The Angular Speed Distribution of Randomly Moving Particles

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Research Article

Keywords: Angular Speed Distribution, Randomly Moving Particles, Spin

Posted Date: July 7th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-653710/v1

Title:

The Angular Speed Distribution of Randomly Moving Particles

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Abstract

Spin is a common phenomenon in nature, but the detailed statistical principle behind it is seldom discussed. This article aims to inspire people to think about such problems. In this article, probability theory and the corresponding modules of Mathematica software are used to study the distribution of angular speed generated by a population of equal speed particles moving randomly in space. The probability density expression of this distribution is derived successfully, and then its accuracy is verified by simulation data. Finally, it is confirmed that the angular speed follows the Maxwell distribution with a specified scale parameter. This research may be helpful for understanding why most celestial bodies or micro particles in the universe spin.

1. Introduction

As a powerful symbolic computing tool, Mathematica (Wolfram Research Inc.) can provide strong support for many aspects of scientific research. In particular, the module for probability theory and statistics in Mathematica v8.0 (2010) provides unprecedented assistance to the research of symbolic statistics. However, Mathematica is not omnipotent. It cannot take over all the steps in complex and profound problems.

In some key links, researchers need to have a thorough understanding of the principles and essence of specific work and split the work into several units by manual intervention. Therefore, many problems that were difficult to solve before can be solved by combining them with manual work.

The virous behavior of a large number of randomly moving particles can only be studied by probability theory and statistical methods, including static probability theory and dynamic probability theory (stochastic process). Previous researchers have performed much research on randomly moving particles and found many meritorious laws This work includes the study of the behavior of the average motion (translational) rule of particles, such as the Maxwell distribution(Maxwell, 1860; Boltzmann, 1872) and the study of the probability of a dynamic diffusion system, such as Brownian diffusion. However, there are few studies on the spin behavior of a system formed by randomly moving particles. Based on Mathematica software and probability theory, this article studies and discusses some topics related to random spin. This study may be helpful for understanding some random spin phenomena in nature.

2. Methods

Mathematica 11.2 for Mac (*Wolfram Research Inc.*) was used for all of the mathematical calculations, and the operating system was macOS High Sierra 10.13.6.

3. Results and Discussions

Given a number of particles (assume they uniformly distribute in spherical space) with the same speed (assume it is c) and randomly moving in 3-dimensional space, there must be a corresponding movement component to produce a spin effect on the overall centroid at a certain moment in time. To illustrate this problem, the particles with equal speed and random directions are regarded as random vectors with equal norms, and the analysis is divided into the following two steps: first, the distribution of the norm of the angular velocity generated by a single random vector relative to the total centroid is obtained, and then, this distribution is extended to the distribution of the norm of the angular velocity generated by a number of random vectors relative to their centroid.

First, let us study the distribution of the norm $\left|\boldsymbol{\Omega}_{\mathrm{S}}\right|$ of the random angular velocity generated by a random vector $\boldsymbol{V}_{\mathrm{S}}$ with a given norm (the linear velocity of the random points on the sphere) uniformly distributed on a unit sphere S. The contribution of $\boldsymbol{V}_{\mathrm{S}}$

to the random angular velocity of the center of the sphere is in all possible directions in space. How can multiple rotation contributions be added together to specify the overall rotation? The vector product $\boldsymbol{\omega}_{\mathrm{S}}$ of the linear velocity $\boldsymbol{v}_{\mathrm{S}}$ of a point on the sphere and the radius $\boldsymbol{r}$ of the unit sphere S on which this point located can easily explain the total rotation contribution, or the distribution of the contribution to the angular velocity by a single random vector (here, $|\boldsymbol{r}| = 1$; if $|\boldsymbol{r}|$ 1, then $\frac{\boldsymbol{\omega}_{\mathrm{S}}}{|\boldsymbol{r}|}$ is the contribution of the linear velocity $\boldsymbol{v}_{\mathrm{S}}$ to the angular velocity $\boldsymbol{\omega}_{\mathrm{S}}$), namely,

$$\boldsymbol{\omega}_{\mathrm{S}} = \boldsymbol{r} \times \boldsymbol{v}_{\mathrm{S}}. \tag{1}$$

The process of solving the above problem can be divided into two independent steps. The first step is to determine the direction of the unit vector $\boldsymbol{r}$ in space, that is, to determine the position of the end point of the vector $\boldsymbol{r}$ on the unit sphere S. This position is uniformly distributed over the whole sphere S and is represented by a random vector $\boldsymbol{R}$. The second step is to determine the direction of the linear velocity $\boldsymbol{v}_{\mathrm{S}}$ of the point at this position, which is also uniformly distributed throughout the whole space and is represented by the random vector $\boldsymbol{V}_{\mathrm{S}}$. Suppose that there is a sphere S of radius $|\boldsymbol{v}_{\mathrm{S}}|$ at the end of $\boldsymbol{r}$ (specifically, the end that lies on the sphere S). Then, the random vector

$\boldsymbol{V}_S$ is equivalent to the vector formed by connecting the uniformly distributed points on the sphere S to the center of S. Considering that $|\boldsymbol{r}|=1$ and the definition of the cross product, when the direction of $\boldsymbol{r}$ is determined, $|\boldsymbol{r} \quad \boldsymbol{v}_{\mathrm{S}}|$ is equivalent to the norm of the vector obtained by connecting the center of the sphere S to the projection of the uniformly distributed points on the sphere S along a direction parallel to $\boldsymbol{r}$ onto a tangent disk D that passes through the center of the sphere S and is perpendicular to $\boldsymbol{r}$; this vector norm is denoted by $|\boldsymbol{\omega}'|$ (Fig. 1).

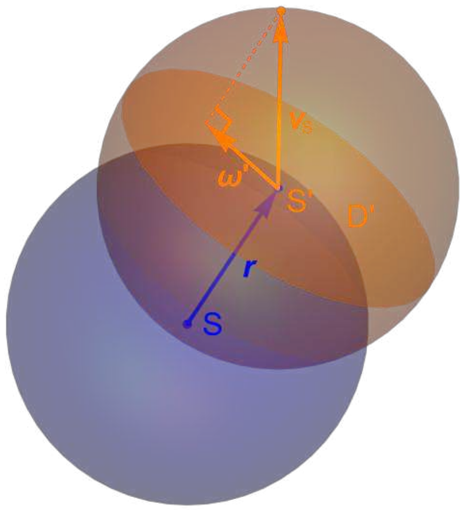

Figure 1 | Schematic diagram of the generation method for the vector $\boldsymbol{\omega}'$.

When the random vector $\boldsymbol{R}$ changes, it is equivalent to driving the tangent disk D on the unit sphere S to move. Since the sum of two uniform distributions is also a uniform distribution, the result of $\boldsymbol{R} \quad \boldsymbol{V}_{\mathrm{S}}$ can be regarded as the uniform distribution of the random vector $\boldsymbol{\Omega}'$ throughout the entire space. Thus, one can find the distribution of the norm $|\boldsymbol{\Omega}'|$ of the random vector on the disk D and assign it a random direction in space to obtain the distribution of the norm $|\boldsymbol{\Omega}_{\mathrm{S}}|$ of the angular velocity generated by the contribution of $\boldsymbol{V}_{\mathrm{S}}$ to the center of the sphere S.

Under the assumption that the random variables $N_1 \sim N(0, 1)$, $N_2 \sim N(0, 1)$ and $N_3 \sim N(0, 1)$ are independent of each other, one coordinate X that is equivalent to the three coordinates of $\boldsymbol{R}$, which is the unit random vector, can be written as(Muller, 1959; Marsaglia, 1972)

$$X = \frac{N_1}{\sqrt{N_1^2 + N_2^2 + N_3^2}}, \tag{2}$$

the probability density of which is

$$f_X(x) = \begin{cases} \frac{1}{2}, & 1 < x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Let us establish a 3-dimensional Cartesian coordinate system for S. Suppose that the disk D is perpendicular to the z-axis and consider the random variables $\Theta \sim U(-1, 1)$ and $H \sim U(-\pi, \pi)$. Then, the coordinates of the random vector $\boldsymbol{\Omega}'$ obtained by projecting the uniformly distributed points on sphere S onto disc D are $X = c \times \sin\cos^{1} \times \cos$, $Y = c \times \sin\cos^{1} \times \cos$ and $Z = 0$, and the norm of this random vector is

$$\left|\boldsymbol{\Omega}'\right| = \sqrt{X^2+Y^2+Z^2} = c\cdot\sqrt{1-\Theta^2}. \tag{4}$$

Therefore, the probability of $\left|\boldsymbol{\Omega}'\right|$ is

$$\left|\boldsymbol{\omega}'\right|(x) = \begin{cases} \dfrac{x}{c\cdot\sqrt{c^2-x^2}}, & 0<x<c, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Note that the distribution of the random variable $\left|\boldsymbol{\Omega}_{\mathrm{S}}\right|$ is the random distribution generated by the random motion of $\boldsymbol{\Omega}'$ in space driven by $\boldsymbol{R}$. Therefore, by taking a product $F_X(x)\cdot\left|\boldsymbol{\Omega}'\right|(x)$ of random variables, we can obtain the probability density of X, which represents one of the three equivalent coordinates of the angular velocity $\boldsymbol{\Omega}_{\mathrm{S}}$ contributed by the random vector $\boldsymbol{V}_{\mathrm{S}}$, namely,

$$\boldsymbol{\omega}_{\mathrm{S},X}(x) = \begin{cases} \dfrac{1}{2c}\sin^{-1}\dfrac{x}{c}+\dfrac{\pi}{4c}, & -c<x<0, \\ \dfrac{1}{2c}\cos^{-1}\dfrac{x}{c}, & 0\le x<c, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

The angular velocity $\boldsymbol{\Omega}_{\mathrm{B}}$ contribution of the random vector $\boldsymbol{V}_{\mathrm{B}}$, which is uniformly distributed throughout the whole unit ball enclosed by the sphere S, is also multiplied by the reciprocal $\dfrac{1}{r}$ of the norm r of the radius $\boldsymbol{r}$ at which the starting point of this vector $\boldsymbol{V}_{\mathrm{B}}$ is located within the ball. Therefore, the contribution of $\boldsymbol{V}_{\mathrm{B}}$ to the equivalent coordinate $\boldsymbol{\Omega}_{\mathrm{B},X}$ of $\boldsymbol{\Omega}_{\mathrm{B}}$ is calculated as follows:

$$\boldsymbol{\Omega}_{\mathrm{B},X}(x,r) = \frac{1}{r} \cdot \boldsymbol{\Omega}_{\mathrm{S},X}(x). \tag{7}$$

Hence, the new probability density is

$$\boldsymbol{\omega}_{\mathrm{B},X}(x,r) = \begin{cases} \dfrac{1}{2c} r \sin^{-1}\dfrac{rx}{c} + \dfrac{\pi r}{4c}, & -\dfrac{c}{r} < x < 0, \\ \dfrac{1}{2c} r \cos^{-1}\dfrac{rx}{c}, & 0 \le x < \dfrac{c}{r}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Thus, the distribution function $\boldsymbol{\Omega}_{\mathrm{B},X}(r)$ of the contribution of $\boldsymbol{V}_{\mathrm{B}}$ to one of the equivalent coordinates of $\boldsymbol{\Omega}_{\mathrm{B}}$ is obtained. Next, $\boldsymbol{\Omega}_{\mathrm{B},X}(r)$ is integrated over the whole unit ball:

$$\int_0^1 4\pi r^2 \cdot \boldsymbol{\Omega}_{\mathrm{B},X}(r)\mathrm{d}r. \tag{9}$$

Eq. 9 describes the case in which the particles are uniformly distributed in the ball. Then, the probability density of the contribution of $\boldsymbol{V}_{\mathrm{B}}$ in the whole unit ball to an equivalent coordinate X of the angular velocity $\boldsymbol{\Omega}_{\mathrm{B}}$ can be obtained by finding the derivative of Eq. 9 with respect to x and normalizing it, as follows:

$$
\boldsymbol{\omega}_{\mathrm{B},X}(x)=\begin{cases}\dfrac{9\pi c^3}{128x^4}, & x>c\vee x\le -c,\\[2ex] \dfrac{3\left(8x^4\sin^{-1}\dfrac{x}{c}+4\pi x^4+U_1\right)}{64cx^4}, & -c<x<0,\\[2ex] \dfrac{3\left(8x^4\cos^{-1}\dfrac{x}{c}-U_1\right)}{64cx^4}, & 0<x\le c,\\[2ex] 0, & \text{otherwise},\end{cases} \tag{10}
$$

where $U_1 = x(2x^2+3c^2)\sqrt{c^2\quad x^2}\quad 3c^4\sin\,{}^{1}\dfrac{x}{c}$. This is the case of a random vector $\boldsymbol{V}_{\mathrm{B}}$ inside the unit sphere S (including S).

Next, we extend the analysis to the case in which the radius of the ball has an arbitrary value R. When the radius of the ball is R, the above situation scales to $\dfrac{\boldsymbol{\Omega}_{\mathrm{B},X}(x)}{R}$. Accordingly, the probability density of the contribution of the random vector $\boldsymbol{V}$ to the single equivalent coordinate X of angular velocity $\boldsymbol{\Omega}$ is

$$
\boldsymbol{\omega}_{X}(x)=\begin{cases}\dfrac{9\pi c^3}{128R^3x^4}, & x>\dfrac{c}{R}\vee x\le -\dfrac{c}{R},\\[2ex] \dfrac{3\left(8R^4x^4\sin^{-1}\dfrac{Rx}{c}+4\pi R^4x^4+U_2\right)}{64cR^3x^4}, & -\dfrac{c}{R}<x<0,\\[2ex] \dfrac{3\left(8R^4x^4\cos^{-1}\dfrac{Rx}{c}-U_2\right)}{64cR^3x^4}, & 0<x\le \dfrac{c}{R},\\[2ex] 0, & \text{otherwise},\end{cases} \tag{11}
$$

where $U_2 = Rx(2R^2x^2+3c^2)\sqrt{c^2\quad R^2x^2}\quad 3c^4\sin\,{}^{1}\dfrac{Rx}{c}$, the standard deviation of which is $\dfrac{\sqrt{6}c}{3R}$. If $R = 3$ and $c = 1$, the above probability density (Eq. 11) of the angular velocity can be plotted as shown in Fig. 2.

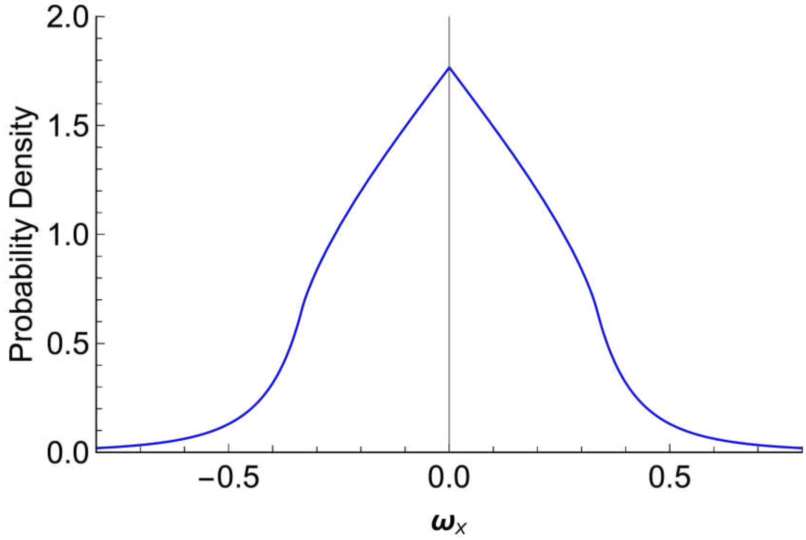

Figure 2 | The contribution of the random vector $\boldsymbol{V}$ to the distribution of the single equivalent coordinate $\boldsymbol{\Omega}_X$ of the angular velocity $\boldsymbol{\Omega}$ when $R = 3$ and $c = 1$.

Therefore, when k independent and identically distributed random vectors $\boldsymbol{V}$ move randomly in space, according to the central limit theorem (when they are grouped together), the norm $|\boldsymbol{\Omega}|$ of the average angular velocity generated by all of their components relative to their total centroid follows the Maxwell distribution with scale parameter $\frac{\sqrt{6}c}{3R\sqrt{k}}$. To verify this conclusion, values of $k = 10^3$, $R = 3$ and $c = 100$ are considered here, and this theoretical distribution is compared with the results of simulating 10^6 samples with the same parameters. The results are illustrated in Fig. 3 (see the description of the process of generating Fig. 3 in Part 2 of the Supplementary

Information for the details of the simulation). It can be seen from this figure that the analytical expression derived in this article is in good agreement with the simulated results. See Part 1 of Supplementary Information for the detailed Mathematica code of the above calculation process.

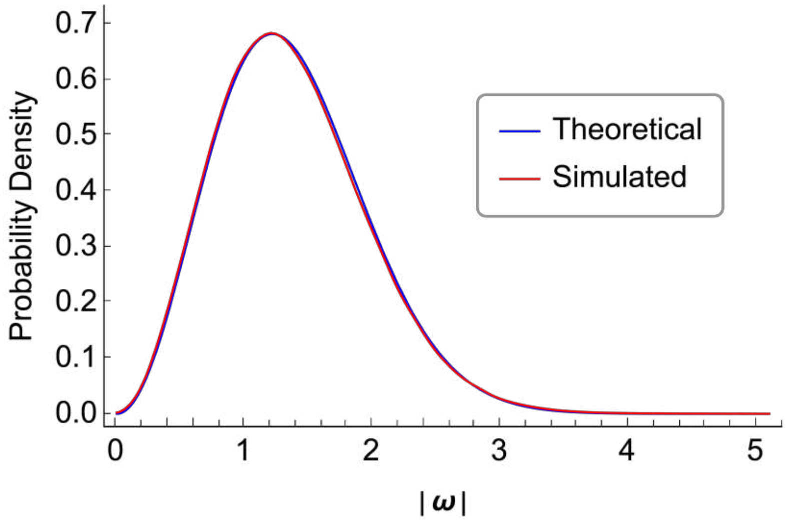

Figure 3 | The distribution of the norm $|\boldsymbol{\Omega}|$ of the total angular velocity when 10^3 random vectors with a norm of 100 are grouped together (with $R = 3$).

4. Conclusions

In this article, I have proven that there are more or less rotational components in a spherical particle swarm generated by the randomly moving particles in space, and the angular velocity of the particle swarm follows the Maxwell distribution with a scale parameter $\dfrac{\sqrt{6}c}{3R\sqrt{k}}$.

Acknowledgments I thank the engineers at *Wolfram Inc*. for technical support.

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