Generalised Geometric Logic

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Generalised Geometric Logic

P. Jana

Abstract. This paper introduces a notion of generalised geometric logic. Connections of generalised geometric logic with L -topological system and L -topological space are established.

Key Words and Phrases: geometric logic, fuzzy geometric logic, generalised geometric logic, L -topology

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1. Introduction

This work is motivated by S. Vickers's work on topology via logic [17]. To show the connection of topology with geometric logic, the notion of topological system played a crucial role. A topological system is a triple $(X, \models, A)$, consisting of a non-empty set X , a frame A and a binary relation $\models$ (known as satisfaction relation) between X and A satisfying certain conditions. The notion of topological system was introduced by S. Vickers in 1989. Topological system is an interesting mathematical structure, which unifies the concepts of topology, algebra, logic in a single framework. In our earlier work [1], we had introduced a notion of fuzzy geometric logic to answer the question viz. "From which logic can fuzzy topology be studied?". For this purpose first of all we introduced the notion of fuzzy topological system [6] which is a triple $(X, \models, A)$ consisting of a non-empty set X , a frame A and a fuzzy relation $\models$ (i.e. $[0, 1]$ valued relation) from X to A . J. Denniston et al. introduced the notion of lattice valued topological system (L -topological system) by considering frame valued relation between X and A . In [3], categorical relationship of Lattice valued topological space (L -topological space) with frame was established using the categorical relationships of them with L -topological system. Moreover categorical equivalence between spatial L -topological system with L -topological space was shown. In this paper

the main focus is to answer the question viz. “From which logic can L -topology be studied?”. From [1], it is clear that the satisfaction relation $\models$ of fuzzy topological system reflects the notion of satisfiability (*sat*) of a geometric formula by a sequence over the domain of interpretation of the corresponding logic. Hence we considered the grade of satisfiability from $[0, 1]$. As for L -topological system the satisfaction relation is an L (frame)-valued relation, the natural tendency is to consider the grade of satisfiability from L . Keeping this in mind, generalised geometric logic (c.f. Section 3) is proposed to provide the answer of the raised question successfully.

The paper is organised as follows. Section 2, includes some of the preliminary definitions and results which are used in the sequel. Generalised geometric logic is proposed and studied in details in Section 3. Section 4, explains the connection of the proposed logic with L -topological system whereas Section 5, contains the study of the connection of the proposed logic with L -topological space. Section 6, concludes the work presented in this article and provides some of the future directions.

2. Preliminaries

In this section we include a brief outline of relevant notions to develop our proposed mathematical structures and results. In [1, 2, 3, 5, 7, 8, 13, 17, 18] one may found the details of the notions stated here.

Definition 1 (Frame). A **frame** is a complete lattice such that,

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}.$$

i.e., the binary meet distributes over arbitrary join.

Definition 2 (Fuzzy topological space). Let X be a set, and τ be a collection of fuzzy subsets of X s.t.

1. $\tilde{\emptyset}, \tilde{X} \in \tau$, where $\tilde{\emptyset}(x) = 0$, for all $x \in X$ and $\tilde{X}(x) = 1$, for all $x \in X$;
2. $\tilde{A}_i \in \tau$ for $i \in I$ implies $\bigcup_{i \in I} \tilde{A}_i \in \tau$, where $\bigcup_{i \in I} \tilde{A}_i(x) = \sup_{i \in I} (\tilde{A}_i(x))$;
3. $\tilde{A}_1, \tilde{A}_2 \in \tau$ implies $\tilde{A}_1 \cap \tilde{A}_2 \in \tau$, where $(\tilde{A}_1 \cap \tilde{A}_2)(x) = \min\{\tilde{A}_1(x), \tilde{A}_2(x)\}$.

Then (X, τ) is called a **fuzzy topological space**. τ is called a **fuzzy topology** over X .

Elements of τ are called **fuzzy open sets** of fuzzy topological space (X, τ) .

Definition 3 (*L*-topological space). Let X be a set, and τ be a collection of *L*-fuzzy subsets of X i.e., $\tilde{A} : X \rightarrow L$, where L is a frame, s.t.

1. $\tilde{\emptyset}, \tilde{X} \in \tau$, where $\tilde{\emptyset}(x) = 0_L$, for all $x \in X$ and $\tilde{X}(x) = 1_L$, for all $x \in X$;
2. $\tilde{A}_i \in \tau$ for $i \in I$ implies $\bigcup_{i \in I} \tilde{A}_i \in \tau$, where $\bigcup_{i \in I} \tilde{A}_i(x) = \sup_{i \in I} (\tilde{A}_i(x))$;
3. $\tilde{A}_1, \tilde{A}_2 \in \tau$ implies $\tilde{A}_1 \cap \tilde{A}_2 \in \tau$, where $(\tilde{A}_1 \cap \tilde{A}_2)(x) = \tilde{A}_1(x) \wedge \tilde{A}_2(x)$.

Then (X, τ) is called an ***L*-topological space**. τ is called an ***L*-topology** over X .

Elements of τ are called ***L*-open sets** of *L*-topological space (X, τ) .

Definition 4. [17] A **topological system** is a triple, $(X, \models, A)$, consisting of a non empty set X , a frame A and a binary relation $\models \subseteq X \times A$ from X to A such that:

1. for **any finite subset** S of A , $x \models \bigwedge S$ if and only if $x \models a$ for all $a \in S$;
2. for **any subset** S of A , $x \models \bigvee S$ if and only if $x \models a$ for some $a \in S$.

Definition 5 (*L*-topological system). An ***L*-topological system** is a triple $(X, \models, A)$, where X is a non-empty set, A is a frame and $\models$ is an *L*-valued relation from X to A ($\models : X \times A \rightarrow L$) such that

1. if S is a finite subset of A , then $\models(x, \bigwedge S) = \inf\{\models(x, s) \mid s \in S\}$;
2. if S is any subset of A , then $\models(x, \bigvee S) = \sup\{\models(x, s) \mid s \in S\}$.

Note that if $L = [0, 1]$ then the triple is known as fuzzy topological system. Similarly considering $L = \{0, 1\}$, we arrive at the notion of a topological system.

Definition 6 (Spatial). An *L*-topological system $(X, \models, A)$ is said to be **spatial** if and only if (for any $x \in X$, $\models(x, a) = \models(x, b)$) imply $(a = b)$, for any $a, b \in A$.

Theorem 1. Category of spatial *L*-topological systems, for a fixed L , is equivalent to the category of *L*-topological spaces.

3. Generalised Geometric Logic

In this section we will introduce the notion of generalised geometric logic which may be considered as a generalisation of fuzzy geometric logic and consequently of so called geometric logic. Detailed studies on fuzzy logic, geometric logic and fuzzy geometric logic may be found in [1, 4, 9, 10, 11, 12, 14, 15, 16, 17].

The **alphabet** of the language $\mathcal{L}$ of generalised geometric logic comprises of the connectives $\wedge, \vee$, the existential quantifier $\exists$, parentheses $)$ and $($ as well as:

- countably many individual constants $c_1, c_2, \dots$;
- denumerably many individual variables $x_1, x_2, \dots$;
- propositional constants $\top, \perp$;
- for each $i > 0$, countably many i -place predicate symbols p_j^i 's, including at least the 2-place symbol “=” for identity;
- for each $i > 0$, countably many i -place function symbols f_j^i 's.

Definition 7 (Term). ***Terms** are recursively defined in the usual way.*

- every constant symbol c_i is a term;
- every variable x_i is a term;
- if f_j is an i -place function symbol, and $t_1, t_2, \dots, t_i$ are terms then $f_j^i t_1 t_2 \dots t_i$ is a term;
- nothing else is a term.

Definition 8 (Geometric formula). ***Geometric formulae** are recursively defined as follows:*

- $\top, \perp$ are geometric formulae;
- if p_j is an i -place predicate symbol, and $t_1, t_2, \dots, t_i$ are terms then $p_j^i t_1 t_2 \dots t_i$ is a geometric formula;
- if t_i, t_j are terms then $(t_i = t_j)$ is a geometric formula;
- if ϕ and ψ are geometric formulae then $(\phi \wedge \psi)$ is a geometric formula;
- if ϕ_i 's ($i \in I$) are geometric formulae then $\bigvee \{\phi_i\}_{i \in I}$ is a geometric formula, when $I = \{1, 2\}$ then the above formula is written as $\phi_1 \vee \phi_2$;
- if ϕ is a geometric formula and x_i is a variable then $\exists x_i \phi$ is a geometric formula;
- nothing else is a geometric formula.

Definition 9 (Interpretation). *An **interpretation** I consists of*

- a set D , called the domain of interpretation;
- an element $I(c_i) \in D$ for each constant c_i ;

- a function $I(f_j^i) : D^i \longrightarrow D$ for each function symbol f_j^i ;
- an L -fuzzy relation $I(p_j^i) : D^i \longrightarrow L$, where L is a frame, for each predicate symbol p_j^i i.e. it is an L -fuzzy subset of D^i .

Definition 10 (Graded Satisfiability). Let s be a sequence over D . Let $s = (s_1, s_2, \dots)$ be a sequence over D where $s_1, s_2, \dots$ are all elements of D . Let d be an element of D . Then $s(d/x_i)$ is the result of replacing i 'th coordinate of s by d i.e., $s(d/x_i) = (s_1, s_2, \dots, s_{i-1}, d, s_{i+1}, \dots)$. Let t be a term. Then s assigns an element $s(t)$ of D as follows:

- if t is the constant symbol c_i then $s(c_i) = I(c_i)$;
- if t is the variable x_i then $s(x_i) = s_i$;
- if t is the function symbol $f_j^i t_1 t_2 \dots t_i$ then $s(f_j^i t_1 t_2 \dots t_i) = I(f_j^i)(s(t_1), s(t_2), \dots, s(t_i))$.

Now we define grade of satisfiability of ϕ by s written as $gr(s \text{ sat } \phi)$, where ϕ is a geometric formula, as follows:

- $gr(s \text{ sat } p_j^i t_1 t_2 \dots t_i) = I(p_j^i)(s(t_1), s(t_2), \dots, s(t_i))$;
- $gr(s \text{ sat } \top) = 1_L$;
- $gr(s \text{ sat } \perp) = 0_L$;
- $gr(s \text{ sat } t_i = t_j) = \begin{cases} 1_L & \text{if } s(t_i) = s(t_j) \\ 0_L & \text{otherwise;} \end{cases}$
- $gr(s \text{ sat } \phi_1 \wedge \phi_2) = gr(s \text{ sat } \phi_1) \wedge gr(s \text{ sat } \phi_2)$;
- $gr(s \text{ sat } \phi_1 \vee \phi_2) = gr(s \text{ sat } \phi_1) \vee gr(s \text{ sat } \phi_2)$;
- $gr(s \text{ sat } \bigvee \{\phi_i\}_{i \in I}) = \sup\{gr(s \text{ sat } \phi_i) \mid i \in I\}$;
- $gr(s \text{ sat } \exists x_i \phi) = \sup\{gr(s(d/x_i) \text{ sat } \phi) \mid d \in D\}$.

Throughout this article $\wedge$ and $\vee$ in L will stand for the meet and join of the frame L respectively. The expression $\phi \vdash \psi$, where ϕ, ψ are wffs, is called a sequent. We now define satisfiability of a sequent.

- Definition 11.**
1. $s \text{ sat } \phi \vdash \psi$ iff $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$.
 2. $\phi \vdash \psi$ is valid in I iff $s \text{ sat } \phi \vdash \psi$ for all s in the domain of I .
 3. $\phi \vdash \psi$ is universally valid iff it is valid in all interpretations.

Theorem 2 (Substitution Theorem). *Let D be the domain of interpretation I :*

1. *Let t and t' be terms. For every sequence s over D ,
 $s(t[t'/x_k]) = s(s(t')/x_k)(t)$.*
2. *Let ϕ be a geometric formula and t be a term. For every sequence s over D , $gr(s \text{ sat } \phi[t/x_k]) = gr(s(s(t)/x_k) \text{ sat } \phi)$.*

Proof. By induction on t and ϕ respectively.

3.1. Rules of Inference

The rules of inference for generalised geometric logic are as follows.

1. $\phi \vdash \phi$,
2. $\frac{\phi \vdash \psi \quad \psi \vdash \chi}{\phi \vdash \chi}$,
3. (i) $\phi \vdash \top$, (ii) $\phi \wedge \psi \vdash \phi$, (iii) $\phi \wedge \psi \vdash \psi$, (iv) $\frac{\phi \vdash \psi \quad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$,
4. (i) $\phi \vdash \bigvee S$ ($\phi \in S$), (ii) $\frac{\phi \vdash \psi \quad \text{all } \phi \in S}{\bigvee S \vdash \psi}$,
5. $\phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}$,
6. $\top \vdash (x = x)$,
7. $((x_1, \dots, x_n) = (y_1, \dots, y_n)) \wedge \phi \vdash \phi[(y_1, \dots, y_n) \mid (x_1, \dots, x_n)]$,
8. (i) $\frac{\phi \vdash \psi[x \mid y]}{\phi \vdash \exists y \psi}$, (ii) $\frac{\exists y \phi \vdash \psi}{\phi[x \mid y] \vdash \psi}$,
9. $\phi \wedge (\exists y) \psi \vdash (\exists y)(\phi \wedge \psi)$.

Theorem 3. *The rules of inference for generalised geometric logic are universally valid.*

Proof. To show the universal validity of the rules of inference is kind of a routine check but for the sake of clarity we would like to provide the proof in details upto certain extent.

1. $gr(s \text{ sat } \phi) = gr(s \text{ sat } \phi)$, for any s . Hence $\phi \vdash \phi$ is valid.

2. Given $\phi \vdash \psi$ and $\psi \vdash \chi$ are valid. So $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$ and $gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \chi)$ for any s . Therefore $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \chi)$ for any s . Hence $\phi \vdash \chi$ is valid when $\phi \vdash \psi$ and $\psi \vdash \chi$ are valid.
3. (i) $gr(s \text{ sat } \phi) \leq 1_L = gr(s \text{ sat } \top)$ for any s . Hence $\phi \vdash \top$ is valid.
(ii) $gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \phi)$ for any s . Hence $\phi \wedge \psi \vdash \phi$ is valid.
(iii) $gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \psi)$ for any s . Hence $\phi \wedge \psi \vdash \psi$ is valid.
(iv) Given $\phi \vdash \psi$ and $\phi \vdash \chi$ are valid. So $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$ and $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \chi)$ for any s . So $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi) \wedge gr(s \text{ sat } \chi) = gr(s \text{ sat } \psi \wedge \chi)$ for any s . Hence $\phi \vdash \psi \wedge \chi$ is valid when $\phi \vdash \psi$ and $\phi \vdash \chi$ are valid.
4. (i) $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \bigvee S(\phi \in S))$ for any s . Hence $\phi \vdash \bigvee S(\phi \in S)$ is valid. (ii) Given $\phi \vdash \psi$ is valid for all $\phi \in S$. So $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$ for all $\phi \in S$ and any s . So, $\sup_{\phi \in S} \{gr(s \text{ sat } \phi)\} \leq gr(s \text{ sat } \psi)$ for any s . Hence $gr(s \text{ sat } \bigvee S) \leq gr(s \text{ sat } \psi)$ for any s . So, $\bigvee S \vdash \psi$ is valid when $\phi \vdash \psi$ is valid for all $\phi \in S$.
5. We have, $gr(s \text{ sat } \phi \wedge \bigvee S) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \bigvee S) = gr(s \text{ sat } \phi) \wedge \sup_{\psi \in S} \{gr(s \text{ sat } \psi)\} = \sup_{\psi \in S} \{gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi)\} = \sup \{gr(s \text{ sat } \phi \wedge \psi) \mid \psi \in S\}$, for any s . Hence $\phi \wedge \bigvee S \vdash \sup \{\phi \wedge \psi \mid \psi \in S\}$ is valid.
6. $gr(s \text{ sat } \top) = 1_L = gr(s \text{ sat } x = x)$, for any s . Hence $\top \vdash x = x$ is valid.
7. $gr(s \text{ sat } ((x_1, \dots, x_n) = (y_1, \dots, y_n)) \wedge \phi)$
 $= gr(s \text{ sat } ((x_1, \dots, x_n) = (y_1, \dots, y_n))) \wedge gr(s \text{ sat } \phi)$.
Now $gr(s \text{ sat } \phi[(y_1, \dots, y_n)/(x_1, \dots, x_n)])$
 $= gr(s(s((y_1, \dots, y_n))/(x_1, \dots, x_n)) \text{ sat } \phi)$.
When $s((y_1, \dots, y_n)) = s((x_1, \dots, x_n))$
then $gr(s(s((y_1, \dots, y_n))/(x_1, \dots, x_n)) \text{ sat } \phi) = gr(s \text{ sat } \phi)$.
Hence, $gr(s \text{ sat } ((x_1, \dots, x_n) = (y_1, \dots, y_n)) \wedge \phi)$
 $\leq gr(s \text{ sat } \phi[(y_1, \dots, y_n)/(x_1, \dots, x_n)])$, for any s .
So, $((x_1, \dots, x_n) = (y_1, \dots, y_n)) \wedge \phi \vdash \phi[(y_1, \dots, y_n)/(x_1, \dots, x_n)]$ is valid.
8. (i) $\phi \vdash \psi[x \mid y]$ is valid so, $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \psi[x \mid y])$, for any s . Using Theorem 2(2) $gr(s \text{ sat } \phi) \leq gr(s(s(x)/y) \text{ sat } \psi)$, for any s , which implies that $gr(s \text{ sat } \phi) \leq \sup \{gr(s(d/y) \text{ sat } \psi) \mid d \in D\}$, for any s . So, $gr(s \text{ sat } \phi) \leq gr(s \text{ sat } \exists y \psi)$ and hence $\phi \vdash \exists y \psi$ is valid.
(ii) $\exists y \phi \vdash \psi$ is valid if and only if $gr(s \text{ sat } \exists y \phi) \leq gr(s \text{ sat } \psi)$, for any s . Hence $\sup \{gr(s(d/y) \text{ sat } \phi) \mid d \in D\} \leq gr(s \text{ sat } \psi)$, for any s . So,

$gr(s(s(x)/y) \text{ sat } \phi) \leq gr(s \text{ sat } \psi)$, for any s , using Theorem 2(2). Therefore $gr(s \text{ sat } \phi[x/y]) \leq gr(s \text{ sat } \psi)$, for any s and hence $\phi[x/y] \vdash \psi$ is valid provided $\exists y \phi \vdash \psi$ is valid.

9. $gr(s \text{ sat } \phi \wedge (\exists y)\psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \exists y \psi) = gr(s \text{ sat } \phi) \wedge \sup_{d \in D} \{gr(s(d/y) \text{ sat } \psi)\}$
 $= \sup_{d \in D} \{gr(s \text{ sat } \phi) \wedge gr(s(d/y) \text{ sat } \psi)\} \leq \sup_{d \in D} \{gr(s(d/y) \text{ sat } \phi) \wedge$
 $gr(s(d/y) \text{ sat } \psi)\} = \sup_{d \in D} \{gr(s \text{ sat } \phi \wedge \psi)\} = gr(s \text{ sat } (\exists y)\phi \wedge \psi)$, for any s . Hence $\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)$ is valid.

3.2. Generalised logic of finite observations

In this subsection we will consider the propositional fragment of the proposed generalised geometric logic and call it generalised logic of finite observations. Throughout this part the justification to choose the name will be provided. In [17], one may notice that the logic of finite observations is nothing but the logic of affirmative assertions. Recall that an assertion is affirmative if and only if the assertion is true precisely in the circumstances where it can be affirmed. Note that we need to do the job in finite time, finite amount of work and on the basis of what we can actually observe. In [17], it is nicely explained why logic of affirmative assertions allows the connectives $\wedge$, $\vee$, $\top$, $\perp$ but not $\neg$ and $\rightarrow$. If we wish to make the idea of logic of affirmative assertions more close to real life situations then discussing the validity (truth value) of affirmative assertions upto some extent instead of, whether affirmative assertions are valid (true) or not valid (false), is a better idea. To address this issue we need to concentrate on the notion of valuation function. In this stage it is better to quickly recapture the propositional part of our proposed generalised logic for better understanding. Let Φ be a set of propositional variables. The language $GGL(\Phi)$ of generalised logic of finite observations of propositional generalised geometric formula is given by

$$\phi ::= \top \mid \perp \mid p \mid \phi_1 \wedge \phi_2 \mid \bigvee \{\phi_i\}_{i \in I}$$

where $p \in \Phi$ and I is some index set. The rules are given by

1. $\phi \vdash \phi$,
2. $\frac{\phi \vdash \psi \quad \psi \vdash \chi}{\phi \vdash \chi}$,
3. (i) $\phi \vdash \top$, (ii) $\phi \wedge \psi \vdash \phi$, (iii) $\phi \wedge \psi \vdash \psi$, (iv) $\frac{\phi \vdash \psi \quad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$,
4. (i) $\phi \vdash \bigvee S$ ($\phi \in S$), (ii) $\frac{\phi \vdash \psi \quad \text{all } \phi \in S}{\bigvee S \vdash \psi}$,

5. $\phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}$.

Proposition 1. $\bigvee \{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S$ is derivable.

Proof.

$$\frac{\frac{\phi \vdash \phi \quad \phi \vdash \bigvee S \ (\phi \in S)}{\phi \vdash \phi \wedge \bigvee S} \quad \frac{\phi \wedge \psi \vdash \phi}{\bigvee \{\phi \wedge \psi \mid \psi \in S\} \vdash \phi}}{\bigvee \{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S}$$

The valuation function $v : \Phi \rightarrow L$ can be extended to $\hat{v} : GGL(\Phi) \rightarrow L$ defined by

1. $\hat{v}(\top) = 1_L$;
2. $\hat{v}(\perp) = 0_L$;
3. $\hat{v}(\phi \wedge \psi) = \hat{v}(\phi) \wedge \hat{v}(\psi)$;
4. $\hat{v}(\bigvee \{\phi_i\}_{i \in I}) = \sup\{\hat{v}(\phi_i) \mid i \in I\}$.

Now notice that if we consider the range of the valuation function a frame (L) instead of $\{0, 1\}$ then mathematically we can reach our goal. Consideration of the range as any frame allows us to think about the incomparable truth values of affirmative assertions, which is a natural phenomenon to arise in our daily life situations. If we think in this line then it is not very hard to understand how the definition of the extended valuation function considered here is the expected one. In this sense we will be able to generalise the notion of the so-called logic of affirmative assertions or logic of finite observations to address real life situations in a better way. Moreover this generalised version of the logic connects the desired mathematical structures (L-topological space, L-topological system and frame) as well.

Definition 12. $\phi \vdash \psi$ is valid if and only if $\hat{v}(\phi) \leq \hat{v}(\psi)$ for all $\hat{v} : GGL(\Phi) \rightarrow L$.

Proposition 2. The rules of inference are valid.

The proposition stated above implies that $\bigvee \{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S$ is valid. We will use this piece of information in the next section.

4. L -Topological System via Generalised Geometric Logic

In this section the way to get an L -topological system from generalised geometric logic is provided. In this respect it is to be noted that the propositional fragment of the proposed generalised geometric logic is enough to serve our purpose. Let us consider the triplet $(X, \models, A)$ where X is the non empty set of extended valuation functions, A is the set of geometric formulae and $\models$ defined as $\models (\hat{v}, \phi) = \hat{v}(\phi)$.

Proposition 3. (i) $\models (\hat{v}, \phi \wedge \psi) = \models (\hat{v}, \phi) \wedge \models (\hat{v}, \psi)$.
(ii) $\models (\hat{v}, \bigvee \{\phi_i\}_{i \in I}) = \sup_{i \in I} \{\models (\hat{v}, \phi_i)\}$.

Proof. (i) $\models (\hat{v}, \phi \wedge \psi) = \hat{v}(\phi \wedge \psi) = \hat{v}(\phi) \wedge \hat{v}(\psi) = \models (\hat{v}, \phi) \wedge \models (\hat{v}, \psi)$.
(ii) $\models (\hat{v}, \bigvee \{\phi_i\}_{i \in I}) = \hat{v}(\bigvee \{\phi_i\}_{i \in I}) = \sup_{i \in I} \{\hat{v}(\phi_i)\} = \sup_{i \in I} \{\models (\hat{v}, \phi_i)\}$.

Definition 13. $\phi \approx \psi$ iff $\models (\hat{v}, \phi) = \models (\hat{v}, \psi)$ for any $\hat{v} \in X$ and $\phi, \psi \in A$.

The above defined “ $\approx$ ” is an equivalence relation. Thus we get $A/\approx$.

Proposition 4. $(A/\approx, \leq, \wedge, \bigvee)$ is a frame, where $[\phi] \leq [\psi]$ holds when $\hat{v}(\phi) \leq \hat{v}(\psi)$ for all $\hat{v} : GGL(\Phi) \rightarrow L$, $[\phi] \wedge [\psi] = [\phi \wedge \psi]$ and $\bigvee \{[\phi_i]\}_{i \in I} = [\bigvee \{\phi_i\}_{i \in I}]$.

Proof. First of all $\hat{v}(\phi) = \hat{v}(\phi)$, for all $\hat{v}$. So $[\phi] \leq [\phi]$ holds for any $[\phi] \in A/\approx$. Let $[\phi] \leq [\psi]$ and $[\psi] \leq [\phi]$. Then $\hat{v}(\phi) = \hat{v}(\psi)$, for all $\hat{v}$. Hence $\phi \approx \psi$, which indicates that $[\phi] = [\psi]$ whenever $[\phi] \leq [\psi]$ and $[\psi] \leq [\phi]$ holds. Similarly, if $[\phi] \leq [\psi]$ and $[\psi] \leq [\chi]$ holds then $[\phi] \leq [\chi]$ holds. Hence $(A/\approx, \leq)$ is a poset. It is easy to observe that $A/\approx$ is closed under $\wedge$ and $\bigvee$ (follows from Proposition 3). Moreover from the previous section we have $\bigvee \{\phi \wedge \psi \mid \psi \in S\} \vdash \phi \wedge \bigvee S$ and $\phi \wedge \bigvee S \vdash \bigvee \{\phi \wedge \psi \mid \psi \in S\}$ are valid. Hence $\hat{v}(\phi \wedge \bigvee S) = \hat{v}(\bigvee \{\phi \wedge \psi \mid \psi \in S\})$ for all $\hat{v} : GGL(\Phi) \rightarrow L$. Therefore $\phi \wedge \bigvee S \approx \bigvee \{\phi \wedge \psi \mid \psi \in S\}$. Consequently we have $[\phi \wedge \bigvee S] = [\bigvee \{\phi \wedge \psi \mid \psi \in S\}]$, i.e., $[\phi] \wedge \bigvee \{[\psi]\}_{\psi \in S} = \bigvee \{[\phi \wedge \psi]\}_{\psi \in S}$. Consequently we arrive at the conclusion that finite meet distributes over arbitrary join, i.e., the frame distributive property holds good.

Proposition 3 and Proposition 4 provides the following theorem.

Theorem 4. $(X, \models', A/\approx)$ is an L -topological system, where $\models'$ is defined by $\models' (\hat{v}, [\phi]) = \models (\hat{v}, \phi)$.

Proposition 5. The L -topological system $(X, \models', A/\approx)$ defined as above, is spatial.

Proof. Let for any $\hat{v} \in X$, $\models (\hat{v}, a) = \models (\hat{v}, b)$. Then $\hat{v}(a) = \hat{v}(b)$ for all $\hat{v} \in X$. Which implies that $a = b$. Therefore for any $\hat{v} \in X$, $\models (\hat{v}, a) = \models (\hat{v}, b)$ imply $(a = b)$, for any $a, b \in A$.

5. L -Topology via Generalised Geometric Logic

We first construct the L -topological system $(X, \models', A/\approx)$ from generalised geometric logic. Then $(X, \text{ext}(A/\approx))$ is constructed as follows:

$\text{ext}(A/\approx) = \{\text{ext}([\phi])\}_{[\phi] \in A/\approx}$ where $\text{ext}([\phi]) : X \rightarrow L$ is such that, for each $[\phi] \in A/\approx$, $\text{ext}([\phi])(\hat{v}) = \models'(\hat{v}, [\phi]) = \models(\hat{v}, \phi)$.

It can be shown that $\text{ext}(A/\approx)$ forms an L -topology on X as follows.

Let $\text{ext}([\phi]), \text{ext}([\psi]) \in \text{ext}(A/\approx)$. Then $(\text{ext}([\phi]) \cap \text{ext}([\psi]))(\hat{v}) = (\text{ext}([\phi])(\hat{v}) \wedge \text{ext}([\psi])(\hat{v})) = \models'(\hat{v}, [\phi]) \wedge \models'(\hat{v}, [\psi]) = \models(\hat{v}, \phi) \wedge \models(\hat{v}, \psi) = \models(\hat{v}, \phi \wedge \psi) = \models'(\hat{v}, [\phi \wedge \psi]) = \text{ext}([\phi \wedge \psi])(\hat{v})$. Hence $\text{ext}([\phi]) \cap \text{ext}([\psi]) = \text{ext}([\phi \wedge \psi]) \in \text{ext}(A/\approx)$. Similarly it can be shown that $\text{ext}(A/\approx)$ is closed under arbitrary union. Hence $(X, \text{ext}(A/\approx))$ is an L -topological space obtained via generalised geometric logic.

Proposition 5 indicates that $(X, \models', A/\approx)$ is a spatial L -topological system and hence from Theorem 1 we arrive at the conclusion that $(X, \models', A/\approx)$, $(A, \text{ext}(A/\approx))$ are equivalent to each other. That is, $(X, \models', A/\approx)$ and $(X, \in, \text{ext}(A/\approx))$ represent the same L -topological system. The following diagram summarize all the facts what we have proved till this stage:

Let X be an L -topological space, τ is its L -topology. Then the corresponding generalised geometric theory can be defined as follows:

- for each L -open set $\tilde{T}$, a proposition $P_{\tilde{T}}$
- if $\tilde{T}_1 \subseteq \tilde{T}_2$, then an axiom

$$P_{\tilde{T}_1} \vdash P_{\tilde{T}_2}$$

- if S is a family of L -open sets, then an axiom

$$P_{\bigcup S} \vdash \bigvee_{\tilde{T} \in S} P_{\tilde{T}}$$

- if S is finite collection of L -open sets, then an axiom

$$\bigwedge_{\tilde{T} \in S} P_{\tilde{T}} \vdash P_{\bigcap S}$$

All other axioms for the (propositional) generalised geometric logic will follow from the above clauses.

If $x \in X$, then x gives a model of the theory in which the truth value of the interpretation of $P_{\tilde{T}}$ will be $\tilde{T}(x)$.

Hence one may study L -topology via generalised geometric logic.

6. Concluding Remarks

In this paper the notion of generalised geometric logic is introduced and studied in details. Using the connection between L -topological system and L -topological space, the strong connection between the proposed logic and L -topological space is established. The interpretation of the predicate symbols for the generalised geometric logic are L (frame)-valued relations and so the proposed logic is more expressible. That is, the proposed logic has the capacity to interpret the situation where the truth values are incomparable. Generalising the proposed logic considering graded consequence relation is in future goal.

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