat{h}_1 = 7.4736 \times 10^{-4}$. The blue curve represents stable equilibrium and a black dot marked with "Hopf" ($R_0 = 5.0738$ ($\Lambda = 1.0249$) and $R_0 = 30.1368$ ($\Lambda = 6.0878$)) indicates that the system (3.5) goes through Hopf bifurcation. (b-e): The time series of the system (3.5) with different R_0 , and in (b-e), R_0 is 0.4950, 2.4752, 14.8511, 34.6526 (the corresponding values of Λ are 0.1, 0.5, 3, 7) and the initial values are (1609, 80, 52, 90), (475.4021, 11.5078, 10.0900, 23.0157), (2901.6825, 36.9450, 61.3725, 73.8900), (6841.3621, 50.0298, 100.6080, 100.0596), respectively.

A backward bifurcation is shown in Fig.2, and Fig.2(a) illustrates the saddle-node bifurcation at $R_0 = 0.1532$ and backward bifurcations at $R_0 = 1$. Fig.2(b) indicates that the system (3.5) has two endemic equilibria for $\Lambda = 3$, denoted as $E_{2i} = (S_{2i}^*, E_{2i}^*, I_{2i}^*, M_{2i}^*)$, where $i = 1, 2$ and $I_{21}^* < I_{22}^*$, since functions $f_1(I)$ and $f_2(I)$ have two intersections for $I > 0$ in such scenario. When $R_0 < 0.1532$ ($\Lambda < 0.7115$), the system (3.5) exists a unique globally stable disease-free equilibrium, shown in Fig.1(c). As parameter Λ increases the DFE E_0 coexists with a stable endemic state E_{22} for $0.1532 < R_0 < 1$ (i.e., $0.7115 < \Lambda < 4.6453$), shown Fig.1(d), and as further increasing parameter Λ leads to system (3.5) stabilizes to the globally stable endemic equilibrium for $R_0 > 1$ ($\Lambda > 4.6453$), shown in Fig.2(e).

Fig. 2 (a): Bifurcation diagram in (R_0, I) plane with R_0 , where $\sigma = 0.2$, $d = 10^{-3}$, $\beta = 4.5 \times 10^{-5}$, $\gamma = 0.207$, $h_1 = 0.0396$, $h_2 = 0$, $\mu = 0.05$, $\delta = 0.1$, $\alpha = 0.0014$. The blue (blue dashed) curves represent stable (unstable) equilibrium and a black dot marked with "SN" ($R_0 = 0.1532$) indicates that the system (3.5) goes through saddle-node bifurcation. (b): The curves of functions $f_1(I)$ and $f_2(I)$ with $\Lambda = 3$. It shows that functions $f_1(I)$ and $f_2(I)$ have two intersections for $I > 0$, so system (3.5) has two endemic equilibria for $\Lambda = 3$. (c-e): The time series of the system (3.5) with different parameters Λ , and in (c-e), the values of parameter Λ are 0.5, 3, 5 respectively.

The forward-backward bifurcation is illustrated in Fig.3(a) where saddle-node bifurcation is denoted by "SN" as a black dot. It is interesting to note that three endemic equilibria (denoted as $E_{3i} = (S_{3i}^*, E_{3i}^*, I_{3i}^*, M_{3i}^*)$, where $i = 1, 2, 3$ and $I_{31}^* < I_{32}^* < I_{33}^*$) are feasible, shown in Fig.3(c-d), where functions $f_1(I)$ and $f_2(I)$ have three intersections for $I > 0$. It follows that when $R_0 < 1$ (i.e., $0 < \Lambda < 2.5909$), the system (3.5) only exists the disease-free equilibrium, which is globally stable; when $1 < R_0 < 1.0218$ (i.e., $2.5909 < \Lambda < 2.6474$) and $R_0 > 1.6565$ ($\Lambda > 4.2918$), the system (3.5) has a stable endemic equilibrium; when

$1.0218 < R_0 < 1.6565$ (i.e., $1.0218 < \Lambda < 4.2918$), the system (3.5) has an unstable endemic equilibrium and two stable endemic equilibria, shown in Fig.3(b). It follows from Fig.3(b) that E_{31} and E_{33} are locally stable, solutions of $I(t)$ initiating from near the equilibrium E_{32} converge to either I_{31}^* or I_{33}^* .

Fig. 3 (a): Bifurcation diagram in (R_0, I) plane with R_0 , where $\sigma = 0.2$, $d = 1.345 \times 10^{-3}$, $\beta = 10^{-3}$, $\gamma = 0.19$, $h_1 = 0.029$, $h_2 = 0$, $\mu = 0.09$, $\delta = 0.1$, $\alpha = 0.3122$, the blue (blue dashed) curves represent stable (unstable) equilibrium and a black dot marked with "SN" ($R_0 = 1.0218(\Lambda = 1.0218)$, $R_0 = 1.6565(\Lambda = 4.2918)$) indicates the system (3.5) goes through saddle-node bifurcation. (b): The time series of the system (3.5) for $\Lambda = 3.6962$ with different initial value. (c): The curves of functions $f_1(I)$ and $f_2(I)$. (d): A partial enlargement of subplot (c).

Another pattern of the forward-backward bifurcation is illustrated in Fig.4 (a) where Saddle-Node bifurcation occurs for $R_0 < 1$ and Hopf bifurcation may also occur, and Fig.4(b) is a partial enlargement of subplot Fig.4(a). We find two Hopf bifurcation points at $R_0 = 0.782390$ and $R_0 = 1.001113$ and two Saddle-Node bifurcation points at $R_0 = 0.752654$ and $R_0 = 1.417937$. The first Lyapunov coefficients at those two Hopf points are both positive, which means that the Hopf bifurcations are subcritical and the bifurcated periodic orbits are born unstable, shown in Fig.5, where some orbits (blue curves) initiating from near the equilibrium approach the stable endemic equilibrium (represented by red dot), while others (black curves) starting the value far away the equilibrium leave away the equilibrium, and consequently there is an unstable limit cycle between two kinds of orbits. To further analyze how periodic orbits change, we plot local bifurcation diagrams (Fig.4(c) and (d)) and the variation in period of cycles versus R_0 (Fig.4 (e) and (f)). Fig.4(c) (or (d)) shows that once the periodic orbit appears, the minimum value of $I(t)$ of the periodic orbit is infinitely close to the value of $I(t)$ at the unstable saddle point E_{21}^* (or E_0) at $R_0 = 0.783296$

and $R_0 = 1.001112$. Fig.4 (e) and (f) show that the periods eventually go to infinity at $R_0 = 0.783296$ and $R_0 = 1.001112$, at which the homoclinic bifurcations occur and the limit cycle is replaced by a homoclinic orbit [24].

In summary, we have illustrated various bifurcations including forward bifurcation, backward bifurcation, and two patterns of forward-backward bifurcations with Saddle-node bifurcation and Hopf bifurcations. Theoretical analysis indicates that three or four positive equilibria are feasible for $R_0 < 1$, but unfortunately we can not give the numerical illustrations.

In order to analyze the influence of media coverage on occurrence of backward bifurcation, we set

$$k = \frac{\alpha\delta}{\mu}, \quad (3.17)$$

which can be regarded as a parameter to measure the effect of psychological impact of media reported numbers of infectious individuals. On the one hand, recall the expression of function $f_2(I)$, we have $f_2(I) = d \frac{d+\sigma}{\sigma\beta} e^{kA(I)I}$. Algebraic calculation yields $\frac{\partial f_2(I)}{\partial k} = -\frac{\partial f_2(I)}{\partial k} < 0$, which implies that we can increase k to make f_{max} be less than 0, and hence multiple equilibria may be infeasible for $R_0 < 1$. On the other hand, $\frac{\partial \tilde{h}_1}{\partial k} > 0$, we can increase k to make $h_1 < \tilde{h}_1$ holds true to avoid the occurrence of backward bifurcation when $R_0 = 1$ on the basis of Theorem 1. Recall that when $h_1 > \hat{h}_1$ and $f_{max} > 0$, the system (2.1) may occur backward bifurcation and the basic reproduction number no longer act as the threshold to distinguish whether the disease dies out or not, but we can improve the effect of psychological impact of media reported numbers of infectious individuals to avoid the occurrence of backward bifurcation, and consequently eliminating the infectious diseases becomes relatively easy.

In general, when only considering nonlinear recovery function, system (3.5) may have multiple endemic states due to backward bifurcation and forward-backward bifurcations. In particular, the system can exhibit rich dynamic behaviors including the existence of multiple endemic equilibria, Hopf bifurcation, saddle-node bifurcation, homoclinic bifurcation and unstable cycle. Comparing with the corresponding system with linear recovery function [13, 17, 27] we can conclude that nonlinear recovery may induce multiple stable endemic states, coexistence of the stable periodic solution (or endemic state) and the disease-free equilibrium.

We then investigate whether nonlinear media response induce those complex dynamics in the next section by briefly analyzing the case when considering system (2.1) with a linearly recovery term but saturated media coverage.

4 Dynamical behaviors of the system (2.1) with $h_1 = 0$

Considering only linear recovery in system (2.1), we have the following model equations

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta e^{-\alpha M} SI - dS + \gamma I \\ \frac{dE}{dt} = \beta e^{-\alpha M} SI - (d + \sigma)E \\ \frac{dI}{dt} = \sigma E - dI - \gamma I \\ \frac{dM}{dt} = \frac{\delta}{1+h_2\sigma E} \sigma E - \mu M \end{cases} \quad (4.18)$$

The existence and stability of disease-free equilibrium for system (4.18) are same as those for the system (2.1). We let $E_1 = (S^*, E^*, I^*, M^*)$ be the possible endemic equilibrium, then

Fig. 4 (a): Bifurcation diagram in (R_0, I) plane with R_0 , where $\sigma = 0.2$, $d = 1.345 \times 10^{-3}$, $\beta = 7 \times 10^{-3}$, $\gamma = 0.19$, $h_1 = 0.029$, $h_2 = 0$, $\mu = 0.0001$, $\delta = 0.1$, $\alpha = 0.0003069$. The blue curves represent stable equilibria, the red curves represent unstable equilibria and a black dot marked with "SN" ($R_0 = 0.752654$ and $R_0 = 1.417937$) or "H" ($R_0 = 0.782390$ and $R_0 = 1.001113$) indicates that the system (3.5) goes through Saddle-Node bifurcation or Hopf bifurcation. (c) and (d): Local bifurcation diagram, where "Hom" ($R_0 = 0.783296$ and $R_0 = 1.001112$) indicates that the system (3.5) goes through homoclinic bifurcation. (d) and (e): The variation in period of cycles versus R_0 .

Fig. 5 Local phase portraits of system (3.5) with different R_0 ($R_0 = 0.7827$ in (a), $R_0 = 1.001127$ in (b)), other parameters are the same in Fig. 4(a). The black dots represent the initial points and red dots represent stable equilibria. Here some orbits (blue curves) initiating from near the equilibrium approach the stable endemic equilibrium, while others (black curves) starting the value far away the equilibrium leave away the equilibrium, and consequently there is a unstable limit cycle between two kinds of orbits.

it satisfies

$$\begin{aligned} E^* &= \frac{1}{\sigma}(d + \gamma)I^*, \quad M^* = \frac{\delta}{\mu} \frac{(d + \gamma)I^*}{1 + h_2(d + \gamma)I^*}, \\ S^* &= \frac{(d + \sigma)(d + \gamma)}{\sigma\beta} e^{\alpha \frac{\delta}{\mu} \frac{(d + \gamma)I^*}{1 + h_2(d + \gamma)I^*}}, \\ \Lambda - [d + \frac{d}{\sigma}(d + \gamma)]I^* - d \frac{(d + \sigma)(d + \gamma)}{\sigma\beta} e^{\alpha \frac{\delta}{\mu} \frac{(d + \gamma)I^*}{1 + h_2(d + \gamma)I^*}} &= 0. \end{aligned} \quad (4.19)$$

Denote

$$\eta(I) = \Lambda - [d + \frac{d}{\sigma}(d + \gamma)]I - d \frac{(d + \sigma)(d + \gamma)}{\sigma\beta} e^{\alpha \frac{\delta}{\mu} \frac{(d + \gamma)I}{1 + h_2(d + \gamma)I}}.$$

Algebraic calculation yields

$$\begin{aligned} \eta'(I) &= -[d + \frac{d}{\sigma}(d + \gamma)] - d \frac{(d + \sigma)(d + \gamma)}{\sigma\beta} e^{\alpha \frac{\delta}{\mu} \frac{(d + \gamma)I}{1 + h_2(d + \gamma)I}} \alpha \frac{\delta}{\mu} (d + \gamma) \frac{1}{[1 + h_2(d + \gamma)I]^2} \\ &< 0, \end{aligned}$$

for $\forall I > 0$, Further, we can testify that

$$\eta(0) = \frac{d(d + \sigma)(d + \gamma)}{\sigma\beta} (R_0 - 1) > 0$$

if and only if $R_0 > 1$ and

$$\eta(\tilde{I}) = -d \frac{(d + \sigma)(d + \gamma)}{\sigma\beta} e^{\alpha \frac{\delta}{\mu} \frac{(d + \gamma)\tilde{I}}{1 + h_2(d + \gamma)\tilde{I}}} < 0,$$

where $\tilde{I} = \frac{\Lambda}{d + \sigma + \gamma}$. Combining the monotonicity of $\eta(I)$, we have function $\eta(I)$ has no zero solution on the interval $[0, \infty)$ when $R_0 \leq 1$, and there exists a unique I such that $\eta(I) = 0$, denoted by I^* ($0 < I^* < \tilde{I}$), for $R_0 > 1$. However, the concrete form of I^* cannot be represented.

According to the above analysis, we may come to the following conclusion.

Theorem 7 When $R_0 > 1$, the system (4.18) has a unique endemic equilibrium, denoted by $E_1 = (S^*, E^*, I^*, M^*)$, which satisfies (4.19) and $0 < I^* < \tilde{I}$; when $R_0 \leq 1$, the system (4.18) has no endemic equilibrium.

Similarly we can investigate the globally stability of the disease-free equilibrium E_0 and this unique endemic equilibrium when it exists. In the following we only give the main conclusions and the detailed proof processes are given in Appendix B.

Theorem 8 The disease-free equilibrium E_0 of the system (4.18) is globally asymptotically stable for $R_0 \leq 1$. The unique endemic equilibrium of the system (4.18) is locally asymptotically stable for $R_0 > 1$, and it is globally asymptotically stable if $\max\{\frac{\partial \sigma}{\partial + \mu}, \frac{\partial \sigma}{\partial + \gamma + \mu - 2\sigma}\} < \frac{d}{\alpha\lambda}$, $d + \gamma + \mu > 2\sigma$ and $R_0 > 1$.

When only considering the saturated media coverage, we find that the basic reproduction number actually act as the threshold to distinguish whether the disease dies out or not. Although we get the global stability of the unique endemic state under $R_0 > 1$ and certain conditions, numerical simulations imply that the system (4.18) has no other complex dynamics for $R_0 > 1$. These results are also consistent with the conclusion in [27], indicating that considering the saturation phenomenon of media coverage, the model will not appear more complicated dynamics, it just affects the prevalence of infection, since it is easy to testify that $\frac{\partial I^*}{\partial h_2} > 0$ for $R_0 > 1$. Therefore, the saturated media impact hardly affects the dynamics in terms of inducing more complicated dynamics behaviors, but does affect the disease prevalence.

5 Conclusion and discussion

During the early stage of emerging and re-emerging infectious diseases (say, COVID-19 infection) massive news coverage significantly influences the disease spread by generating psychological/behavioural changes, while shortage of media recourses worsens the disease infection. In this study, we proposed a mathematical model with double nonlinear functions to investigate the impacts of saturated media coverage and limited medical resources on disease transmission. We found that the system (2.1) may occur backward bifurcation, which is mainly associated with nonlinear recovery due to limited medical recourses, instead of saturated media impact. Then we first considered a subsystem with only nonlinear recovery (i.e., $h_2 = 0$) considering great difficulties caused by high nonlinearity in theoretical analysis. Theoretical analysis shows that backward bifurcation occurs under certain parameters, and multiple equilibria can coexist in such system. And numerical simulations reveal some rich dynamic behaviors, including forward-backward bifurcation, Hopf bifurcation, Saddle-Node bifurcation, Homoclinic bifurcation and unstable limit cycle. We further consider the other special case where the recovery term is linear (i.e., $h_1 = 0$) to clearly investigate the contribution of only saturated media coverage on those complex dynamics, we proved the subsystem (4.18) has a unique endemic equilibrium which is locally stable for $R_0 > 1$ and a stable disease-free equilibrium for otherwise. The global stability of the endemic state under certain conditions is also obtained. This indicates that the basic reproduction number actually acts as the threshold to distinguish whether the disease dies out or not, and the saturated media impact slightly affects the dynamics, compared with system with linear media impact [13, 17, 27].

In general, comparing the system with linear recovery, the nonlinear recovery may induce backward bifurcation and other rich dynamics. By comparing with system with nonlinear recovery but without media impact [28], we can conclude that though saturation recovery

leads to multiple endemic equilibria and backward bifurcation, including media coverage as an independent variable may induce Hopf bifurcation, Homoclinic bifurcation and unstable limit cycle. This means a combination of nonlinear recovery and media coverage significantly affect disease transmission in terms of causing oscillations. We also numerically analyzed effect of key parameter on occurrence of backward bifurcation, and obtained that increasing the effect of psychological impact of media reported numbers of infectious individuals can make the backward bifurcation less likely. Then improving media response to infection and individuals' response to mass media may be beneficial for disease control.

6 Declarations

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Conflicts of interest/Competing interests

The authors declare no competing financial interests.

Availability of data and material

Not applicable

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Appendices

Appendix A.1 The proof of Theorem 2

Proof Define the set

$$X = \{(S, E, I, M) | S \geq 0, E \geq 0, I \geq 0, M \geq 0\},$$

$$X_0 = \{(S, E, I, M) | S \geq 0, E > 0, I > 0, M > 0\},$$

and

$$\partial X_0 = X \setminus X_0,$$

Then X is the positive invariant set of X_0 , and X_0 is relatively closed in X . In order to prove that the disease is uniformly persistent, it suffices to show that ∂X repels uniformly the solutions of (2.1) in X_0 .

First, we can see the system (2.1) is point dissipative since the attracting domain $\Gamma = \{(S, E, I, M) \in \mathbb{R}_+^4 : 0 < S + E + I \leq \frac{\Lambda}{d}, 0 < M < \frac{\delta\sigma}{\mu} \frac{\Lambda}{d}\}$ is a positive invariant set.

Define

$$M_\partial = \{(S(t), E(t), I(t), M(t)) | S(t), E(t), I(t), M(t) \text{ satisfy system (2.1), } (S(t), E(t), I(t), M(t)) \in \partial X_0\}.$$

It suffices to show $M_{\partial} = \{(S, 0, 0, 0) | S > 0\}$. Suppose not. There is $\phi_0 = (S(t), E(t), I(t), M(t)) \in M_{\partial}$ and $t_0 \geq 0$, such that at least one of the terms in $E(t)$, $I(t)$, $M(t)$ is greater than 0 at $t = t_0$. If $E(t_0) > 0$, $I(t_0) = M(t_0) = 0$, then $I'(t_0) = \sigma E(t_0) > 0$, that is to say that there is $t_1 > 0$, such that $E(t) > 0$ and $I(t) > 0$ when $t_0 < t < t_0 + t_1$, so $(S(t), E(t), I(t), M(t)) \notin \partial X_0$, which contradicts $\phi_0 \in M_{\partial}$. The case, $I(t_0) > 0$, $E(t_0) = M(t_0) = 0$, can be proved similarly. If $M(t_0) > 0$, $I(t_0) = E(t_0) = 0$, then $M'(t_0) = -\mu M(t_0) < 0$. So $M(t)$ will end up being 0. So $M_{\partial} = \{(S, 0, 0, 0) | S > 0\}$. If the initial value $\psi_0 \in M_{\partial}$, $U_{\psi_0 \in M_{\partial}} = E_0$ since there is only one equilibrium E_0 in M_{∂} . Thus E_0 is the compact isolated invariant set of M_{∂} over the initial value ψ_0 .

Since $R_0 = \beta \frac{\Lambda}{(d+\sigma)(d+\gamma)} > 1$, there is a enough small δ_0 such that $(d+\sigma)(d+\gamma) < \sigma \beta e^{-\alpha \delta_0} (\frac{\Lambda}{d} - \delta_0)$. Then we can prove that for any solution $u(t, x_0)$, $x_0 \in X_0$, $\limsup_{t \rightarrow \infty} \|u(t, x_0) - E_0\| \leq \delta_0$, that is $W^S(E_0) \cap X_0 = \emptyset$, where $W^S(E_0)$ is the stable manifold of E_0 . Let's prove it by proof by contraction. If $\limsup_{t \rightarrow \infty} \|u(t, x_0) - E_0\| < \delta_0$, i.e., there is a positive constant η_0 , for any $t > \eta_0$, such that

$$\frac{\Lambda}{d} - \delta_0 \leq S(t) \leq \frac{\Lambda}{d} + \delta_0, \quad 0 \leq E(t), I(t), M(t) \leq \delta_0.$$

then when $t > \eta_0$, we have

$$\begin{cases} \frac{dE}{dt} > \beta e^{-\alpha \delta_0} (\frac{\Lambda}{d} - \delta_0) I - (d + \sigma) E \\ \frac{dI}{dt} = \sigma E - dI - \gamma I \end{cases} \quad (\text{A.1})$$

The characteristic equation of the Jacobian matrix of (A.1) at $(0, 0)$ is

$$\lambda^2 + v_1 \lambda + v_2 = 0,$$

where $v_1 = 2d + \sigma + \gamma > 0$ and $v_2 = (d + \sigma)(d + \gamma) - \sigma \beta e^{-\alpha \delta_0} (\frac{\Lambda}{d} - \delta_0) < 0$, which implies that the characteristic equation has a positive root, then any positive solution of (A.1) tends to infinity as t tends infinity, this contradicts $0 < S + E < I < \frac{\Lambda}{d}$. Thus, E_0 is an isolated invariant set in set X , and $W^S(E_0) \cap X_0 = \emptyset$.

In conclusion, each of the forward orbital in M_{∂} converges to E_0 , and E_0 is aperiodic in M_{∂} . By Theorem 4.6 of [19] or Theorem 4.3 and Remark 4.3 of [6], we can conclude that ∂X repels uniformly the solutions of (2.1) in X_0 .

Appendix A.2 Convexity analysis of function $f_2(I)$

Algebraic calculation yields

$$f_2''(I) = (\alpha \frac{\delta}{\mu})^2 \frac{f_2(I)}{(1+h_1 I)^4} [dC^2(I) + \gamma + \sqrt{\frac{2\gamma\mu h_1}{\alpha\delta}} \sqrt{C(I)}] [dC^2(I) + \gamma - \sqrt{\frac{2\gamma\mu h_1}{\alpha\delta}} \sqrt{C(I)}]$$

for $\forall I > 0$, where $C(I) = 1 + h_1 I$.

Denoting $P_1(C) = dC^2 + \gamma$ and $P_2(C) = m\sqrt{C}$, where $m = \sqrt{\frac{2\gamma\mu h_1}{\alpha\delta}}$, and we only need to consider the case $C > 1$. We can get $\lim_{C \rightarrow +\infty} P_1(C) - P_2(C) = +\infty$ and $P_1(1) - P_2(1) = d + \gamma - m \doteq Q$. Algebraic calculation yields

$$P_1'(C) > 0, \quad P_1''(C) > 0, \quad P_2'(C) > 0 \quad \text{and} \quad P_2''(C) < 0,$$

which means that $P_1(C)$ is a strictly concave function of monotonically increasing, and $P_2(C)$ is a strictly convex function of monotonically increasing for $C > 1$.

If $Q > 0$, we can show that functions $P_1(C)$ and $P_2(C)$ intersect at one point if and only if there is a constant $\hat{C} > 1$ such that $P_1'(\hat{C}) = P_2'(\hat{C})$ and $P_1(\hat{C}) = P_2(\hat{C})$, that is equations

$$\begin{cases} d\hat{C}^2 + \gamma = m\sqrt{\hat{C}} \\ 2d\hat{C} = \frac{1}{2}m\frac{1}{\sqrt{\hat{C}}} \end{cases} \quad (\text{A.2})$$

have a unique solution for $\hat{C} > 1$. We can further testify that equations (A.2) have a unique solution $\hat{C} = (\frac{m}{4d})^{\frac{2}{3}} > 1$ if and only if

$$d(\frac{m}{4d})^{\frac{4}{3}} + \gamma = m(\frac{m}{4d})^{\frac{1}{3}} \quad \text{and} \quad (\frac{m}{4d})^{\frac{2}{3}} > 1. \quad (\text{A.3})$$

Similarly, we can show that functions $P_1(C)$ and $P_2(C)$ intersect at two points if and only if there is a constant $\hat{C} > 1$ such that $P_1'(\hat{C}) = P_2'(\hat{C})$ and $P_1(\hat{C}) > P_2(\hat{C})$, that is there is a constant $\hat{C} > 1$ such that equations

$$\begin{cases} d\hat{C}^2 + \gamma > m\sqrt{\hat{C}} \\ 2d\hat{C} = \frac{1}{2}m\frac{1}{\sqrt{\hat{C}}}. \end{cases} \quad (\text{A.4})$$

We can further testify that (A.4) holds true if and only if

$$d\left(\frac{m}{4d}\right)^{\frac{4}{3}} + \gamma > m\left(\frac{m}{4d}\right)^{\frac{1}{3}} \quad \text{and} \quad \left(\frac{m}{4d}\right)^{\frac{2}{3}} > 1. \quad (\text{A.5})$$

Let's call these two points of intersection $\hat{C}_1$ and $\hat{C}_2$ ($\hat{C}_1 < \hat{C}_2$) in this case. We can further testify that when

$$d\left(\frac{m}{4d}\right)^{\frac{4}{3}} + \gamma < m\left(\frac{m}{4d}\right)^{\frac{1}{3}} \quad \text{or} \quad \left(\frac{m}{4d}\right)^{\frac{2}{3}} < 1 \quad (\text{A.6})$$

there is no intersection between $P_1(C)$ and $P_2(C)$.

Denoting $C = 1 + h_1 I > 1$, and considering the positive linear correlation between C and I and the expression for the function $f_2''(I)$, we can tell whether $f_2''(I)$ is positive or negative by the number of points where $P_1(C)$ and $P_2(C)$ intersect. So, if the functions $P_1(C)$ and $P_2(C)$ intersect at two points $\hat{C}_1 > 1$ and $\hat{C}_2 > 1$, let $\tilde{I}_i = \frac{\hat{C}_i - 1}{h_1}$, ($i = 1, 2$), we can get

$$\begin{aligned} f_2''(I) &> 0 \quad \text{for} \quad 0 < I < \tilde{I}_1 \quad \text{and} \quad I > \tilde{I}_2 \\ f_2''(I) &< 0 \quad \text{for} \quad \tilde{I}_1 < I < \tilde{I}_2 \end{aligned}.$$

If the functions $P_1(C)$ and $P_2(C)$ intersect at one point $\hat{C} > 1$, let $\tilde{I} = \frac{\hat{C} - 1}{h_1}$, we can get

$$\begin{aligned} f_2''(I) &> 0 \quad \text{for} \quad 0 < I < \tilde{I} \quad \text{and} \quad I > \tilde{I} \\ f_2''(I) &= 0 \quad \text{for} \quad I = \tilde{I} \end{aligned}.$$

If there is no intersection between $P_1(C)$ and $P_2(C)$,

$$f_2''(I) > 0 \quad \text{for} \quad I > 0.$$

On the other hand, if $Q < 0$, combining the monotonicity and concavity of functions $P_1(C)$ and $P_2(C)$, we can testify functions $P_1(C)$ and $P_2(C)$ have a unique intersection for $C > 1$, denoted as $\hat{C}$. Similarly, let $\tilde{I} = \frac{\hat{C} - 1}{h_1}$, we can get

$$\begin{aligned} f_2''(I) &< 0 \quad \text{for} \quad I < \tilde{I} \\ f_2''(I) &> 0 \quad \text{for} \quad I > \tilde{I}. \end{aligned}$$

Combining that $0 < I < \frac{\Lambda}{d}$, we can get to the concavity of $f_2(I)$, which is shown in Table A1 or the Table 1.

Appendix A.3 The number of endemic equilibrium of the case C

The number of endemic equilibrium of the case C is given here.

Appendix B The proof of Theorem 8

Proof We first testify the global stability of the disease-free equilibrium E_0 for $R_0 \leq 1$, let's consider a Lyapunov function

$$V = \sigma E + (d + \sigma)I.$$

Table A1 The concavity of $f_2(I)$.

Cases	Conditions	Conclusion
Case A	(A.3) holds	$f_2''(I) > 0$ for $0 < I < \tilde{I}$ and $\tilde{I} < I < \frac{\Lambda}{d}$
	$Q > 0$ and $\tilde{I} < \frac{\Lambda}{d}$	$f_2''(I) = 0$ for $I = \tilde{I}$
Case B	(A.3) holds	$f_2''(I) > 0$ for $0 < I < \frac{\Lambda}{d}$
	$Q > 0$ and $\tilde{I} > \frac{\Lambda}{d}$	
Case C	(A.5) holds	$f_2''(I) > 0$ for $0 < I < \tilde{I}_1$ and $\tilde{I}_2 < I < \frac{\Lambda}{d}$
	$Q > 0$ and $\tilde{I}_2 < \frac{\Lambda}{d}$	$f_2''(I) < 0$ for $\tilde{I}_1 < I < \tilde{I}_2$
Case D	(A.5) holds	$f_2''(I) > 0$ for $0 < I < \tilde{I}_1$
	$Q > 0$ and $\tilde{I}_1 < \frac{\Lambda}{d} < \tilde{I}_2$	$f_2''(I) < 0$ for $\tilde{I}_1 < I < \frac{\Lambda}{d}$
Case E	(A.5) holds	$f_2''(I) > 0$ for $0 < I < \frac{\Lambda}{d}$
	$Q > 0$ and $\frac{\Lambda}{d} < \tilde{I}_1$	
Case F	(A.6) holds and $Q > 0$	$f_2''(I) > 0$ for $0 < I < \frac{\Lambda}{d}$
Case G	$Q < 0$ and $\tilde{I} < \frac{\Lambda}{d}$	$f_2''(I) < 0$ for $0 < I < \tilde{I}$
		$f_2''(I) > 0$ for $\tilde{I} < I < \frac{\Lambda}{d}$
Case H	$Q < 0$ and $\tilde{I} > \frac{\Lambda}{d}$	$f_2''(I) < 0$ for $0 < I < \frac{\Lambda}{d}$

Table A2 The number of endemic equilibria of the case 2.

Conditions	Amount	Range
$I_1 \leq \tilde{I}_1 < \tilde{I}_2$	One	$I^* > I_1$ for $f_1(I_1) > f_2(I_1)$
		$I^* < I_1$ for $f_1(I_1) < f_2(I_1)$
		$I^* = I_1$ for $f_1(I_1) = f_2(I_1)$
$\tilde{I}_1 < I_1 < \tilde{I}_2, f_2(\tilde{I}_1) > f_1(\tilde{I}_1), f_1(I_1) > f_2(I_1)$	Three	$I_1^* < \tilde{I}_1 < I_2^* < I_1 < I_3^* < \tilde{I}_2$
$\tilde{I}_1 < I_1 < \tilde{I}_2, f_2(\tilde{I}_1) > f_1(\tilde{I}_1), f_1(I_1) < f_2(I_1)$	One	$I^* < \tilde{I}_1$
$\tilde{I}_1 < I_1 < \tilde{I}_2, f_2(\tilde{I}_1) > f_1(\tilde{I}_1), f_1(I_1) = f_2(I_1)$	Two	$I_1^* < \tilde{I}_1 < I_2^* = I_1$
$\tilde{I}_1 < I_1 < \tilde{I}_2, f_2(\tilde{I}_1) = f_1(\tilde{I}_1), f_1(I_1) > f_2(I_1)$	Two	$I_1^* = \tilde{I}_1 < I_1 < I_2^*$
$\tilde{I}_1 < I_1 < \tilde{I}_2, f_2(\tilde{I}_1) = f_1(\tilde{I}_1), f_1(I_1) < f_2(I_1)$	One	$I^* = \tilde{I}_1$
$\tilde{I}_1 < I_1 < \tilde{I}_2, f_2(\tilde{I}_1) = f_1(\tilde{I}_1), f_1(I_1) = f_2(I_1)$	Two	$I_1^* = \tilde{I}_1 < I_2^* = I_1$
$\tilde{I}_1 < I_1 < \tilde{I}_2, f_2(\tilde{I}_1) < f_1(\tilde{I}_1)$	One	$I^* > I_1$
$\tilde{I}_1 < \tilde{I}_2 < I_1, f_2(\tilde{I}_1) > f_1(\tilde{I}_1), f_1(\tilde{I}_2) > f_2(\tilde{I}_2)$	Three	$I_1^* < \tilde{I}_1 < I_2^* < \tilde{I}_2 < I_3^*$
$\tilde{I}_1 < \tilde{I}_2 < I_1, f_2(\tilde{I}_1) > f_1(\tilde{I}_1), f_1(\tilde{I}_2) < f_2(\tilde{I}_2)$	One	$I^* < \tilde{I}_1$
$\tilde{I}_1 < \tilde{I}_2 < I_1, f_2(\tilde{I}_1) > f_1(\tilde{I}_1), f_1(\tilde{I}_2) = f_2(\tilde{I}_2)$	Two	$I_1^* < \tilde{I}_1 < I_2^* = \tilde{I}_2 < I_1$
$\tilde{I}_1 < \tilde{I}_2 < I_1, f_2(\tilde{I}_1) = f_1(\tilde{I}_1), f_1(\tilde{I}_2) > f_2(\tilde{I}_2)$	Two	$I_1^* = \tilde{I}_1 < I_2^*$
$\tilde{I}_1 < \tilde{I}_2 < I_1, f_2(\tilde{I}_1) = f_1(\tilde{I}_1), f_1(\tilde{I}_2) < f_2(\tilde{I}_2)$	One	$I^* = \tilde{I}_1$
$\tilde{I}_1 < \tilde{I}_2 < I_1, f_2(\tilde{I}_1) = f_1(\tilde{I}_1), f_1(\tilde{I}_2) = f_2(\tilde{I}_2)$	Two	$I_1^* = \tilde{I}_1 < I_2^* = \tilde{I}_2$
$\tilde{I}_1 < \tilde{I}_2 < I_1, f_2(\tilde{I}_1) < f_1(\tilde{I}_1)$	One	$I^* > I_1$

We have

$$\begin{aligned} \frac{dV}{dt} &= \sigma\beta e^{-\alpha M} SI - (d + \sigma)(d + \gamma)I, \\ &\leq \frac{1}{(d + \sigma)(d + \gamma)} (R_0 e^{-\alpha M} - 1)I. \end{aligned}$$

So, if $R_0 < 1$, we have $\frac{dV}{dt} \leq 0$, and $\frac{dV}{dt} = 0$ holds if and only if $I = 0$. Applying LaSalle's invariance principle, we get that $\lim_{t \rightarrow \infty} I(t) = 0$. Then applying comparison principle, we can testify $\lim_{t \rightarrow \infty} E(t) = \lim_{t \rightarrow \infty} M(t) = 0$ and $\lim_{t \rightarrow \infty} S(t) = \frac{\Lambda}{d}$. If $R_0 = 1$, we have

$$\frac{dV}{dt} = \sigma \beta I (e^{-\alpha M} S - \frac{\Lambda}{d}),$$

then $\frac{dV}{dt} \leq 0$, and $\frac{dV}{dt} = 0$ holds if and only if $I = 0$, we can prove $\lim_{t \rightarrow \infty} I(t) = \lim_{t \rightarrow \infty} E(t) = \lim_{t \rightarrow \infty} M(t) = 0$ and $\lim_{t \rightarrow \infty} S(t) = \frac{\Lambda}{d}$ using LaSalle's invariance principle similarly in this case. So the disease-free equilibrium E_0 of the system (4.18) is globally asymptotically stable if $R_0 \leq 1$. That completes the proof.

Then, we prove the stability of this unique endemic when it exists. According to the Theorem 7, the system (4.18) has a unique endemic equilibrium E_1 for $R_0 > 1$. The Jacobian matrix concerned the linearization of system (4.18) at E_1 is

$$J_{E_1} = \begin{pmatrix} -\beta e^{-\alpha M^*} I^* - d & 0 & -\beta e^{-\alpha M^*} S^* + \gamma & \alpha \beta e^{-\alpha M^*} S^* I^* \\ \beta e^{-\alpha M^*} I^* & -(d + \sigma) & \beta e^{-\alpha M^*} S^* & -\alpha \beta e^{-\alpha M^*} S^* I^* \\ 0 & \sigma & -(d + \gamma) & 0 \\ 0 & \frac{\delta \sigma}{(1 + h_2 \sigma E^*)^2} & 0 & -\mu \end{pmatrix},$$

and the characteristic equation yields

$$|\lambda E - J_{E_1}| = (\lambda + d) \begin{vmatrix} \lambda + \beta I^* e^{-\alpha M^*} + (d + \sigma) & -\beta e^{-\alpha M^*} (S^* - I^*) & \alpha \beta e^{-\alpha M^*} S^* I^* \\ -\sigma & \lambda + (d + \gamma) & 0 \\ -\frac{\delta \sigma}{(1 + h_2 \sigma E^*)^2} & 0 & \lambda + \mu \end{vmatrix} \quad (\text{B.1})$$

$$\doteq (\lambda + d)(\lambda^3 + n_2 \lambda^2 + n_1 \lambda + n_0) = 0$$

Further, we can get

$$\begin{aligned} \beta e^{-\alpha M^*} (S^* - I^*) &= \beta e^{-\alpha M^*} \left[\frac{(d + \sigma)(d + \gamma)}{\sigma \beta} e^{\alpha M^*} - I^* \right] \\ &= \frac{(d + \sigma)(d + \gamma)}{\sigma} - \beta I^* e^{-\alpha M^*} \end{aligned}$$

and

$$\begin{aligned} \alpha \beta e^{-\alpha M^*} S^* I^* &= \alpha \beta e^{-\alpha M^*} \frac{(d + \sigma)(d + \gamma)}{\sigma \beta} e^{\alpha M^*} I^* \\ &= \alpha \frac{(d + \sigma)(d + \gamma)}{\sigma} I^* \end{aligned}$$

Combining $\beta I^* e^{-\alpha M^*} = \frac{(d + \gamma)(d + \sigma) I^*}{\sigma \frac{\Lambda}{d} - (d + \gamma + \sigma) I^*} := g(I^*)$, we can testify that

$$\begin{aligned} n_0 &= \mu(d + \gamma + \sigma)g(I^*) + \alpha \delta (d + \gamma)^2 (d + \sigma) \frac{I^*}{[1 + h_2(d + r(I^*))I^*]^2} > 0, \\ n_1 &= g(I^*)(d + \gamma + \mu + \sigma) + \mu(2d + \sigma + \gamma) + \alpha \delta (d + \sigma)(d + \gamma) \frac{I^*}{[1 + h_2(d + \gamma)I^*]^2} > 0, \\ n_2 &= \mu + 2d + \gamma + \sigma + g(I^*) > 0. \end{aligned} \quad (\text{B.2})$$

It's easy to show that the characteristic equation (B.1) has a negative root $-d$, and other roots are the roots of the polynomial $\lambda^3 + n_2 \lambda^2 + n_1 \lambda + n_0 = 0$. Further, we have

$$\begin{aligned} n_2 n_1 - n_0 &= \mu^2 [g(I^*) + 2d + \sigma + \gamma] + \mu [2d + \gamma + \sigma + g(I^*)]^2 \\ &\quad + [2d + \gamma + \sigma + g(I^*)](d + \gamma + \sigma)g(I^*) \\ &\quad + \alpha \delta (d + \sigma)(d + \gamma)[d + \sigma + \mu + g(I^*)] \frac{I^*}{[1 + h_2(d + r)I^*]^2}. \end{aligned} \quad (\text{B.3})$$

$$> 0$$

According to the Hurwitz criterion, the roots of the polynomial $\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0 = 0$ all have negative real parts, then the unique endemic equilibrium of the system (4.18) is locally asymptotically stable if it is feasible ($R_0 > 1$).

Finally, we use compound matrices and geometric ideas to develop global results under mild restrictions as well. The Jacobian matrix of system (4.18) reads

$$J = \begin{pmatrix} -\beta e^{-\alpha M} I - d & 0 & -\beta e^{-\alpha M} S + \gamma & \alpha \beta e^{-\alpha M} SI \\ \beta e^{-\alpha M} I & -(d + \sigma) & \beta e^{-\alpha M} S & -\alpha \beta e^{-\alpha M} SI \\ 0 & \sigma & -(d + \gamma) & 0 \\ 0 & \frac{\delta \sigma}{(1 + h_2 \sigma E)^2} & 0 & -\mu \end{pmatrix}.$$

The third additive compound matrix of J is

$$J^{[3]} = \begin{pmatrix} a_{11} & 0 & \alpha \beta e^{-\alpha M} SI & \alpha \beta e^{-\alpha M} SI \\ 0 & a_{22} & \beta e^{-\alpha M} S & \beta e^{-\alpha M} S - \gamma \\ -\frac{\delta \sigma}{(1 + h_2 \sigma E)^2} & \sigma & a_{33} & 0 \\ 0 & 0 & \beta e^{-\alpha M} I & a_{44} \end{pmatrix},$$

and the corresponding linear compound system reads

$$(X, Y, Z, W)^T = J^{[3]}(X, Y, Z, W)^T \quad (\text{B.4})$$

where

$$\begin{aligned} a_{11} &= -\beta e^{-\alpha M} I - d - (d + \sigma) - (d + \gamma), & a_{22} &= -\beta e^{-\alpha M} I - d - (d + \sigma) - \mu, \\ a_{33} &= -\beta e^{-\alpha M} I - d - (d + \gamma) - \mu, & a_{44} &= -(d + \sigma) - (d + \gamma) - \mu. \end{aligned}$$

Then we just need to testify the global stability of this linear compound system according to [8, 9]. To do this, we choose the Lyapunov functional as

$$V(S, E, I, R, X, Y, Z, W) = \begin{cases} \max\{t_1 \frac{|X|}{I}, \frac{|Y| + |W|}{E}, \frac{|Z| + |W|}{I}\} & \text{if } E \geq I, \\ \max\{t_2 \frac{|X|}{E}, \frac{|Y| + |W|}{E}, \frac{|Z| + |W|}{E}\} & \text{if } E < I, \end{cases}$$

where t_1, t_2 will be determined later. It is easy to show the system (4.18) admits a unique nonnegative bounded solution for all $t \geq 0$ and any non-negative initial value. Combing the Theorem 2, there exist two positive constants C_1, C_2 such that

$$C_1(|X| + |Y| + |Z| + |W|) \leq V \leq C_2(|X| + |Y| + |Z| + |W|).$$

Recalling that

$$\frac{\dot{E}}{E} = \frac{\beta e^{-\alpha M} SI}{E} - (d + \sigma), \quad \frac{\dot{I}}{I} = \frac{\sigma \sigma E}{I} - (d + \gamma).$$

Then, discussing the derivative of V along the trajectory of the linear compound system (B.4) for different conditions and denoting D_+ the right-hand (total) derivative with respect to t .

Case (1): $E \geq I$ and $V = t_1 \frac{|X|}{I}$, which implies $t_1 \frac{|X|}{I} > \{\frac{|Y| + |W|}{E}, \frac{|Z| + |W|}{I}\}$. Algebraic calculation yields

$$\begin{aligned} D_+ V &\leq t_1 \frac{[-\beta e^{-\alpha M} I - d - (d + \sigma) - (d + \gamma)]|X| + \alpha \beta e^{-\alpha M} SI[|Z| + |W|]}{I} - \frac{I'}{I} t_1 \frac{|X|}{I} \\ &= t_1 \frac{|X|}{I} [-\beta e^{-\alpha M} I - d - (d + \sigma) - (d + \gamma) - \frac{I'}{I}] + t_1 \alpha \beta e^{-\alpha M} SI \frac{|Z| + |W|}{I} \\ &\leq [(t_1 \alpha S - 1) \beta e^{-\alpha M} I - d - (d + \sigma) - \sigma \frac{E}{I}] V. \end{aligned}$$

Case (2): $E \geq I$ and $V = \frac{|Y|+|W|}{E}$, which implies $\frac{|Y|+|W|}{E} > \{t_1 \frac{|X|}{I}, \frac{|Z|+|W|}{I}\}$. Algebraic calculation yields

$$\begin{aligned} D_+V &\leq \frac{[-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu]|Y| + \beta e^{-\alpha M}(S+I)|Z| + [\beta e^{-\alpha M}S - (d + \sigma) - d - \mu]|W|}{E} \\ &\quad - \frac{E'}{E} \frac{|Y|+|W|}{E} \\ &= \frac{|Y|+|W|}{E} [-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu - \frac{E'}{E}] + I \frac{\beta e^{-\alpha M}(S+I)}{E} \frac{|Z|+|W|}{I} \\ &\leq [-\beta e^{-\alpha M}I - d - \mu - \frac{\beta e^{-\alpha M}SI}{E} + \frac{\beta e^{-\alpha M}(SI+I^2)}{E}]V \\ &= [-\beta e^{-\alpha M}I(1 - \frac{I}{E}) - d - \mu]V. \end{aligned}$$

Since $E \geq I$, $1 - \frac{I}{E} \geq 0$ must hold, which means $D_+V \leq -(d + \mu)V$.

Case (3): $E \geq I$ and $V = \frac{|Z|+|W|}{I}$, which implies $\frac{|Z|+|W|}{I} > \{\frac{|Y|+|W|}{E}, t_1 \frac{|X|}{I}\}$. Algebraic calculation yields

$$\begin{aligned} D_+V &\leq \frac{\delta\sigma|X| + \sigma|Y| + [-d - (d + \gamma) - \mu]|Z| + [-(d + \sigma) - (d + \gamma) - \mu]|W|}{I} - \frac{I'}{I} \frac{|Z|+|W|}{I} \\ &= t_1 \frac{|X|}{I} \frac{\delta\sigma}{t_1} + \sigma \frac{E}{I} \frac{|Y|+|W|}{E} + \frac{|Z|+|W|}{I} [-d - (d + \gamma) - \mu - \frac{I'}{I}] \\ &\leq [\frac{\delta\sigma}{t_1} - d - \mu]V. \end{aligned}$$

Case (4): $E < I$ and $V = t_2 \frac{|X|}{E}$, which implies $t_2 \frac{|X|}{E} > \{\frac{|Y|+|W|}{E}, \frac{|Z|+|W|}{E}\}$. Algebraic calculation yields

$$\begin{aligned} D_+V &\leq t_2 \frac{[-\beta e^{-\alpha M}I - d - (d + \sigma) - (d + \gamma)]|X| + \alpha\beta e^{-\alpha M}SI[|Z|+|W|]}{E} - \frac{E'}{E} t_2 \frac{|X|}{E} \\ &= t_2 \frac{|X|}{E} [-\beta e^{-\alpha M}I - d - (d + \sigma) - (d + \gamma) - \frac{E'}{E}] + t_2 \alpha\beta e^{-\alpha M}SI \frac{|Z|+|W|}{E} \\ &\leq [(t_2\alpha S - 1)\beta e^{-\alpha M}I - d - (d + \gamma) - \frac{\beta e^{-\alpha M}SI}{E}]V. \end{aligned}$$

Case (5): $E < I$ and $V = \frac{|Y|+|W|}{E}$, which implies $\frac{|Y|+|W|}{E} > \{t_2 \frac{|X|}{E}, \frac{|Z|+|W|}{E}\}$. Algebraic calculation yields

$$\begin{aligned} D_+V &\leq \frac{[-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu]|Y| + \beta e^{-\alpha M}(S+I)|Z| + [\beta e^{-\alpha M}S - (d + \sigma) - d - \mu]|W|}{E} \\ &\quad - \frac{E'}{E} \frac{|Y|+|W|}{E} \\ &= \frac{|Y|+|W|}{E} [-\beta e^{-\alpha M}I - d - (d + \sigma) - \mu - \frac{E'}{E}] + \beta e^{-\alpha M}(S+I) \frac{|Z|+|W|}{E} \\ &\leq [-\beta e^{-\alpha M}I - d - \mu - \frac{\beta e^{-\alpha M}SI}{E} + \beta e^{-\alpha M}(S+I)]V \\ &= [-\beta e^{-\alpha M}S(\frac{I}{E} - 1) - d - \mu]V. \end{aligned}$$

Since $E < I$, $\frac{I}{E} - 1 > 0$ must hold, which means $D_+V \leq -(d + \mu)V$.

Case (6): $E < I$ and $V = \frac{|Z|+|W|}{E}$, which implies $\frac{|Z|+|W|}{E} > \{\frac{|Y|+|W|}{E}, t_2 \frac{|X|}{E}\}$. Algebraic calculation yields

$$\begin{aligned} D_+V &\leq \frac{\delta\sigma|X| + \sigma|Y| + [-d - (d + \gamma) - \mu]|Z| + [-(d + \sigma) - (d + \gamma) - \mu]|W|}{E} + \frac{E'}{E} \frac{|Z|+|W|}{E} \\ &= t_2 \frac{|X|}{E} \frac{\delta\sigma}{t_2} + \sigma \frac{|Y|}{E} + \frac{|Z|+|W|}{E} [-d - (d + \gamma) - \mu - \frac{\beta e^{-\alpha M}SI}{E} + (d + \sigma)] \\ &\leq [\frac{\delta\sigma}{t_2} + 2\sigma - (d + \gamma) - \mu - \frac{\beta e^{-\alpha M}SI}{E}]V. \end{aligned}$$

Since $\max\{\frac{\delta\sigma}{d+\mu}, \frac{\delta\sigma}{d+\gamma+\mu-2\sigma}\} < \frac{1}{\alpha} \frac{d}{\Lambda}$ and $d + \gamma + \mu > 2\sigma$, we can choose two positive constants ι_1 and ι_2 such that $\frac{\delta\sigma}{d+\mu} < \iota_1 < \frac{1}{\alpha} \frac{d}{\Lambda}$ and $\frac{\delta\sigma}{d+\gamma+\mu-2\sigma} < \iota_2 < \frac{1}{\alpha} \frac{d}{\Lambda}$ respectively, which means that $\iota_1 \alpha S - 1 < 0$, $\frac{\delta\sigma}{\iota_1} - d - \mu < 0$, $\iota_2 \alpha S - 1 < 0$ and $\frac{\delta\sigma}{\iota_2} + 2\sigma - d - \gamma - \mu < 0$. Therefore, there exists a positive constant κ such that $-\kappa > \max\{-d, \frac{\delta\sigma}{\iota_1} - d - \mu, \frac{\delta\sigma}{\iota_2} + 2\sigma - d - \gamma - \mu < 0\}$, in this case, $D_+ V \leq -\kappa V$ must hold. By Corollary 3.2 in [9], E_1 of the system (4.18) is globally asymptotically stable. That completes the proof.