Comparison of Twelve Types of Rough Approximations Based on j-Neighborhood Space and j-Adhesion Neighborhood Space

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Comparison of twelve types of rough approximations based on j -neighborhood space and j -adhesion neighborhood space

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Abstract In this paper, we generalize six kinds of rough set models based on j -neighborhood space (i.e., reflexive 1 j -neighborhood rough set, reflexive 2 j -neighborhood rough set, reflexive 3 j -neighborhood rough set, similarity 4 j -neighborhood rough set, similarity 5 j -neighborhood rough set, and similarity 6 j -neighborhood rough set), and investigate some of their basic properties. Further, we propose a new neighborhood space called j -adhesion neighborhood based on six types of rough set models (i.e., reflexive 7 j -adhesion neighborhood rough set, reflexive 8 j -adhesion neighborhood rough set, reflexive 9 j -adhesion neighborhood rough set, similarity 10 j -adhesion neighborhood rough set, similarity 11 j -adhesion neighborhood rough set, and similarity 12 j -neighborhood rough set) to reduce the boundary region and the accuracy. The fundamental properties of approximation operators based on j -adhesion neighborhood space are investigated. The relationship between the properties of these types is explained. Finally, we give comparisons between the proposed approach with the previous approach (i.e., Abo-Tabl's approach and Dai et al.'s approach)

from six types of rough set models. Consequently, the accuracy from the proposed approach is improved.

Keywords Rough sets. Lower and upper approximations. j -Neighborhood space. j -Adhesion neighborhood space. Accuracy measure

1 Introduction

The original concept of a rough set is proposed by Pawlak [1, 2]. The rough set theory has been applied to many different fields, such as economics, engineering, medicine, biology, chemistry, banking, market research, speech recognition, information analysis, data analysis, material science, data mining, networking, linguistics, and other fields (see, e.g., [3-15]). A framework for the formulation, interpretation, and comparison of neighborhood systems and rough set approximations (lower / upper) using the more familiar notion of binary relations is given by Yao [16]. He also introduced a special class of neighborhood systems, it is called 1-neighborhood systems and analyzed three extensions of Pawlak's lower and upper approximations. Later, many researchers presented some concepts based on a rough set, for example, similarity relation [17], the tolerance relation [18], and the arbitrary binary relation [16, 19-22]. Abu-Donia [23] discussed three types of lower and upper approximations of any set concerning any relationship based on the right neighborhood and generalized these three types of approximations into two ways by using a finite number of any binary relations. Abo-Tabl [24] gave a comparison of two types of definitions of the rough approximations based on a similarity relation (i.e., reflexivity and symmetry). Besides, he defined three new types of generalized rough approximations. Dai et al. [25] determined the generalized rough set models by multiple neighborhoods generated from a similarity relation.

In 2014, Abd El-Monsef et al. [26] presented the main ideas about the concept of " j -neighborhood space" and studied eight approaches for approximating rough sets. Abbas et al. [27] introduced the new concepts of j -near closure operators induced from binary relation and its applications. Hosny [28] gave a generalization of rough sets by using two different methods. Abd El-Monsef [29] presented the cover rough sets based on j -neighborhoods by approximation operations as a new type of extended covering

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rough set models. Amer [30] proposed the concepts of “ j -near” in rough sets with some applications.

Regarding these developments, as the motivation of this paper, we will generalize the three types of Abo-Tabl’s lower and upper approximations and three types of Dai et al.’s lower and upper approximations of any set concerning for j -neighborhood space. Further, we will define six new types of approximations based on j -adhesion neighborhood space. Properties of approximation operators based on j -adhesion neighborhood space are explained and discussed. The relationship between these types of the j -adhesion neighborhood space is introduced. Finally, we will give a comparison between the previous approach (i.e., Abo-Tabl’s approach [24] and Dai et al.’s approach [25]) and our approach by using j -adhesion neighborhood space.

The rest of this paper is organized as follows. In Section 2, we review some basic concepts, such as types of rough set approximations based on binary relation, j -neighborhood of element x , j -neighborhood space, Abo-Tabl’s approach, and Dai et al.’s approach. In Section 3, we generalize six types of rough sets based on j -neighborhood space. Several of the existing properties of six types of rough sets are explained. A new six types of rough set based on j -adhesion neighborhood space are defined and investigated their properties are shown in Section 4. A comparison between the previous approach (i.e., Abo-Tabl’s approach [24] and Dai et al.’s approach [25]) and our approach is discussed in Section 5. Finally, conclusions are given in Section 6.

2 Preliminaries

In this section, we recall several basic notions and results which will be used in this paper as indicated below.

2.1. Types of rough set approximations based on binary relation

The concept of a rough set to any relation $\mathcal{R}$ extended by Yao [16] as follows:

Definition 2.1 (cf. [16]). Let $\mathcal{R}$ be an arbitrary binary relation on an universe set $\mathcal{U}$ and $(\mathcal{U}, \mathcal{R})$ be an approximation space. Then

$$\mathcal{R}(u) = \{v \in \mathcal{U} : u\mathcal{R}v\}.$$

Definition 2.2 (cf. [16]). Let $(\mathcal{U}, \mathcal{R})$ be an approximation space and $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$. The lower and the upper approximations of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathcal{R})$ or with respect to $\mathcal{R}$ is a pair $(\mathcal{R}^-(\hat{X}), \mathcal{R}^+(\hat{X}))$, defined by

$$\mathcal{R}^-(\hat{X}) = \{u \in \mathcal{U} : \mathcal{R}(u) \subseteq \hat{X}\}$$

(called the lower approximation of $\hat{X}$) and

$$\mathcal{R}^+(\hat{X}) = \{u \in \mathcal{U} : \mathcal{R}(u) \cap \hat{X} \neq \emptyset\}$$

(called the upper approximation of $\hat{X}$).

We list the properties that are of interest in the theory of rough sets, let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$:

- (L1) $\mathcal{R}^-(\hat{X}) = (\mathcal{R}^+(\hat{X}^c))^c$.
- (L2) $\mathcal{R}^-(\mathcal{U}) = \mathcal{U}$.
- (L3) $\mathcal{R}^-(\hat{X}_1 \cap \hat{X}_2) = \mathcal{R}^-(\hat{X}_1) \cap \mathcal{R}^-(\hat{X}_2)$.
- (L4) $\mathcal{R}^-(\hat{X}_1) \cup \mathcal{R}^-(\hat{X}_2) \subseteq \mathcal{R}^-(\hat{X}_1 \cup \hat{X}_2)$.
- (L5) If $\hat{X}_1 \subseteq \hat{X}_2$, then $\mathcal{R}^-(\hat{X}_1) \subseteq \mathcal{R}^-(\hat{X}_2)$.

- (L6) $\mathcal{R}^-(\emptyset) = \emptyset$.
- (L7) $\mathcal{R}^-(\hat{X}) \subseteq \hat{X}$.
- (L8) $\hat{X} \subseteq \mathcal{R}^-(\mathcal{R}^+(\hat{X}))$.
- (L9) $\mathcal{R}^-(\mathcal{R}^-(\hat{X})) = \mathcal{R}^-(\hat{X})$.
- (L10) $\mathcal{R}^+(\hat{X}) \subseteq \mathcal{R}^-(\mathcal{R}^+(\hat{X}))$.
- (U1) $\mathcal{R}^+(\hat{X}) = (\mathcal{R}^-(\hat{X}^c))^c$.
- (U2) $\mathcal{R}^+(\emptyset) = \emptyset$.
- (U3) $\mathcal{R}^+(\hat{X}_1 \cap \hat{X}_2) \subseteq \mathcal{R}^+(\hat{X}_1) \cap \mathcal{R}^+(\hat{X}_2)$.
- (U4) $\mathcal{R}^+(\hat{X}_1 \cup \hat{X}_2) = \mathcal{R}^+(\hat{X}_1) \cup \mathcal{R}^+(\hat{X}_2)$.
- (U5) If $\hat{X}_1 \subseteq \hat{X}_2$, then $\mathcal{R}^+(\hat{X}_1) \subseteq \mathcal{R}^+(\hat{X}_2)$.
- (U6) $\mathcal{R}^+(\mathcal{U}) = \mathcal{U}$.
- (U7) $\hat{X} \subseteq \mathcal{R}^+(\hat{X})$.
- (U8) $\mathcal{R}^+(\mathcal{R}^-(\hat{X})) \subseteq \hat{X}$.
- (U9) $\mathcal{R}^+(\mathcal{R}^+(\hat{X})) = \mathcal{R}^+(\hat{X})$.
- (U10) $\mathcal{R}^+(\mathcal{R}^-(\hat{X})) \subseteq \mathcal{R}^-(\hat{X})$.
- (K) $\mathcal{R}^-(\hat{X}_1^c \cup \hat{X}_2) \subseteq (\mathcal{R}^-(\hat{X}_1))^c \cup \mathcal{R}^-(\hat{X}_2)$.
- (LU) $\mathcal{R}^-(\hat{X}) \subseteq \mathcal{R}^+(\hat{X})$.

2.2. Types of rough set approximations based on Abo-Tabl’s approach [24]

Abo-Tabl [24] presented the following definitions:

Definition 2.3. Let $\mathcal{R}$ be any reflexive relation on a universe set $\mathcal{U}$ and $(\mathcal{U}, \mathcal{R})$ be an approximation space. The set $\mathcal{R}^\square(u)$ ($\forall u \in \mathcal{U}$) (The intersection of all right neighborhoods containing u) as follows:

$$\mathcal{R}^\square(u) = \begin{cases} \bigcap_{v \in \mathcal{R}(u)} \mathcal{R}(v), & \text{if } \exists v, u \in \mathcal{R}(v) \\ \emptyset, & \text{otherwise} \end{cases}.$$

Definition 2.4. Let $(\mathcal{U}, \mathcal{R})$ be a reflexive approximation space and $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$. The first type of the reflexive lower and reflexive upper approximations of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathcal{R})$ or with respect to $\mathcal{R}$ is a pair $(\mathcal{R}^-(\hat{X}), \mathcal{R}^+(\hat{X}))$, defined by

$$\mathcal{R}^-(\hat{X}) = \{u \in \mathcal{U} : \mathcal{R}^\square(u) \subseteq \hat{X}\}$$

(called the first reflexive lower approximation of $\hat{X}$) and

$$\mathcal{R}^+(\hat{X}) = \{u \in \mathcal{U} : \mathcal{R}^\square(u) \cap \hat{X} \neq \emptyset\}$$

(called the first reflexive upper approximation of $\hat{X}$).

Definition 2.5. Let $(\mathcal{U}, \mathcal{R})$ be a reflexive approximation space and $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$. The second type of the reflexive lower and reflexive upper approximations of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathcal{R})$ or with respect to $\mathcal{R}$ is a pair $(\mathcal{R}^-(\hat{X}), \mathcal{R}^+(\hat{X}))$, defined by

$$\mathcal{R}^-(\hat{X}) = \bigcap \{\mathcal{R}^\square(u) : \mathcal{R}^\square(u) \subseteq \hat{X}\}$$

(called the second reflexive lower approximation of $\hat{X}$)

and

$$\mathcal{R}^+(\hat{X}) = [\mathcal{R}^-(\hat{X}^c)]^c$$

(called the second reflexive upper approximation of $\hat{X}$).

Definition 2.6. Let $(\mathcal{U}, \mathcal{R})$ be a reflexive approximation space and $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$. The third type of the reflexive lower and reflexive upper approximations of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathcal{R})$ or with respect to $\mathcal{R}$ is a pair $(\mathcal{R}^-(\hat{X}), \mathcal{R}^+(\hat{X}))$, defined by

$$\mathcal{R}^-(\hat{X}) = [\mathcal{R}^+(\hat{X}^c)]^c$$

(called the third reflexive lower approximation of $\hat{X}$)
and

$$\mathfrak{R}^+(\hat{X}) = \bigcup \{\mathfrak{R}^\cap(u) : \mathfrak{R}^\cap(u) \cap \hat{X} \neq \emptyset\}$$

(called the third reflexive upper approximation of $\hat{X}$).

2.3. Types of rough set approximations based on Dai et al.'s approach [25]

Dai et al. [25] introduced the following definitions:

Definition 2.7. Let $\mathfrak{R}$ be a similarity relation on a universe set $\mathcal{U}$ and $(\mathcal{U}, \mathfrak{R})$ be an approximation space. The set $\mathfrak{R}^\sqcup(u)$ ($\forall u \in \mathcal{U}$) (The union of all right **neighborhoods** containing u) as follows:

$$\mathfrak{R}^\sqcup(u) = \begin{cases} \bigcup_{v \in \mathfrak{R}(u)} \mathfrak{R}(v), & \text{if } \exists v, u \in \mathfrak{R}(v) \\ \emptyset, & \text{otherwise} \end{cases}.$$

Definition 2.8. Let $(\mathcal{U}, \mathfrak{R})$ be a similarity approximation space and $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$. The first type of the similarity lower and similarity upper approximations of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R})$ or with respect to $\mathfrak{R}$ is a pair $(\mathfrak{R}^-(\hat{X}), \mathfrak{R}^+(\hat{X}))$, defined by

$$\mathfrak{R}^-(\hat{X}) = \{u \in \mathcal{U} : \mathfrak{R}^\sqcup(u) \subseteq \hat{X}\}$$

(called the first similarity lower approximation of $\hat{X}$) and

$$\mathfrak{R}^+(\hat{X}) = \{u \in \mathcal{U} : \mathfrak{R}^\sqcup(u) \cap \hat{X} \neq \emptyset\}$$

(called the first similarity upper approximation of $\hat{X}$).

Definition 2.9. Let $(\mathcal{U}, \mathfrak{R})$ be a similarity approximation space and $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$. The second type of the similarity lower and similarity upper approximations of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R})$ or with respect to $\mathfrak{R}$ is a pair $(\mathfrak{R}^-(\hat{X}), \mathfrak{R}^+(\hat{X}))$, defined by

$$\mathfrak{R}^-(\hat{X}) = \bigcup \{\mathfrak{R}^\sqcup(u) : \mathfrak{R}^\sqcup(u) \subseteq \hat{X}\}$$

(called the second similarity lower approximation of $\hat{X}$)
and

$$\mathfrak{R}^+(\hat{X}) = [\mathfrak{R}^-(\hat{X}^c)]^c$$

(called the second similarity upper approximation of $\hat{X}$).

Definition 2.10. Let $(\mathcal{U}, \mathfrak{R})$ be a similarity approximation space and $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$. The third type of the similarity lower and similarity upper approximations of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R})$ or with respect to $\mathfrak{R}$ is a pair $(\mathfrak{R}^-(\hat{X}), \mathfrak{R}^+(\hat{X}))$, defined by

$$\mathfrak{R}^-(\hat{X}) = [\mathfrak{R}^+(\hat{X}^c)]^c$$

(called the third similarity lower approximation of $\hat{X}$)
and

$$\mathfrak{R}^+(\hat{X}) = \bigcup \{\mathfrak{R}^\sqcup(u) : \mathfrak{R}^\sqcup(u) \cap \hat{X} \neq \emptyset\}$$

(called the third similarity upper approximation of $\hat{X}$).

2.4. j -Neighborhood of element u and j -neighborhood space

Abd El-Monsef et al. [26] proposed the concepts of j -neighborhood of an element u and j -neighborhood space as indicated below.

Definition 2.11. Let $\mathfrak{R}$ be an arbitrary binary relation on $\mathcal{U}$. The j -neighborhood of an element $u \in \mathcal{U}$ are defined as follows:
 $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$

- (1) $\mathfrak{N}_{j_1}(u) = \{v \in \mathcal{U} | u\mathfrak{R}v\}.$
- (2) $\mathfrak{N}_{j_2}(u) = \{v \in \mathcal{U} | v\mathfrak{R}u\}.$
- (3) $\mathfrak{N}_{j_3}(u) = \left\{ v \in \mathcal{U} \mid \bigcap_{u \in \mathfrak{N}_{j_1}(v)} \mathfrak{N}_{j_1}(v) \right\}.$
- (4) $\mathfrak{N}_{j_4}(u) = \left\{ v \in \mathcal{U} \mid \bigcap_{u \in \mathfrak{N}_{j_2}(v)} \mathfrak{N}_{j_2}(v) \right\}.$
- (5) $\mathfrak{N}_{j_5}(u) = \mathfrak{N}_{j_1}(u) \sqcup \mathfrak{N}_{j_2}(u).$
- (6) $\mathfrak{N}_{j_6}(u) = \mathfrak{N}_{j_1}(u) \cap \mathfrak{N}_{j_2}(u).$
- (7) $\mathfrak{N}_{j_7}(u) = \mathfrak{N}_{j_3}(u) \sqcup \mathfrak{N}_{j_4}(u).$
- (8) $\mathfrak{N}_{j_8}(u) = \mathfrak{N}_{j_3}(u) \cap \mathfrak{N}_{j_4}(u).$

Definition 2.12. Let $\mathfrak{R}$ be an arbitrary binary relation on $\mathcal{U}$, $2^{\mathcal{U}}$ the set of all subset of $\mathcal{U}$, and $\vartheta_j : \mathcal{U} \rightarrow 2^{\mathcal{U}}$ be a mapping which assigns for each u in $\mathcal{U}$ its j -neighborhood in $2^{\mathcal{U}}$. The triple $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ is called a j -neighborhood space (briefly, j -NS).

Remark 2.13. (1) If $\mathfrak{R}$ be any reflexive relation on a universe set $\mathcal{U}$, then the triple $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ is called a reflexive j -neighborhood space (briefly, j -RNS).

(2) If $\mathfrak{R}$ be any similarity relation on a universe set $\mathcal{U}$, then the triple $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ is called a similarity j -neighborhood space (briefly, j -SNS).

3 Generalized rough set models by Abo-Tabl's approach [24] and Dai et al.'s approach [25] based on j -neighborhood space

3.1. First kind: Generalized rough set models by Abo-Tabl's approach [24] based on j -reflexive neighborhood space

In this subsection, we will extend the Definitions 2.3, 2.4, 2.5 and 2.6 to j -reflexive neighborhood space as indicated below.

Definition 3.1. Let $\mathfrak{R}$ be any reflexive relation on a universe set $\mathcal{U}$ and $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -RNS. We define a set $\mathfrak{N}_j^\cap(u)$ ($\forall u \in \mathcal{U}$) (The intersection of all j -neighborhoods containing u) as follows:

$$\mathfrak{N}_j^\cap(u) = \begin{cases} \bigcap_{u \in \mathfrak{N}_j(v)} \mathfrak{N}_j(v), & \text{if } \exists v, u \in \mathfrak{N}_j(v) \\ \emptyset, & \text{otherwise} \end{cases},$$

Example 3.2. Let $\mathcal{U} = \{1, 2, 3, 4\}$ and $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$ be a reflexive relation on $\mathcal{U}$ defined by

$$\mathfrak{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 3), (3, 4), (4, 2)\}.$$

Firstly, we calculate the j -neighborhood according to Definition 2.11 as follows ($\forall u \in \mathcal{U}, \forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$):

(1) Take $j \in \{j_1, j_4\}$, we get

$$\begin{aligned} \mathfrak{N}_{j_1}(1) &= \{1, 2, 3\}, & \mathfrak{N}_{j_1}(2) &= \{2, 3\}, \\ \mathfrak{N}_{j_1}(3) &= \{3, 4\}, & \mathfrak{N}_{j_1}(4) &= \{2, 4\}. \end{aligned}$$

(2) Take $j \in \{j_2, j_3\}$, we get

$$\begin{aligned} \mathfrak{N}_{j_2}(1) &= \{1\}, & \mathfrak{N}_{j_2}(2) &= \{1, 2, 4\}, \\ \mathfrak{N}_{j_2}(3) &= \{1, 2, 3\}, & \mathfrak{N}_{j_2}(4) &= \{3, 4\}. \end{aligned}$$

(3) Take $j \in \{j_5, j_7\}$, we get

$$\mathfrak{N}_j(1) = \{1, 2, 3\}, \quad \mathfrak{N}_j(2) = \mathcal{U},$$

$$\mathfrak{N}_j(3) = \mathcal{U}, \quad \mathfrak{N}_j(4) = \{2, 3, 4\}.$$

(4) Take $j \in \{j_6, j_8\}$, we get

$$\mathfrak{N}_j(1) = \{1\}, \quad \mathfrak{N}_j(2) = \{2\},$$

$$\mathfrak{N}_j(3) = \{3\}, \quad \mathfrak{N}_j(4) = \{4\}.$$

Secondly, we calculate the $\mathfrak{N}_j^\square(u)$ according to Definition 3.1 as follows ($\forall u \in U, \forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$):

(1) Take $j \in \{j_1, j_4\}$, we get

$$\mathfrak{N}_j^\square(1) = \{1, 2, 3\}, \quad \mathfrak{N}_j^\square(2) = \{2\},$$

$$\mathfrak{N}_j^\square(3) = \{3\}, \quad \mathfrak{N}_j^\square(4) = \{4\}.$$

(2) Take $j \in \{j_2, j_3\}$, we get

$$\mathfrak{N}_j^\square(1) = \{1\}, \quad \mathfrak{N}_j^\square(2) = \{1, 2\},$$

$$\mathfrak{N}_j^\square(3) = \{3\}, \quad \mathfrak{N}_j^\square(4) = \{4\}.$$

(3) Take $j \in \{j_5, j_7\}$, we get

$$\mathfrak{N}_j^\square(1) = \{1, 2, 3\}, \quad \mathfrak{N}_j^\square(2) = \{2, 3\},$$

$$\mathfrak{N}_j^\square(3) = \{2, 3\}, \quad \mathfrak{N}_j^\square(4) = \{2, 3, 4\}.$$

(4) Take $j \in \{j_6, j_8\}$, we get

$$\mathfrak{N}_j^\square(1) = \{1\}, \quad \mathfrak{N}_j^\square(2) = \{2\},$$

$$\mathfrak{N}_j^\square(3) = \{3\}, \quad \mathfrak{N}_j^\square(4) = \{4\}.$$

Definition 3.3. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -RNS. A reflexive 1 j -neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}) = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \subseteq \hat{X}\}$$

(called the reflexive 1-lower approximation of $\hat{X}$)
and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}) = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \cap \hat{X} \neq \emptyset\}$$

(called the reflexive 1-upper approximation of $\hat{X}$).

Example 3.4 (Continued from Example 3.2). Let $\hat{X} = \{1, 3\}$. Then

(1) Take $j \in \{j_1, j_4\}$, we get

$$\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}) = \{3\}, \quad \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}) = \{1, 3\}.$$

(2) Take $j \in \{j_2, j_3\}$, we get

$$\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}) = \{1, 3\}, \quad \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}) = \{1, 2, 3\}.$$

(3) Take $j \in \{j_5, j_7\}$, we get

$$\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}) = \emptyset, \quad \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}) = \mathcal{U}.$$

(4) Take $j \in \{j_6, j_8\}$, we get

$$\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}) = \{1, 3\}, \quad \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}) = \{1, 3\}.$$

Remark 3.5. When $j = j_1$ in Definition 3.3. Then we obtain on Definition 2.4.

Proposition 3.6. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L7), (L9), (U1)-(U7), (U9), (K), and (LU) will be satisfy based on Definition 3.3.

Proof. The following properties (L2), (L4), (L6), (L7), (U1)-(U7), (U9) and (LU) follows from Definition 3.3.

(L1) As $[\mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}^c)]^c = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \cap \hat{X}^c \neq \emptyset\}^c = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \cap \hat{X} = \emptyset\} = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \subseteq \hat{X}\} = \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})$.

(L3) As $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1 \cap \hat{X}_2) = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_1 \cap \hat{X}_2\} = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_1 \wedge \mathfrak{N}_j^\square(u) \subseteq \hat{X}_2\} = \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_1\} \cap \{u \in \mathcal{U} : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_2\} = \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1) \cap \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$.

(L5) Let $\hat{X}_1 \subseteq \hat{X}_2$ and $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1)$. Then $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_1$ and so $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_2$ implies $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$.

(L9) By (L7), we obtain $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})$. Now, we show $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}))$. As $\mathfrak{N}_j^\square(u) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})$, let $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})$. Since $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})$ implies $\mathfrak{N}_j^\square(u) \subseteq \hat{X}$. Again, let $v \in \mathfrak{N}_j^\square(u)$. Then $\mathfrak{N}_j^\square(v) \subseteq \mathfrak{N}_j^\square(u)$ implies $\mathfrak{N}_j^\square(v) \subseteq \hat{X}$. Thus $v \in \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})$ and so $\mathfrak{N}_j^\square(u) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X})$ implies $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}))$. Therefore, (L9) is holding.

(K) Let $u \notin (\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$. Then $u \notin (\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1))^c \wedge u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$ and $u \in (\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1)) \wedge u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$, that implies $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_1$ and $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_2$. Then $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_1^c$ and $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_2$. Hence, $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_1^c \sqcup \hat{X}_2$ and $u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1^c \sqcup \hat{X}_2)$. Therefore, $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1^c \sqcup \hat{X}_2) \subseteq (\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$.

The equality of (L4), (U3), and (K) in Proposition 3.6 does not hold as shown in the following example.

Example 3.7. (Continued from Example 3.2). Take $j = j_4$ (and also $j \in \{j_1, j_2, j_3, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1) = \{2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2) = \{3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}_1) = \{1, 2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}_2) = \{1, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}_1 \cap \hat{X}_2) = \emptyset$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}_1 \cap \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}_1) \cap \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1 \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2) = \{1, 3, 4\}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1 \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}_2)$.

Definition 3.8. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -RNS. A reflexive 2 j -neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}) = \bigsqcup \left\{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \subseteq \hat{X} \right\}$$

(called the reflexive 2-lower approximation of $\hat{X}$)
and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}) = [\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}^c)]^c$$

(called the reflexive 2-upper approximation of $\hat{X}$).

Example 3.9 (Continued from Example 3.2). Take $j = j_2$ (and also $j \in \{j_1, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $X = \{1, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}) = \{1, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}) = \{1, 2, 3\}$.

Remark 3.10. When $j = j_1$ in Definition 3.8, then we obtain on Definition 2.5.

Proposition 3.11. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L7), (L9), (U1)-(U7), (U9), (K), and (LU) will be satisfy based on Definition 3.8.

Proof. The following properties (L2), (L4), (L6), (L7), (U1)-(U7), (U9) and (LU) follows from Definition 3.8.

$$(L1) \text{ As } [\mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}^c)]^c = \left[[\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}^c)]^c \right]^c = \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}).$$

$$(L3) \text{ As } \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1 \sqcap \hat{X}_2) = \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_1 \sqcap \hat{X}_2 \} = \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_1 \wedge \mathfrak{N}_j^\square(u) \subseteq \hat{X}_2 \} = \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_1 \} \sqcap \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \subseteq \hat{X}_2 \} = \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1) \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2).$$

(L5) Let $\hat{X}_1 \subseteq \hat{X}_2$ and $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1)$. Then $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_1$ and so $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_2$ implies $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$.

(L9) By (L7), we obtain $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X})) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X})$. Now, we show $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}))$. As $\mathfrak{N}_j^\square(u) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X})$, let $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X})$. Since $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X})$ implies $\mathfrak{N}_j^\square(u) \subseteq \hat{X}$. Again, let $v \in \mathfrak{N}_j^\square(u)$. Then $\mathfrak{N}_j^\square(v) \subseteq \mathfrak{N}_j^\square(u)$ implies $\mathfrak{N}_j^\square(v) \subseteq \hat{X}$. Thus $v \in \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X})$ and so $\mathfrak{N}_j^\square(u) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X})$ implies $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}))$. Therefore, (L9) is holding.

(K) Let $u \notin (\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$. Then $u \notin (\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1))^c \wedge u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$ and $u \in (\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1)) \wedge u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$, that implies $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_1$ and $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_2$. Then $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_1^c$ and $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_2$. Hence, $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_1^c \sqcup \hat{X}_2$ and $u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1^c \sqcup \hat{X}_2)$. Therefore, $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1^c \sqcup \hat{X}_2) \subseteq (\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$.

The equality of (L4), (U3), and (K) in Proposition 3.11 does not hold as shown in the following example.

Example 3.12. (Continued from Example 3.2). Take $j = j_4$ (and also $j \in \{j_1, j_2, j_3, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1) = \{2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2) = \{3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$. Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3, 4\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}_1) = \{1, 2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}_2) = \{1, 3, 4\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}_1 \sqcap \hat{X}_2) = \emptyset$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}_1 \sqcap \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}_1) \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1^c \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2) = \{1, 3, 4\}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1^c \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}_2)$.

Definition 3.13. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -RNS. A reflexive 3 j -neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}) = [\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}^c)]^c$$

(called the reflexive 3-lower approximation of $\hat{X}$)

and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}) = \sqcup \left\{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \sqcap \hat{X} \neq \emptyset \right\}$$

(called the reflexive 3-upper approximation of $\hat{X}$).

Example 3.14 (Continued from Example 3.2). Take $j = j_2$ (and also $j \in \{j_1, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $X = \{1, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}) = \{3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}) = \{1, 2, 3\}$.

Remark 3.15. When $j = j_1$ in Definition 3.13, then we obtain on Definition 2.6.

Proposition 3.16. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L8), (U1)-(U8), (K), and (LU) will be satisfy based on Definition 3.13.

Proof. The following properties (L1)-(L8), (U2), (U4), (U6), (U7) and (LU) follows from Definition 3.13.

$$(U1) \text{ As } [\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}^c)]^c = \left[[\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}^c)]^c \right]^c = \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}).$$

$$(U3) \text{ As } \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1 \sqcup \hat{X}_2) = \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \sqcap (\hat{X}_1 \sqcup \hat{X}_2) \neq \emptyset \} = \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \sqcap \hat{X}_1 \neq \emptyset \vee \mathfrak{N}_j^\square(u) \sqcap \hat{X}_2 \neq \emptyset \} = \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \sqcap \hat{X}_1 \neq \emptyset \} \sqcup \sqcup \{ \mathfrak{N}_j^\square(u) : \mathfrak{N}_j^\square(u) \sqcap \hat{X}_2 \neq \emptyset \} = \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_2).$$

(U5) Let $\hat{X}_1 \subseteq \hat{X}_2$ and $u \in \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1)$. Then $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_1$ and so $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_2$ implies $u \in \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_2)$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_2)$.

(U8) Let $u \in \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}))$. Then there exist $v \in \mathcal{U}$ such that $v \in \mathfrak{N}_j^\square(u)$ and $\mathfrak{N}_j^\square(v) \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}) \neq \emptyset$. Hence, there exist $w \in \mathcal{U}$ such that $w \in \mathfrak{N}_j^\square(v)$ and $w \in \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X})$, thus $w \notin \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}^c)$ and $\mathfrak{N}_j^\square(u) \sqcap \hat{X}^c \neq \emptyset$, then $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}$, hence $w \in \hat{X}$. Therefore $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X})) \subseteq (\hat{X})$.

(K) Let $u \notin (\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2)$. Then $u \notin (\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1))^c \wedge u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2)$ and $u \in (\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1)) \wedge u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2)$, that implies $\mathfrak{N}_j^\square(u) \subseteq \hat{X}_1$ and $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_2$. Then $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_1^c$ and $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_2$. Hence, $\mathfrak{N}_j^\square(u) \not\subseteq \hat{X}_1^c \sqcup \hat{X}_2$ and $u \notin \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1^c \sqcup \hat{X}_2)$. Therefore, $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1^c \sqcup \hat{X}_2) \subseteq (\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2)$.

The equality of (L4), (L8), (U3), (U8) and (K) in Proposition 3.16 does not hold as shown in the following example.

Example 3.17. (Continued from Example 3.2). Take $j = j_4$ (and also $j \in \{j_1, j_2, j_3, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{1, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1) = \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2) = \emptyset$ and $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2)$. Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3, 4\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1) = \{1, 2, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_2) = \mathcal{U}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1 \sqcap \hat{X}_2) = \emptyset$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1 \sqcap \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1) \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_2)$. $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1)) = \{1, 2, 3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2)) = \emptyset$, then $\hat{X}_1 \neq \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X}_1))$ and $\hat{X}_2 \neq \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2))$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1^c \sqcup \hat{X}_2) = \{4\}$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1^c \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_1))^c \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}_2)$.

3.2 Second kind: Generalized rough set models by Dai et al.'s approach [25] based on j -similarity neighborhood space

In this subsection, we will extend the Definitions 2.7, 2.8, 2.9 and 2.10 to j -similarity neighborhood space as indicated below.

Definition 3.18. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -neighborhood space, we define a set $\mathfrak{N}_j^\sqcup(u)$ ($\forall u \in \mathcal{U}$) (The union of all j -neighborhoods containing u) as follows:

$$\mathfrak{N}_j^\sqcup(u) = \begin{cases} \sqcup_{v \in \mathfrak{N}_j(u)} \mathfrak{N}_j(v), & \text{if } \exists v, u \in \mathfrak{N}_j(v) \\ \emptyset, & \text{otherwise} \end{cases}.$$

Example 3.19. Let $\mathcal{U} = \{1, 2, 3, 4\}$ and $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$ be a similarity relation on $\mathcal{U}$ defined by

$\mathfrak{R} = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (3, 3), (3, 2), (4, 1), (4, 4)\}$. Then the $\mathfrak{N}_j(u) (\forall u \in \mathcal{U}, \forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\})$ is

$$\begin{aligned}\mathfrak{N}_j(1) &= \{1, 2, 4\}, \quad \mathfrak{N}_j(2) = \{1, 2, 3\}, \\ \mathfrak{N}_j(3) &= \{2, 3\}, \quad \mathfrak{N}_j(4) = \{1, 4\}\end{aligned}$$

and the $\mathfrak{N}_j^\sqcup(u) (\forall u \in \mathcal{U}, \forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\})$ is

$$\begin{aligned}\mathfrak{N}_j^\sqcup(1) &= \mathcal{U}, \quad \mathfrak{N}_j^\sqcup(2) = \mathcal{U} \\ \mathfrak{N}_j^\sqcup(3) &= \{1, 2, 3\}, \quad \mathfrak{N}_j^\sqcup(4) = \{1, 2, 4\}.\end{aligned}$$

Definition 3.20. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -SNS. A similarity 4 j -neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}) = \{u \in \mathcal{U} : \mathfrak{N}_j^\sqcup(u) \subseteq \hat{X}\}$$

(called the similarity 4-lower approximation of $\hat{X}$)
and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}) = \{u \in \mathcal{U} : \mathfrak{N}_j^\sqcup(u) \cap \hat{X} \neq \emptyset\}$$

(called the similarity 4-upper approximation of $\hat{X}$).

Example 3.21 (Continued from Example 3.19). Let $\hat{X} = \{1, 3\}$ and take $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$. Then

$$\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}) = \emptyset \text{ and } \mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}) = \mathcal{U}.$$

Remark 3.22. If we consider $j = j_1$ in Definition 3.20, then we obtain on Definition 2.8.

Proposition 3.23. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L8), (U1)-(U8), (K), and (LU) will be satisfy based on Definition 3.20.

Proof. Similar to Proposition 3.8.

The equality of (L4), (U3), and (K) in Proposition 3.23 does not hold as shown in the following example.

Example 3.24. (Continued from Example 3.19). Take $j = j_3$ (and also $j \in \{j_1, j_2, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_2) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}_1) = \mathcal{U}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}_2) = \{1, 2, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1 \sqcup \hat{X}_2) = \{3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}_1 \sqcup \hat{X}_2) = \emptyset$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_2)$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}_1) \cap \mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1 \sqcup \hat{X}_2) = \emptyset$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1 \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}_2)$.

Definition 3.25. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -SNS. A similarity 5 j -neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+5}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}) = \bigsqcup \left\{ \mathfrak{N}_j^\sqcup(u) : \mathfrak{N}_j^\sqcup(u) \subseteq \hat{X} \right\}$$

(called the similarity 5-lower approximation of $\hat{X}$)
and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+5}(\hat{X}) = [\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}^c)]^c$$

(called the similarity 5-upper approximation of $\hat{X}$).

Example 3.26 (Continued from Example 3.19). Take $j = j_3$ (and also $j \in \{j_1, j_2, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $X = \{1, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{+5}(\hat{X}) = \mathcal{U}$.

Remark 3.27. If we consider $j = j_1$ in Definition 3.25, then we obtain on Definition 2.9.

Proposition 3.28. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L2), (L4)-(L7), (L9) (U1)-(U2), (U4)-(U7), (U9) (K), and (LU) will be satisfy based on Definition 3.25.

Proof. Similar to Proposition 3.11.

The equality of (L4) and (K) in Proposition 3.28 does not hold as shown in the following example.

Example 3.29 (Continued from Example 3.19). Take $j = j_3$ (and also $j \in \{j_1, j_2, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_2) = \emptyset$ and $\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1^c \sqcup \hat{X}_2) = \emptyset$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1^c \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}_2)$.

Definition 3.30. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -SNS. A similarity 6 j -neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}) = [\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}^c)]^c$$

(called the similarity 6-lower approximation of $\hat{X}$) and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}) = \bigsqcup \left\{ \mathfrak{N}_j^\sqcup(u) : \mathfrak{N}_j^\sqcup(u) \cap \hat{X} \neq \emptyset \right\}$$

(called the similarity 6-upper approximation of $\hat{X}$).

Example 3.31 (Continued from Example 3.19). Take $j = j_2$ (and also $j \in \{j_1, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $X = \{1, 3\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}) = \mathcal{U}$.

Remark 3.32. When $j = j_1$ in Definition 3.30, then we obtain on Definition 2.10.

Proposition 3.33. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L8), (U1)-(U8), (K), and (LU) will be satisfy based on Definition 3.30.

Proof. Similar to Proposition 3.16.

The equality of (L4), (L8), (U3), (U8) and (K) in Proposition 3.33 does not hold as shown in the following example.

Example 3.34 (Continued from Example 3.19). Take $j = j_3$ (and also $j \in \{j_1, j_2, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3, 4\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1) = \mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_2) = \emptyset$ and $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1 \sqcup \hat{X}_2) = \mathcal{U}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_2)$. Let $\hat{X}_1 = \{1, 2\}$ and $\hat{X}_2 = \{3, 4\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_1) = \mathcal{U}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_2) = \mathcal{U}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_1 \sqcup \hat{X}_2) = \emptyset$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_1) \cap \mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_2)$. $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_1)) = \mathcal{U}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_2)) = \emptyset$, then $\hat{X}_1 \neq \mathfrak{R}_{\mathfrak{N}_j}^{-6}(\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X}_1))$ and $\hat{X}_2 \neq \mathfrak{R}_{\mathfrak{N}_j}^{-6}(\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_2))$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1^c \sqcup \hat{X}_2) = \emptyset$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1^c \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}_2)$.

4 New generalized rough set models based on j -adhesion neighborhood space

In this section, we will extent Section 3 to j -adhesion neighborhood space as indicated below.

4.1 Third kind: Generalized rough set models by based on j -reflexive adhesion neighborhood space

Definition 4.1. Let $\mathcal{R}$ be reflexive on $\mathcal{U}$. The j -reflexive adhesion neighborhood of element $u \in \mathcal{U}$ are defined as follows:
 $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$

- (1) $h_{j_1}(u) = \{v \in \mathcal{U} : \mathcal{N}_{j_1}^\square(u) = \mathcal{N}_{j_1}^\square(v)\}$.
- (2) $h_{j_2}(u) = \{v \in \mathcal{U} : \mathcal{N}_{j_2}^\square(u) = \mathcal{N}_{j_2}^\square(v)\}$.
- (3) $h_{j_3}(u) = \{v \in \mathcal{U} : \bigcap_{u \in \mathcal{N}_{j_1}^\square(v)} \mathcal{N}_{j_1}^\square(v) = \bigcap_{v \in \mathcal{N}_{j_1}^\square(x)} \mathcal{N}_{j_1}^\square(u)\}$.
- (4) $h_{j_4}(u) = \{v \in \mathcal{U} : \bigcap_{u \in \mathcal{N}_{j_2}^\square(v)} \mathcal{N}_{j_2}^\square(v) = \bigcap_{v \in \mathcal{N}_{j_2}^\square(u)} \mathcal{N}_{j_2}^\square(u)\}$.
- (5) $h_{j_5}(u) = h_{j_1}(u) \sqcup h_{j_2}(u)$.
- (6) $h_{j_6}(u) = h_{j_1}(u) \sqcap h_{j_2}(u)$.
- (7) $h_{j_7}(u) = h_{j_3}(u) \sqcup h_{j_4}(u)$.
- (8) $h_{j_8}(u) = h_{j_3}(u) \sqcap h_{j_4}(u)$.

According to above Definition 4.1, we call $(\mathcal{U}, \mathcal{R}, \vartheta_j)$ be a j -reflexive adhesion neighborhood space (briefly, a j -RANS).

Example 4.2 (Continued from Example 3.2). Take $j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, Then the j -reflexive adhesion neighborhood is

$$\begin{aligned} h_j(1) &= \{1\}, \quad h_j(2) = \{2\}, \\ h_j(3) &= \{3\}, \quad h_j(4) = \{4\}. \end{aligned}$$

Definition 4.3. Let $(\mathcal{U}, \mathcal{R}, \vartheta_j)$ be a j -RANS. A reflexive 7 j -adhesion neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathcal{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}), \mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}))$, defined by

$$\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}) = \{u \in \mathcal{U} : h_j^\square(u) \subseteq \hat{X}\}$$

(called the reflexive 7-lower approximation of $\hat{X}$)
and

$$\mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}) = \{u \in \mathcal{U} : h_j^\square(u) \cap \hat{X} \neq \emptyset\}$$

(called the reflexive 7-upper approximation of $\hat{X}$).

Example 4.4 (Continued from Example 4.2). Let $\hat{X} = \{1, 3\}$ and take $j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$. Then

$$\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}) = \{1, 3\} \text{ and } \mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}) = \{1, 3\}.$$

Proposition 4.5. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L10), (U1)-(U10), (K), and (LU) will be satisfy based on Definition 4.3.

Proof. We only prove (L9), (L10), (U9), and (U10).

(L9) Let $u \in \hat{X}$, for $v \in h_j^\square(u)$, then we have $u \in h_j^\square(v)$. Hence, $\hat{X} \cap h_j^\square(u) \neq \emptyset$. It follows that $v \in \mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X})$. Thus, $h_j^\square(u) \subseteq \mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X})$ and $u \in \mathcal{R}_{\mathcal{N}_j}^{-7}(\mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}))$.

(L10) From (L7) and (L9).

(U9) Let $u \in \mathcal{R}_{\mathcal{N}_j}^{-7}(\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}))$. Then there exist $v \in \mathcal{U}$ such that $v \in h_j^\square(u)$ and $h_j^\square(u) \cap \mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}) \neq \emptyset$. Hence, there exist $w \in \mathcal{U}$ such that $w \in h_j^\square(u)$ and $w \in \mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X})$, thus $w \notin \mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}^c)$ and $h_j^\square(u) \cap \hat{X}^c \neq \emptyset$, then $h_j^\square(u) \subseteq \hat{X}$, hence $w \in \hat{X}$. Therefore $\mathcal{R}_{\mathcal{N}_j}^{+7}(\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X})) \subseteq \hat{X}$.

(U10) From (U7) and (U9).

The equality of (L4), (U3), and (K) in Proposition 4.5 does not hold as shown in the following example.

Example 4.6. Let $\mathcal{U} = \{1, 2, 3, 4\}$ and $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$ be a reflexive relation on $\mathcal{U}$ defined by

$\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 2)\}$. Take $j = j_2$ (and also $j \in \{j_1, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Then

$$h_{j_2}(1) = h_{j_2}(2) = \{1, 2\},$$

$$h_{j_2}(3) = \{3\}, \quad h_{j_2}(4) = \{4\}.$$

Let $\hat{X}_1 = \{1, 3\}$ and $\hat{X}_2 = \{2\}$. Then $\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1) = \{3\}$, $\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_2) = \emptyset$, $\mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}_1) = \{1, 2, 3\}$, $\mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}_2) = \{1, 2\}$, $\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$ and $\mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}_1 \cap \hat{X}_2) = \emptyset$. Then $\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1) \sqcup \mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_2)$ and $\mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}_1 \cap \hat{X}_2) \neq \mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}_1) \cap \mathcal{R}_{\mathcal{N}_j}^{+7}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1 \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1))^c \sqcup \mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_2) = \mathcal{U}$. Thus $\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1 \sqcup \hat{X}_2) \neq (\mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_1))^c \sqcup \mathcal{R}_{\mathcal{N}_j}^{-7}(\hat{X}_2)$.

Definition 4.7. Let $(\mathcal{U}, \mathcal{R}, \vartheta_j)$ be a j -RANS. A reflexive 8 j -adhesion neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathcal{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}), \mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}))$, defined by

$$\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}) = \bigsqcup \{h_j^\square(u) : h_j^\square(u) \subseteq \hat{X}\}$$

(called the reflexive 8-lower approximation of $\hat{X}$)
and

$$\mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}) = [\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}^c)]^c$$

(called the reflexive 8-upper approximation of $\hat{X}$).

Example 4.8 (Continued from Example 4.2). Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). If $\hat{X} = \{1, 3\}$, then

$$\mathcal{R}_{\mathcal{N}_{j_1}}^{-8}(\hat{X}) = \mathcal{R}_{\mathcal{N}_{j_1}}^{+8}(\hat{X}) = \{1, 3\}.$$

Proposition 4.9. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L10), (U1)-(U10), (K), and (LU) will be satisfy based on Definition 4.7.

Proof. Similar to Proposition 4.5.

The equality of (L4), (U3), and (K) in Proposition 4.7 does not hold as shown in the following example.

Example 4.10 (Continued from Example 4.6). Take $j = j_2$ (and also $j \in \{j_1, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 3\}$ and $\hat{X}_2 = \{2\}$. Then $\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1) = \{3\}$, $\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_2) = \emptyset$, $\mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}_1) = \{1, 2, 3\}$, $\mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}_2) = \{1, 2\}$, $\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$ and $\mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}_1 \cap \hat{X}_2) = \emptyset$. Then $\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1) \sqcup \mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_2)$ and $\mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}_1 \cap \hat{X}_2) \neq \mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}_1) \cap \mathcal{R}_{\mathcal{N}_j}^{+8}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1 \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1))^c \sqcup \mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_2) = \mathcal{U}$. Thus $\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1 \sqcup \hat{X}_2) \neq (\mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_1))^c \sqcup \mathcal{R}_{\mathcal{N}_j}^{-8}(\hat{X}_2)$.

Definition 4.11. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -RANS. A reflexive 9 j -adhesion neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}) = [\mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}^c)]^c$$

(called the reflexive 9-lower approximation of $\hat{X}$)

and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}) = \bigsqcup \left\{ h_j^\square(u) : h_j^\square(u) \cap \hat{X} \neq \emptyset \right\}$$

called the reflexive 9-upper approximation of $\hat{X}$).

Example 4.12 (Continued from Example 4.2). Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). If $\hat{X} = \{1, 3\}$, then we have

$$\mathfrak{R}_{\mathfrak{N}_{j_1}}^{-9}(\hat{X}) = \{1, 3\}, \quad \mathfrak{R}_{\mathfrak{N}_{j_1}}^{+9}(\hat{X}) = \{1, 3\}.$$

Proposition 4.13. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L10), (U1)-(U10), (K), and (LU) will be satisfy based on Definition 4.11.

Proof. Similar to Proposition 4.5.

The equality of (L4), (U3), and (K) in Proposition 4.13 does not hold as shown in the following example.

Example 4.14. (Continued from Example 4.6). Take $j = j_2$ (and also $j \in \{j_1, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 3\}$ and $\hat{X}_2 = \{2\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1) = \{3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_2) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}_1) = \{1, 2, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}_2) = \{1, 2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}_1 \cap \hat{X}_2) = \emptyset$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_2)$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}_1 \cap \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}_1) \cap \mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1 \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1 \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}_2)$.

In the following, we will give the comparison between the properties of rough sets based on Definitions 3.3, 3.8, and 3.13 by using j - reflexive neighborhood space and Definitions 4.3, 4.7, and 4.11 by using j -reflexive adhesion neighborhood space as shown in Table 1.

Table 1 Comparison between the properties of rough sets based on Definitions 3.3, 3.8, and 3.13 by using j -reflexive neighborhood space and Definitions 4.3, 4.7, and 4.11 by using j -reflexive adhesion neighborhood space.

Property	Definition 3.3	Definition 3.8	Definition 3.13	Definition 4.3	Definition 4.7	Definition 4.11
L1	×	×	×	×	×	×
L2	×	×	×	×	×	×
L3	×	×	×	×	×	×
L4	×	×	×	×	×	×
L5	×	×	×	×	×	×
L6	×	×	×	×	×	×
L7	×	×	×	×	×	×
L8			×	×	×	×
L9	×	×		×	×	×
L10				×	×	×
U1	×	×	×	×	×	×
U2	×	×	×	×	×	×
U3	×	×	×	×	×	×
U4	×	×	×	×	×	×
U5	×	×	×	×	×	×
U6	×	×	×	×	×	×
U7	×	×	×	×	×	×
U8			×	×	×	×
U9	×	×		×	×	×
U10				×	×	×
K	×	×	×	×	×	×
LU	×	×	×	×	×	×

A cross (×) indicates that the property is satisfied.

4.2. Fourth kind: Generalized rough set models by based on j -similarity adhesion neighborhood space

Definition 4.15. Let $\mathfrak{R}$ be similarity on $\mathcal{U}$. The j -similarity adhesion neighborhood of element $u \in \mathcal{U}$ are defined as follows:
 $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$

- (1) $h_{j_1}(u) = \{v \in \mathcal{U} : \mathfrak{N}_{j_1}^{\sqcup}(u) = \mathfrak{N}_{j_1}^{\sqcup}(v)\}$.
- (2) $h_{j_2}(u) = \{v \in \mathcal{U} : \mathfrak{N}_{j_2}^{\sqcup}(u) = \mathfrak{N}_{j_2}^{\sqcup}(v)\}$.
- (3) $h_{j_3}(u) = \{v \in \mathcal{U} : \bigcap_{u \in \mathfrak{N}_{j_1}^{\sqcup}(v)} \mathfrak{N}_{j_1}^{\sqcup}(v) = \bigcap_{v \in \mathfrak{N}_{j_1}^{\sqcup}(u)} \mathfrak{N}_{j_1}^{\sqcup}(u)\}$.
- (4) $h_{j_4}(u) = \{v \in \mathcal{U} : \bigcap_{u \in \mathfrak{N}_{j_2}^{\sqcup}(v)} \mathfrak{N}_{j_2}^{\sqcup}(v) = \bigcap_{v \in \mathfrak{N}_{j_2}^{\sqcup}(u)} \mathfrak{N}_{j_2}^{\sqcup}(u)\}$.
- (5) $h_{j_5}(u) = h_{j_1}(u) \sqcup h_{j_2}(u)$.
- (6) $h_{j_6}(u) = h_{j_1}(u) \sqcap h_{j_2}(u)$.
- (7) $h_{j_7}(u) = h_{j_3}(u) \sqcup h_{j_4}(u)$.
- (8) $h_{j_8}(u) = h_{j_3}(u) \sqcap h_{j_4}(u)$.

According to above Definition 4.15, we call $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -similarity adhesion neighborhood space (briefly, a j -SANS).

Example 4.16 (Continued from Example 4.2). Take $j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, Then the j -similarity adhesion neighborhood is

$$\begin{aligned} h_j(1) &= h_j(2) = \{1, 2\}, \\ h_j(3) &= \{3\}, \quad h_j(4) = \{4\}. \end{aligned}$$

Definition 4.17. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -SANS. A similarity 10 j -adhesion neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}) = \{u \in \mathcal{U} : h_j^{\sqcup}(u) \subseteq \hat{X}\}$$

(called the similarity 10-lower approximation of $\hat{X}$)
and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}) = \{u \in \mathcal{U} : h_j^{\sqcap}(u) \sqcap \hat{X} \neq \emptyset\}$$

(called the similarity 10-upper approximation of $\hat{X}$).

Example 4.18 (Continued from Example 4.16). Take $j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$. If $\hat{X} = \{1, 3\}$, then

$$\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}) = \{3\}, \quad \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}) = \{1, 2, 3\}.$$

Proposition 4.19. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L10), (U1)-(U10), (K), and (LU) will be satisfy based on Definition 4.17.

Proof. We only prove (L9), (L10), (U9), and (U10).

(L9) Let $u \in \hat{X}$, for $v \in h_j^{\sqcup}(u)$, then we have $u \in h_j^{\sqcup}(v)$. Hence, $\hat{X} \cap h_j^{\sqcup}(u) \neq \emptyset$. It follows that $v \in \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X})$. Thus, $h_j^{\sqcup}(u) \subseteq \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X})$ and $u \in \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}))$.

(L10) From (L7) and (L9).

(U9) Let $u \in \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}))$. Then there exist $v \in \mathcal{U}$ such that $v \in h_j^{\sqcup}(u)$ and $h_j^{\sqcup}(u) \cap \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}) \neq \emptyset$. Hence, there exist $w \in \mathcal{U}$ such that $w \in h_j^{\sqcup}(u)$ and $w \in \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X})$, thus $w \notin \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}^c)$ and $h_j^{\sqcup}(u) \cap \hat{X}^c \neq \emptyset$, then $h_j^{\sqcup}(u) \subseteq \hat{X}$, hence $w \in \hat{X}$. Therefore $\mathfrak{R}_{\mathfrak{N}_j}^{+10}(\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X})) \subseteq \hat{X}$.

(U10) From (U7) and (U9).

The equality of (L4), (U3), and (K) in Proposition 4.19 does not hold as shown in the following example.

Example 4.20. (Continued from Example 4.16). Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 3\}$ and $\hat{X}_2 = \{2\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1) = \{3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_2) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}_1) = \{1, 2, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}_2) = \{1, 2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}_1 \sqcap \hat{X}_2) = \emptyset$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_2)$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}_1 \sqcap \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}_1) \sqcap \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1^c \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1^c \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}_2)$.

Definition 4.21. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -SANS. A similarity 11 j -adhesion neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}) = \bigsqcup \{h_j^{\sqcup}(u) : h_j^{\sqcup}(u) \subseteq \hat{X}\}$$

(called the similarity 11-lower approximation of $\hat{X}$)

and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}) = [\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}^c)]^c$$

called the similarity 11-upper approximation of $\hat{X}$.

Example 4.22 (Continued from Example 4.16). Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). If $\hat{X} = \{1, 3\}$, then

$$\mathfrak{R}_{\mathfrak{N}_{j_1}}^{-11}(\hat{X}) = \{3\}, \quad \mathfrak{R}_{\mathfrak{N}_{j_1}}^{+11}(\hat{X}) = \{1, 2, 3\}.$$

Proposition 4.23. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L10), (U1)-(U10), (K), and (LU) will be satisfy based on Definition 4.21.

Proof. Similar to Proposition 4.19.

The equality of (L4), (U3), and (K) in Proposition 4.23 does not hold as shown in the following example.

Example 4.24. (Continued from Example 4.16). Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 3\}$ and $\hat{X}_2 = \{2\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1) = \{3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_2) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}_1) = \{1, 2, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}_2) = \{1, 2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}_1 \cap \hat{X}_2) = \emptyset$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_2)$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}_1 \cap \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}_1) \cap \mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1^c \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1^c \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}_2)$.

Definition 4.25. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be a j -SANS. A similarity 12 j -adhesion neighborhood rough set of $\hat{X} \subseteq \mathcal{U}$ in $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ or with respect to ϑ_j is a pair $(\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}), \mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}))$, defined by

$$\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}) = [\mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}^c)]^c$$

(called the similarity 12-lower approximation of $\hat{X}$)

and

$$\mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}) = \bigsqcup \left\{ h_j^{\sqcup}(u) : h_j^{\sqcup}(u) \cap \hat{X} \neq \emptyset \right\}$$

called the similarity 12-upper approximation of $\hat{X}$.

Example 4.26 (Continued from Example 4.16). Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). If $\hat{X} = \{1, 3\}$, then we have

$$\mathfrak{R}_{\mathfrak{N}_{j_1}}^{-12}(\hat{X}) = \{3\}, \quad \mathfrak{R}_{\mathfrak{N}_{j_1}}^{+12}(\hat{X}) = \{1, 2, 3\}.$$

Proposition 4.27. Let $\hat{X}, \hat{X}_1, \hat{X}_2 \subseteq \mathcal{U}$. Then the following properties (L1)-(L10), (U1)-(U10), (K), and (LU) will be satisfy based on Definition 4.25.

Proof. Similar to Proposition 4.19.

The equality of (L4), (U3), and (K) in Proposition 4.27 does not hold as shown in the following example.

Example 4.28. (Continued from Example 4.16). Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). Let $\hat{X}_1 = \{1, 3\}$ and $\hat{X}_2 = \{2\}$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1) = \{3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_2) = \emptyset$, $\mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}_1) = \{1, 2, 3\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}_2) = \{1, 2\}$, $\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1 \sqcup \hat{X}_2) = \{1, 2, 3\}$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}_1 \cap \hat{X}_2) = \emptyset$. Then $\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1 \sqcup \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1) \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_2)$ and $\mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}_1 \cap \hat{X}_2) \neq \mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}_1) \cap \mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}_2)$.

$\mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}_2)$. Let $\hat{X}_1 = \{2, 3\}$ and $\hat{X}_2 = \{3, 4\}$, then $\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1^c \sqcup \hat{X}_2) = \{3, 4\}$ and $(\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_2) = \mathcal{U}$. Thus $\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1^c \sqcup \hat{X}_2) \neq (\mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_1))^c \sqcup \mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}_2)$.

In the following, we will give the comparison between the properties of rough sets based on Definitions 3.20, 3.25, and 3.30 by using j -similarly neighborhood space and Definitions 4.17, 4.21, and 4.25 by using j -similarly adhesion neighborhood space as shown in Table 2.

Table 2 Comparison between the properties of rough sets based on Definitions 3.20, 3.25, and 3.30 by using j -similarly neighborhood space and Definitions 4.17, 4.21, and 4.25 by using j -similarly adhesion neighborhood space.

Property	Definition 3.20	Definition 3.25	Definition 3.30	Definition 4.17	Definition 4.21	Definition 4.25
L1	×	×	×	×	×	×
L2	×	×	×	×	×	×
L3	×	×	×	×	×	×
L4	×		×	×	×	×
L5	×	×	×	×	×	×
L6	×	×	×	×	×	×
L7	×	×	×	×	×	×
L8	×		×	×	×	×
L9		×		×	×	×
L10				×	×	×
U1	×	×	×	×	×	×
U2	×	×	×	×	×	×
U3	×	×	×	×	×	×
U4	×		×	×	×	×
U5	×	×	×	×	×	×
U6	×	×	×	×	×	×
U7	×	×	×	×	×	×
U8	×		×	×	×	×
U9		×		×	×	×
U10				×	×	×
K	×	×	×	×	×	×
LU	×	×	×	×	×	×

A cross (×) indicates that the property is satisfied.

4.3. The relationships among our models

In this subsection, we give the relationships among our models. From the proposed study, we have the following results.

Proposition 4.29. Let $(\mathcal{U}, \mathbb{R}, \vartheta_j)$ be a j -RANS, j -RNS and $\hat{X} \sqsubseteq \mathcal{U}$. Then we have the following results.

- (1) $\mathbb{R}_{\mathbb{N}_j}^{-1}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{-7}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{+7}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+1}(\hat{X})$.
- (2) $\mathbb{R}_{\mathbb{N}_j}^{-2}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{-8}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{+8}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+2}(\hat{X})$.
- (3) $\mathbb{R}_{\mathbb{N}_j}^{-3}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{-9}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{+9}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+3}(\hat{X})$.
- (4) $\mathbb{R}_{\mathbb{N}_j}^{-1}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+1}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{-7}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+7}(\hat{X})$.
- (5) $\mathbb{R}_{\mathbb{N}_j}^{-2}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+2}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{-8}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+8}(\hat{X})$.
- (6) $\mathbb{R}_{\mathbb{N}_j}^{-3}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+3}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{-9}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+9}(\hat{X})$.

Proof. Follows from Definitions 3.3, 3.8, 3.13, 4.3, 4.7 and 4.11.

Proposition 4.30. Let $(\mathcal{U}, \mathbb{R}, \vartheta_j)$ be a j -SANS, j -SNS and $\hat{X} \sqsubseteq \mathcal{U}$. Then we have the following results.

- (1) $\mathbb{R}_{\mathbb{N}_j}^{-4}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{-10}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{+10}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+4}(\hat{X})$.
- (2) $\mathbb{R}_{\mathbb{N}_j}^{-5}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{-11}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{+11}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+5}(\hat{X})$.
- (3) $\mathbb{R}_{\mathbb{N}_j}^{-6}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{-12}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{+12}(\hat{X}) \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+6}(\hat{X})$.
- (4) $\mathbb{R}_{\mathbb{N}_j}^{-4}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+4}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{-10}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+10}(\hat{X})$.
- (5) $\mathbb{R}_{\mathbb{N}_j}^{-5}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+5}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{-11}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+11}(\hat{X})$.
- (6) $\mathbb{R}_{\mathbb{N}_j}^{-6}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+6}(\hat{X}), \mathbb{R}_{\mathbb{N}_j}^{-12}(\hat{X}) \sqsubseteq \hat{X} \sqsubseteq \mathbb{R}_{\mathbb{N}_j}^{+12}(\hat{X})$.

Proof. Follows from Definitions 3.20, 3.25, 3.30, 4.17, 4.21 and 4.25.

The relationships can be illustrated as the following Tables 3 and 4.

Table 3 The relationships between our models which are given in Definitions 3.3, 3.8, 3.13, 4.3, 4.7 and 4.11.

$$\begin{array}{ccccccccc}
\mathfrak{R}_{\mathfrak{N}_j}^{-1}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{-7}(\hat{X}) & \longrightarrow & \hat{X} & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+7}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+1}(\hat{X}) \\
\uparrow & & \uparrow & & \parallel & & \downarrow & & \downarrow \\
\mathfrak{R}_{\mathfrak{N}_j}^{-2}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{-8}(\hat{X}) & \longrightarrow & \hat{X} & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+8}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+2}(\hat{X}) \\
\uparrow & & \uparrow & & \parallel & & \downarrow & & \downarrow \\
\mathfrak{R}_{\mathfrak{N}_j}^{-3}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{-9}(\hat{X}) & \longrightarrow & \hat{X} & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+9}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+3}(\hat{X})
\end{array}$$

Table 4 The relationships between our models which are given in Definitions 3.20, 3.25, 3.30, 4.17, 4.21 and 4.25.

$$\begin{array}{ccccccccc}
\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{-11}(\hat{X}) & \longrightarrow & \hat{X} & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+11}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+5}(\hat{X}) \\
\uparrow & & \uparrow & & \parallel & & \downarrow & & \downarrow \\
\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{-10}(\hat{X}) & \longrightarrow & \hat{X} & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+10}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X}) \\
\uparrow & & \uparrow & & \parallel & & \downarrow & & \downarrow \\
\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{-12}(\hat{X}) & \longrightarrow & \hat{X} & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+12}(\hat{X}) & \longrightarrow & \mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X})
\end{array}$$

5 Comparison between the previous methods [24, 25] and the present method

In this section, we apply the proposed method with a numerical example to describe its performance.

Definition 5.1. Let $(\mathcal{U}, \mathfrak{R}, \vartheta_j)$ be an approximation space and $\hat{X} \subseteq \mathcal{U}$. We define the i -boundary region and i -accuracy measure, denoted by $\mathcal{B}_j^i$ and $\mathcal{A}_j^i$, respectively, defined as follows ($\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ and $\forall i \in \{1, 2, 3, \dots, 12\}$):

$$\begin{aligned}
(1) \mathcal{B}_j^i(\hat{X}) &= \mathfrak{R}_{\mathfrak{N}_j}^i(\hat{X}) - \mathfrak{R}_{\mathfrak{N}_j}^{-i}(\hat{X}). \\
(2) \mathcal{A}_j^i(\hat{X}) &= \frac{|\mathfrak{R}_{\mathfrak{N}_j}^i(\hat{X})|}{|\mathfrak{R}_{\mathfrak{N}_j}^{+i}(\hat{X})|}.
\end{aligned}$$

Based on Definition 5.1, we give the following two examples to illustrate the efficiency for our models.

The main aim of comparison between the previous method and the present method is to increase the accuracy measure and reduce the boundary region of sets by increasing the lower approximations. Moreover, decreasing the upper approximations. Therefore,

(1) We apply Example 5.2 to compare between the Abo Tabl's method [24] (see Definitions 2.4, 2.5, and 2.6) (i.e., Definitions 3.3, 3.8, and 3.13 when $j = j_1$) and the current method (i.e., Definitions 4.3, 4.7, and 4.11 when $j = j_1$), respectively.

Example 5.2. Let $\mathcal{U} = \{1, 2, 3, 4\}$ and $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$ be a reflexive relation on $\mathcal{U}$ defined by

$$\mathfrak{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 3), (3, 4), (4, 2)\}.$$

From Example 5.2, we will compute the reflexive 1 lower approximation, the reflexive 1 upper approximation, the reflexive 1 j -neighborhood boundary region, and the reflexive 1 j -neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in Abo Tabl's method [24] (as in Definition 2.4) (i.e., Definition 3.3 when $j = j_1$) and the reflexive 7 lower approximation, the reflexive 7 upper approximation, the reflexive 7 j -adhesion neighborhood boundary region, and the reflexive 7 j -adhesion neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in the present method in Definition 4.3, respectively, as shown in Table 5. Further, we will compute the reflexive 2 lower approximation, the reflexive 2 upper approxima-

tion, the reflexive 2 j -neighborhood boundary region, and the reflexive 2 j -neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in Abo Tabl's method [24] (as in Definition 2.5) (i.e., Definition 3.8 when $j = j_1$) and the reflexive 8 lower approximation, the reflexive 8 upper approximation, the reflexive 8 j -adhesion neighborhood boundary region, and the reflexive 8 j -adhesion neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in the present method in Definition 4.7, respectively, as shown in Table 6. Also, we will compute the reflexive 3 lower approximation, the reflexive 3 upper approximation, the reflexive 3 j -neighborhood boundary region, and the reflexive 3 j -neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in Abo Tabl's method [24] (as in Definition 2.6) (i.e., Definition 3.13 when $j = j_1$) and the reflexive 9 lower approximation, the reflexive 9 upper approximation, the reflexive 9 j -adhesion neighborhood boundary region, and the reflexive 9 j -adhesion neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in the the present method in Definition 4.11, respectively, as shown in Table 7.

The comparison between the Abo Tabl's method [24] and the present method is shown in Tables 5-7.

Table 5 Comparison between the reflexive 1 j -neighborhood boundary region and the reflexive 1 j -neighborhood accuracy measure of a set $\hat{X}$ by using Abo Tabl's method [24] (as in Definition 2.4) (i.e., Definition 3.3 when $j = j_1$) and the reflexive 7 j -adhesion neighborhood boundary region and the reflexive 7 j -adhesion neighborhood accuracy measure of a set $\hat{X}$ by using Definition 4.3.

$\hat{X}$	Abo Tabl's method [24] (as in Definition 2.4) (i.e., Definition 3.3 when $j = j_1$)				The current method in Definition 4.3			
	$\mathfrak{R}_{N_j}^{-1}(\hat{X})$	$\mathfrak{R}_{N_j}^{+1}(\hat{X})$	$\mathcal{B}_{j_1}^1(X)$	$\mathcal{A}_j^1(X)$	$\mathfrak{R}_{h_j}^{-7}(\hat{X})$	$\mathfrak{R}_{h_j}^{+7}(\hat{X})$	$\mathcal{B}_{j_1}^7(X)$	$\mathcal{A}_j^7(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	0
$\{1\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{1\}$	$\{1\}$	$\emptyset$	1
$\{2\}$	$\{2\}$	$\{1, 2\}$	$\{1\}$	$\frac{1}{2}$	$\{2\}$	$\{2\}$	$\emptyset$	1
$\{3\}$	$\{3\}$	$\{1, 3\}$	$\{1\}$	$\frac{1}{2}$	$\{3\}$	$\{3\}$	$\emptyset$	1
$\{4\}$	$\{4\}$	$\{4\}$	$\emptyset$	1	$\{4\}$	$\{4\}$	$\emptyset$	1
$\{1, 2\}$	$\{2\}$	$\{1, 2\}$	$\{1\}$	$\frac{1}{2}$	$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	1
$\{1, 3\}$	$\{3\}$	$\{1, 3\}$	$\{1\}$	$\frac{1}{2}$	$\{1, 3\}$	$\{1, 3\}$	$\emptyset$	1
$\{1, 4\}$	$\{4\}$	$\{1, 4\}$	$\{1\}$	$\frac{1}{2}$	$\{1, 4\}$	$\{1, 4\}$	$\emptyset$	1
$\{2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{1\}$	$\frac{1}{3}$	$\{2, 3\}$	$\{2, 3\}$	$\emptyset$	1
$\{2, 4\}$	$\{2, 4\}$	$\{1, 2, 4\}$	$\{1\}$	$\frac{1}{3}$	$\{2, 4\}$	$\{2, 4\}$	$\emptyset$	1
$\{3, 4\}$	$\{3, 4\}$	$\{1, 3, 4\}$	$\{1\}$	$\frac{1}{3}$	$\{3, 4\}$	$\{3, 4\}$	$\emptyset$	1
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1
$\{1, 2, 4\}$	$\{2, 4\}$	$\{1, 2, 4\}$	$\{1\}$	$\frac{2}{3}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\emptyset$	1
$\{1, 3, 4\}$	$\{3, 4\}$	$\{1, 3, 4\}$	$\{1\}$	$\frac{2}{3}$	$\{1, 3, 4\}$	$\{1, 3, 4\}$	$\emptyset$	1
$\{2, 3, 4\}$	$\{2, 3, 4\}$	$\mathcal{U}$	$\{1\}$	$\frac{4}{4}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$	$\emptyset$	1
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1

Table 6 Comparison between the reflexive 2 j -neighborhood boundary region and the reflexive 2 j -neighborhood accuracy measure of a set $\hat{X}$ by using Abo Tabl's method [24] (as in Definition 2.5) (i.e., Definition 3.8 when $j = j_1$) and the reflexive 8 j -adhesion neighborhood boundary region and the reflexive 8 j -adhesion neighborhood accuracy measure of a set $\hat{X}$ by using Definition 4.7.

$\hat{X}$	Abo Tabl's method [24] (as in Definition 2.5) (i.e., Definition 3.8 when $j = j_1$)				The current method in Definition 4.7			
	$\mathfrak{R}_{N_j}^{-2}(\hat{X})$	$\mathfrak{R}_{N_j}^{+2}(\hat{X})$	$\mathcal{B}_{j_1}^2(X)$	$\mathcal{A}_j^2(X)$	$\mathfrak{R}_{h_j}^{-8}(\hat{X})$	$\mathfrak{R}_{h_j}^{+8}(\hat{X})$	$\mathcal{B}_{j_1}^8(X)$	$\mathcal{A}_j^8(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	0
$\{1\}$	$\emptyset$	$\{1\}$	$\{1\}$	0	$\{1\}$	$\{1\}$	$\emptyset$	1
$\{2\}$	$\{2\}$	$\{1, 2\}$	$\{1\}$	$\frac{1}{2}$	$\{2\}$	$\{2\}$	$\emptyset$	1
$\{3\}$	$\{3\}$	$\{1, 3\}$	$\{1\}$	$\frac{1}{2}$	$\{3\}$	$\{3\}$	$\emptyset$	1
$\{4\}$	$\{4\}$	$\{4\}$	$\emptyset$	1	$\{4\}$	$\{4\}$	$\emptyset$	1
$\{1, 2\}$	$\{2\}$	$\{1, 2\}$	$\{1\}$	$\frac{1}{2}$	$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	1
$\{1, 3\}$	$\{3\}$	$\{1, 3\}$	$\{1\}$	$\frac{1}{2}$	$\{1, 3\}$	$\{1, 3\}$	$\emptyset$	1
$\{1, 4\}$	$\{4\}$	$\{1, 4\}$	$\{1\}$	$\frac{1}{2}$	$\{1, 4\}$	$\{1, 4\}$	$\emptyset$	1
$\{2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{1\}$	$\frac{1}{3}$	$\{2, 3\}$	$\{2, 3\}$	$\emptyset$	1
$\{2, 4\}$	$\{2, 4\}$	$\{1, 2, 4\}$	$\{1\}$	$\frac{1}{3}$	$\{2, 4\}$	$\{2, 4\}$	$\emptyset$	1
$\{3, 4\}$	$\{3, 4\}$	$\{1, 3, 4\}$	$\{1\}$	$\frac{1}{3}$	$\{3, 4\}$	$\{3, 4\}$	$\emptyset$	1
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	0	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1
$\{1, 2, 4\}$	$\{2, 4\}$	$\{1, 2, 4\}$	$\{1\}$	$\frac{2}{3}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\emptyset$	1
$\{1, 3, 4\}$	$\{3, 4\}$	$\{1, 3, 4\}$	$\{1\}$	$\frac{2}{3}$	$\{1, 3, 4\}$	$\{1, 3, 4\}$	$\emptyset$	1
$\{2, 3, 4\}$	$\{2, 3, 4\}$	$\mathcal{U}$	$\{1\}$	$\frac{4}{4}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$	$\emptyset$	1
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1

Table 7 Comparison between the reflexive 3 j -neighborhood boundary region and the reflexive 3 j -neighborhood accuracy measure of a set $\hat{X}$ by using Abo Tabl's method [24] (as in Definition 2.6) (i.e., Definition 3.13 when $j = j_1$) and the reflexive 9 j -adhesion neighborhood boundary region and the reflexive 9 j -adhesion neighborhood accuracy measure of a set $\hat{X}$ by using Definition 4.11.

$\hat{X}$	Abo Tabl's method [24] (as in Definition 2.6) (i.e., Definition 3.13 when $j = j_1$)				The current method in Definition 4.11			
	$\mathfrak{R}_{N_j}^{-3}(\hat{X})$	$\mathfrak{R}_{N_j}^{+3}(\hat{X})$	$\mathcal{B}_{j_1}^3(X)$	$\mathcal{A}_j^3(X)$	$\mathfrak{R}_{h_j}^{-9}(\hat{X})$	$\mathfrak{R}_{h_j}^{+9}(\hat{X})$	$\mathcal{B}_{j_1}^9(X)$	$\mathcal{A}_j^9(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	0
$\{1\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{1\}$	$\{1\}$	$\emptyset$	1
$\{2\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{2\}$	$\{2\}$	$\emptyset$	1
$\{3\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{3\}$	$\{3\}$	$\emptyset$	1
$\{4\}$	$\{4\}$	$\{4\}$	$\emptyset$	1	$\{4\}$	$\{4\}$	$\emptyset$	1
$\{1, 2\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	1
$\{1, 3\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{1, 3\}$	$\{1, 3\}$	$\emptyset$	1
$\{1, 4\}$	$\{4\}$	$\mathcal{U}$	$\{1, 2, 3\}$	$\frac{1}{4}$	$\{1, 4\}$	$\{1, 4\}$	$\emptyset$	1
$\{2, 3\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{2, 3\}$	$\{2, 3\}$	$\emptyset$	1
$\{2, 4\}$	$\{4\}$	$\mathcal{U}$	$\{1, 2, 3\}$	$\frac{1}{4}$	$\{2, 4\}$	$\{2, 4\}$	$\emptyset$	1
$\{3, 4\}$	$\{4\}$	$\mathcal{U}$	$\{1, 2, 3\}$	$\frac{1}{4}$	$\{3, 4\}$	$\{3, 4\}$	$\emptyset$	1
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1
$\{1, 2, 4\}$	$\{4\}$	$\mathcal{U}$	$\{1, 2, 3\}$	$\frac{1}{4}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\emptyset$	1
$\{1, 3, 4\}$	$\{4\}$	$\mathcal{U}$	$\{1, 2, 3\}$	$\frac{1}{4}$	$\{1, 3, 4\}$	$\{1, 3, 4\}$	$\emptyset$	1
$\{2, 3, 4\}$	$\{4\}$	$\mathcal{U}$	$\{1, 2, 3\}$	$\frac{1}{4}$	$\{2, 3, 4\}$	$\{2, 3, 4\}$	$\emptyset$	1
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1

From Table 5, for example, if we take $\{1, 2\}$, then the reflexive 7 j -adhesion neighborhood boundary and the reflexive 7 j -adhesion neighborhood accuracy by Definition 4.3 are $\emptyset$ and 1, respectively, whereas the reflexive 1 j -neighborhood boundary and the reflexive 1 j -neighborhood accuracy by using Abo Tabl's method [24] (as in Definition 2.4) (i.e., Definition 3.3 when $j = j_1$) are $\{1\}$ and $\frac{1}{2}$, respectively. Also, from Table 6, for example, if we take $\{2, 3\}$, then the reflexive 8 j -adhesion neighborhood boundary and the reflexive 8 j -adhesion neighborhood accuracy by Definition 4.7 are $\emptyset$ and 1, respectively, whereas the reflexive 2 j -neighborhood boundary and the reflexive 2 j -neighborhood accuracy by using Abo Tabl's method [24] (as in Definition 2.5) (i.e., Definition 3.8 when $j = j_1$) are $\{1\}$ and $\frac{2}{3}$, respectively. Additionally, from Table 7, for example, if we take $\{1, 4\}$, then the reflexive 9 j -adhesion neighborhood boundary and the reflexive 9 j -adhesion neighborhood accuracy by Definition 4.11 are $\emptyset$ and 1, respectively, whereas the reflexive 3 j -neighborhood boundary and the reflexive 3 j -neighborhood accuracy by using Abo Tabl's method [24] (as in Definition 2.6) (i.e., Definition 3.13 when $j = j_1$) are $\{1, 2, 3\}$ and $\frac{1}{4}$, respectively.

From Tables 5-7, we can see the reflexive 7-9 of lower and upper approximations based on j -adhesion neighborhood space are achieved to increase the accuracy measure and reduce the boundary region of set $\hat{X}$. Therefore, the results by Definitions 4.3, 4.7, and 4.11 when $j = j_1$ are better than the Abo Tabl's method [24] (as Definitions 2.4, 2.5, and 2.6) (i.e., Definitions 3.3, 3.8, and 3.13 when $j = j_1$), respectively.

(2) We apply Example 5.3 to compare between the Dai et al.'s method [25] (see Definitions 2.7, 2.8, and 2.9) (i.e., Definitions 3.20, 3.25, and 3.30 when $j = j_1$) and the current method (i.e., Definitions 4.17, 4.21, and 4.25 when $j = j_1$), respectively.

Example 5.3. Let $\mathcal{U} = \{1, 2, 3, 4\}$ and $\mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U}$ be a similarity relation on $\mathcal{U}$ defined by

$$\mathfrak{R} = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (3, 3), (3, 2), (4, 1), (4, 4)\}.$$

From Example 5.3, we will compute the similarity 4 lower approximation, the similarity 4 upper approximation, the similarity 4 j -neighborhood boundary region, and the similarity 4 j -neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in Dai et al.'s method [25] (as in Definition 2.8) (i.e., Definition 3.20 when $j = j_1$) and the similarity 10 lower approximation, the similarity 10 upper approximation, the similarity 10 j -adhesion neighborhood boundary region, and the similarity 10 j -adhesion neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in present method in Definition 4.17, respectively, as shown in Table 8. Further, we will compute the similarity 5 lower approximation, the similarity 5 upper approximation, the similarity 5 j -neighborhood boundary region, and the similarity 5 j -neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in Dai et al.'s method [25] (as in Definition 2.9) (i.e., Definition 3.25 when $j = j_1$) and the similarity 11 lower approximation, the similarity 11 upper approximation, the similarity 11 j -adhesion neighborhood boundary region, and the similarity 11 j -adhesion neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in the present method in Definition 4.21, respectively, as shown in Table 9. Finally, we will compute the similarity 6

lower approximation, the similarity 6 upper approximation, the similarity 6 j -neighborhood boundary region, and the similarity 6 j -neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in Dai et al.'s method [25] (as in Definition 2.10) (i.e., Definition 3.30 when $j = j_1$) and the similarity 12 lower approximation, the similarity 12 upper approximation, the similarity 12 j -adhesion neighborhood boundary region, and the similarity 12 j -adhesion neighborhood accuracy measure for subsets of $\hat{X} \subseteq \mathcal{U}$ in the present method in Definition 4.25, respectively, as shown in Table 10.

The comparison between the Dai et al.'s method [25] and the present method is shown in Tables 8-10.

Table 8 Comparison between the similarity 4 j -neighborhood boundary region and the similarity 4 j -neighborhood accuracy measure of a set $\hat{X}$ by using Dai et al.'s method [25] (as in Definition 2.8) (i.e., Definition 3.20 when $j = j_1$) and the similarity 10 j -adhesion neighborhood boundary region and the similarity 10 j -adhesion neighborhood accuracy measure of a set $\hat{X}$ by using Definition 4.17.

$\hat{X}$	Dai et al.'s method [25] (as in Definition 2.8) (i.e., Definition 3.20 when $j = j_1$)				The current method in Definition 4.17			
	$\mathfrak{R}_{\mathfrak{N}_j}^{-4}(\hat{X})$	$\mathfrak{R}_{\mathfrak{N}_j}^{+4}(\hat{X})$	$\mathcal{B}_{j_1}^4(X)$	$\mathcal{A}_j^4(X)$	$\mathfrak{R}_{h_j}^{-10}(\hat{X})$	$\mathfrak{R}_{h_j}^{+10}(\hat{X})$	$\mathcal{B}_{j_1}^{10}(X)$	$\mathcal{A}_j^{10}(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	0
$\{1\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\emptyset$	$\{1, 2\}$	$\{1, 2\}$	0
$\{2\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\emptyset$	$\{1, 2\}$	$\{1, 2\}$	0
$\{3\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{3\}$	$\{3\}$	$\emptyset$	1
$\{4\}$	$\emptyset$	$\{1, 2, 4\}$	$\{1, 2, 4\}$	0	$\{4\}$	$\{4\}$	$\emptyset$	1
$\{1, 2\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	1
$\{1, 3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{1, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{4\}$	$\{1, 2, 4\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{2, 3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{2, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{4\}$	$\{1, 2, 4\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\{3, 4\}$	$\emptyset$	1
$\{1, 2, 3\}$	$\{3\}$	$\mathcal{U}$	$\{1, 2, 4\}$	$\frac{1}{4}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1
$\{1, 2, 4\}$	$\{4\}$	$\mathcal{U}$	$\{1, 2, 3\}$	$\frac{1}{4}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\emptyset$	1
$\{1, 3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\mathcal{U}$	$\{1, 2\}$	$\frac{2}{4}$
$\{2, 3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\mathcal{U}$	$\{1, 2\}$	$\frac{2}{4}$
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1

Table 9 Comparison between the similarity 5 j -neighborhood boundary region and the similarity 5 j -neighborhood accuracy measure of a set $\hat{X}$ by using Dai et al.'s method [25] (as in Definition 2.9) (i.e., Definition 3.25 when $j = j_1$) and the similarity 11 j -adhesion neighborhood boundary region and the similarity 11 j -adhesion neighborhood accuracy measure of a set $\hat{X}$ by using Definition 4.21.

$\hat{X}$	Dai et al.'s method [25] (as in Definition 2.9) (i.e., Definition 3.25 when $j = j_1$)				The current method in Definition 4.21			
	$\mathfrak{R}_{\mathfrak{N}_j}^{-5}(\hat{X})$	$\mathfrak{R}_{\mathfrak{N}_j}^{+5}(\hat{X})$	$\mathcal{B}_{j_1}^5(X)$	$\mathcal{A}_j^5(X)$	$\mathfrak{R}_{h_j}^{-11}(\hat{X})$	$\mathfrak{R}_{h_j}^{+11}(\hat{X})$	$\mathcal{B}_{j_1}^{11}(X)$	$\mathcal{A}_j^{11}(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	0
$\{1\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\emptyset$	$\{1, 2\}$	$\{1, 2\}$	0
$\{2\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\emptyset$	$\{1, 2\}$	$\{1, 2\}$	0
$\{3\}$	$\emptyset$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	0	$\{3\}$	$\{3\}$	$\emptyset$	1
$\{4\}$	$\emptyset$	$\{1, 2, 4\}$	$\{1, 2, 4\}$	0	$\{4\}$	$\{4\}$	$\emptyset$	1
$\{1, 2\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	1
$\{1, 3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{1, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{4\}$	$\{1, 2, 4\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{2, 3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{2, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{4\}$	$\{1, 2, 4\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\{3, 4\}$	$\emptyset$	1
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\mathcal{U}$	$\{4\}$	$\frac{3}{4}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1
$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\mathcal{U}$	$\{3\}$	$\frac{3}{4}$	$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\emptyset$	1
$\{1, 3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\mathcal{U}$	$\{1, 2\}$	$\frac{2}{4}$
$\{2, 3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\mathcal{U}$	$\{1, 2\}$	$\frac{2}{4}$
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1

Table 10 Comparison between the similarity 6 j -neighborhood boundary region and the similarity 6 j -neighborhood accuracy measure of a set $\hat{X}$ by using Dai et al.'s method [25] (as in Definition 2.10) (i.e., Definition 3.30 when $j = j_1$) and the similarity 12 j -adhesion neighborhood boundary region and the similarity 12 j -adhesion neighborhood accuracy measure of a set $\hat{X}$ by using Definition 4.25.

$\hat{X}$	Dai et al.'s method [25] (as in Definition 2.10) (i.e., Definition 3.30 when $j = j_1$)				The current method in Definition 4.25			
	$\mathfrak{R}_{\mathfrak{N}_j}^{-6}(\hat{X})$	$\mathfrak{R}_{\mathfrak{N}_j}^{+6}(\hat{X})$	$\mathcal{B}_{j_1}^6(X)$	$\mathcal{A}_j^6(X)$	$\mathfrak{R}_{h_j}^{-12}(\hat{X})$	$\mathfrak{R}_{h_j}^{+12}(\hat{X})$	$\mathcal{B}_{j_1}^{12}(X)$	$\mathcal{A}_j^{12}(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	0
$\{1\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\emptyset$	$\{1, 2\}$	$\{1, 2\}$	0
$\{2\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\emptyset$	$\{1, 2\}$	$\{1, 2\}$	0
$\{3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3\}$	$\{3\}$	$\emptyset$	1
$\{4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{4\}$	$\{4\}$	$\emptyset$	1
$\{1, 2\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{1, 2\}$	$\{1, 2\}$	$\emptyset$	1
$\{1, 3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{1, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{4\}$	$\{1, 2, 4\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{2, 3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{2, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{4\}$	$\{1, 2, 4\}$	$\{1, 2\}$	$\frac{1}{3}$
$\{3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\{3, 4\}$	$\emptyset$	1
$\{1, 2, 3\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\emptyset$	1
$\{1, 2, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{1, 2, 4\}$	$\{1, 2, 4\}$	$\emptyset$	1
$\{1, 3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\mathcal{U}$	$\{1, 2\}$	$\frac{2}{4}$
$\{2, 3, 4\}$	$\emptyset$	$\mathcal{U}$	$\mathcal{U}$	0	$\{3, 4\}$	$\mathcal{U}$	$\{1, 2\}$	$\frac{2}{4}$
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1	$\mathcal{U}$	$\mathcal{U}$	$\emptyset$	1

From Table 8, for example, if we take $\{1, 2, 3\}$, then the similarity 10 j -adhesion neighborhood boundary and the similarity 10 j -adhesion neighborhood accuracy by Definition 4.17 are $\emptyset$ and 1, respectively, whereas the similarity 4 j -neighborhood boundary and the similarity 4 j -neighborhood accuracy by using Dai et al.'s method [25] (as in Definition 2.8) (i.e., Definition 3.20 when $j = j_1$) are $\{1, 2, 4\}$ and $\frac{1}{4}$, respectively. Also, from Table 9, for example, if we take $\{3\}$, then the similarity 11 j -adhesion neighborhood boundary and the similarity 11 j -adhesion neighborhood accuracy by Definition 4.21 are $\emptyset$ and 1, whereas the similarity 5 j -neighborhood boundary and the similarity 5 j -neighborhood accuracy by using Dai et al.'s method [25] (as in Definition 2.9) (i.e., Definition 3.25 when $j = j_1$) are $\{1, 2, 3\}$ and 0, respectively. Additionally, from Table 10, for example, if we take $\{4\}$, then the similarity 12 j -adhesion neighborhood boundary and the similarity 12 j -adhesion neighborhood accuracy by Definition 4.25 are $\emptyset$ and 1, whereas the similarity 6 j -neighborhood boundary and the similarity 6 j -neighborhood accuracy by using Dai et al.'s method [25] (as in Definition 2.10) (i.e., Definition 3.30 when $j = j_1$) are $\mathcal{U}$ and 0, respectively.

From Tables 8-10, we can see the similarity 10-12 of lower and upper approximations based on j -adhesion neighborhood space are achieved to increase the accuracy measure and reduce the boundary region of set $\hat{X}$. Therefore, the results by Definitions 4.17, 4.21, and 4.25 when $j = j_1$ are better than the Dai et al.'s method [25] (as Definitions 2.8, 2.9, and 2.10) (i.e., Definitions 3.20, 3.25, and 3.30 when $j = j_1$), respectively.

6 Conclusions

In recent decades, many applications of the rough set theory have emerged in various fields. Therefore, in this paper, we aim to reduce the boundary region and increase the accuracy measure of decision-making. Besides, we introduce a generalization for six types of rough set models based on j -neighborhood space. A new six types of rough set models based on j -adhesion neighborhood space as, reflexive 7 j -adhesion neighborhood rough set, reflexive 8 j -adhesion neighborhood rough set, reflexive 9 j -adhesion neighborhood rough set, similarity 10 j -adhesion neighborhood rough set, similarity 11 j -adhesion neighborhood rough set, and similarity 12 j -adhesion neighborhood rough set are defined. Also, some of the fundamental properties of these types of rough j -adhesion neighborhood space are discussed. The relationship between the properties of different types is achieved. Finally, we made a comparison of previous methods (i.e., Abo-Tabl's method and Dai et al.'s method) and our method of finding boundary region and accuracy. In the future, the present work can be extended to present some new rough set approximations under j -adhesion neighborhood space [31, 32].

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Declarations

Conflict of interest

All the authors certify that there is no conflict of interest with any individual or organization for the present work.

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Animal and human rights

This article does not contain any studies with human participants or animals performed by any of the authors.

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