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Tsallis Entropy of Uncertain Random Variables and Its Application

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Abstract

Tsallis entropy is a flexible extension of Shannon (logarithm) entropy. Since, entropy measures indeterminacy of an uncertain random variable, this paper proposes the concept of partial Tsallis entropy for uncertain random variables as a flexible device in chance theory. An approach for calculating partial Tsallis entropy for uncertain random variables, based on Monte-Carlo simulation, is provided. As an application in finance, partial Tsallis entropy is invoked to optimize portfolio selection of uncertain random returns via crow search algorithm.

Keywords: chance theory, uncertain random variable, partial Tsallis entropy, portfolio selection, Monte-Carlo simulation, crow search algorithm.

1 Introduction

In statistics, entropy measures the indeterminacy of a random variable. Firstly, Shannon [22] proposed the concept of entropy of discrete random variables. After that, several authors devoted their studies to the case of entropy of random variables. As an extension of Shannon (logarithm) entropy, Tsallis [25] introduced the concept of Tsallis entropy. In order to measure the indeterminacy of random lifetime variables, several authors proposed different types of cumulative and cumulative residual entropy, for instance [5, 9, 10].

In many situation, we face to lack of data. In order to model such phenomena, Liu [12] founded uncertainty theory. By inception of Shannon entropy for discrete random variables, Liu [12] presented the concept of entropy for uncertain variables. As other types of entropy, Dai [7] and Yao

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and Dai [26] proposed quadratic and sine entropy, respectively. In order to characterize the difference between two uncertainty distributions, Chen et al. [6] established cross (relative) entropy for uncertain variables.

However, there exist many complex phenomena including uncertainty and randomness. These phenomena are modeled via chance theory which proposed by Liu [19]. As a device to measure the indeterminacy of uncertain random variables, Sheng et al. [24] introduced the concept of uncertain random variables and applied it to minimum spanning tree. After that, Ahmadzade et al. [4] presented the concept of partial entropy for uncertain random variables and applied to portfolio selection problems via different models, for instance see [2, 3]. In this work, as an extension of logarithm entropy, we propose the flexible concept of Tsallis entropy and discuss about its properties. As an application of entropy, we optimize the portfolio selection problem via mean-variance-Tsallis entropy model.

This paper is organized as follows. Section 2 reviews some concepts of uncertainty theory and chance theory as they are needed. Section 3 presents a definition of partial Tsallis entropy for uncertain random variable and studies its properties. Section 4 provides Monte-Carlo approach for calculating partial Tsallis entropy of uncertain random variables. In Section 5, portfolio selection problems with uncertain random returns are optimized via meta-heuristic Crow Search Algorithm. Finally, some conclusions are derived in Section 6.

2 Preliminaries

In this section, we recall some basic concepts such as expected value, variance, covariance and entropy for uncertain random variables.

Definition 1 (Liu [20]) *Let ξ be an uncertain random variable. Then its expected value is defined by*

$$E[\xi] = \int_0^\infty \text{Ch}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Ch}\{\xi < r\} dr,$$

provided that at least one of the two integrals is finite.

Let Φ denote the chance distribution of ξ . Liu (2013c) derived a formula to calculate the expected value of uncertain random variable with chance distribution, that is,

$$E[\xi] = \int_0^\infty (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx,$$

Theorem 1 (Liu [19]) *Let $\eta_1, \eta_2, \dots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \dots, \Psi_m$, respectively, and $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables (not necessarily independent). Then the uncertain random variable*

$$\xi = f(\eta_1, \dots, \eta_m, \tau_1, \dots, \tau_n)$$

has an expected value

$$E[\xi] = \int_{\mathbb{R}^m} E[f(y_1, \dots, y_m, \tau_1, \dots, \tau_n)] d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$

where $E[f(y_1, \dots, y_m, \tau_1, \dots, \tau_n)]$ is the expected value of the uncertain variable $f(y_1, \dots, y_m, \tau_1, \dots, \tau_n)$ for any real numbers $y_1, \dots, y_m$.

Definition 2 (Ahmadzade and Gao [2]) Let ξ_1 and ξ_2 be two uncertain random variables. Then the covariance of ξ_1 and ξ_2 is defined by

$$Cov(\xi_1, \xi_2) = E[(\xi_1 - E[\xi_1])(\xi_2 - E[\xi_2])].$$

In special case $V[\xi] = Cov(\xi, \xi)$.

Theorem 2 (Ahmadzade and Gao [2]) Let $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables, and let $\eta_1, \eta_2, \dots, \eta_n$ be independent random variables with probability distribution functions $\Phi_1, \Phi_2, \dots, \Phi_n$, respectively. Suppose

$$\xi_1 = f_1(\eta_1, \tau_1), \xi_2 = f_2(\eta_2, \tau_2), \dots, \xi_n = f_n(\eta_n, \tau_n).$$

Then

$$V[\xi_1 + \dots + \xi_n] = \sum_{i=1}^n V[\xi_i] + 2 \sum_{i=1}^n \sum_{j=i+1}^n Cov(\xi_i, \xi_j).$$

In general,

$$V[a_1\xi_1 + \dots + a_n\xi_n] = \sum_{i=1}^n a_i^2 V[\xi_i] + 2 \sum_{i=1}^n \sum_{j=i+1}^n a_i a_j Cov(\xi_i, \xi_j),$$

where, $a_1, a_2, \dots, a_n$ are constants.

Sheng et al. [24] introduced the concept of entropy for uncertain random variables. First, we recall their definition for uncertain random variables.

Definition 3 (Sheng et al. [24]) Suppose ξ is an uncertain random variable. Then the entropy of ξ is defined by

$$H[\xi] = \int_{-\infty}^{\infty} T(\Phi(x)) dx,$$

where $T(s) = -s \ln s - (1-s) \ln(1-s)$, and $\Phi(x)$ is chance distribution of ξ .

Definition 4 (Ahmadzade et al. [1]) Suppose that $\eta_1, \eta_2, \dots, \eta_m$ are independent random variables and $\tau_1, \tau_2, \dots, \tau_m$ are uncertain variables, also $\xi = f(\eta_1, \dots, \eta_m, \tau_1, \dots, \tau_m)$ is an uncertain random variable. Partial entropy of uncertain random variable ξ is defined as following

$$PH[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} T(F(x, y_1, \dots, y_m)) dx d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$

where $T(s) = -s \ln s - (1-s) \ln(1-s)$ and $F(x, y_1, \dots, y_m)$ is the uncertainty distribution of the uncertain variable

$f(y_1, \dots, y_m, \tau_1, \dots, \tau_n)$ for any real numbers $y_1, \dots, y_m$.

3 Partial Tsallis Entropy of Uncertain Random Variables

In this section, we explain the motivation of definition of Tsallis entropy for uncertain random variable. Rao et al. [21] introduced the concept of cumulative residual entropy as follows:

$$CRE(X) = - \int_{-\infty}^{\infty} \bar{F}(x) \ln \bar{F}(x) dx,$$

where $\bar{F}(\cdot)$ is the survival function of random variable X . Similarly, Di Crescenzo and Longobardi [10] proposed the concept of cumulative entropy of random variable X as follows:

$$CE(X) = - \int_{-\infty}^{\infty} F(x) \ln F(x) dx,$$

where $F(\cdot)$ is the cumulative distribution function of random variable X . Di Crescenzo and Longobardi [10] showed the relation between cumulative and residual cumulative entropy as follows:

$$CE(X) + CRE(X) = - \int_{-\infty}^{\infty} h(x) dx,$$

where $h(x) = -[F(x) \ln F(x) + \bar{F}(x) \ln \bar{F}(x)]$ is the partition entropy of X evaluated at x . As an extension of cumulative entropy, Ahmadi [5] presented the concept of cumulative Tsallis entropy for random variable X as follows:

$$CTE_{\beta}(X) = \frac{1}{\beta - 1} \int_{\mathbb{R}} [F(x) - F^{\beta}(x)] dx, \quad \beta \neq 1, \quad \beta > 0.$$

Cumulative residual Tsallis entropy of random variable X as follows:

$$CRTE_{\beta}(X) = \frac{1}{\beta - 1} \int_{\mathbb{R}} [\bar{F}(x) - \bar{F}^{\beta}(x)] dx, \quad \beta \neq 1, \quad \beta > 0.$$

By inception of relation , we can present the relation between cumulative Tsallis entropy and cumulative residual Tsallis entropy as follows:

$$CRTE_{\beta}(X) + CTE_{\beta}(X) = \frac{1}{\beta - 1} \int_{\mathbb{R}} [1 - F^{\beta}(x) - \bar{F}^{\beta}(x)] dx, \quad \beta \neq 1, \quad \beta > 0. \quad (1)$$

By inception of relation (1), we propose the concept of Tsallis entropy for uncertain random variables as follows:

Definition 5 Let ξ be an uncertain random variable with chance distribution $\Phi(x)$. Then its entropy is defined by

$$TE_\beta[\xi] = \int_{-\infty}^{\infty} T_\beta(\Phi(x))dx,$$

where $T_\beta(s) = \frac{1-s^\beta-(1-s)^\beta}{\beta-1}$.

As an extension of partial entropy, we introduce the flexible concept of partial Tsallis entropy for uncertain random variable.

Definition 6 Suppose that $\eta_1, \eta_2, \dots, \eta_m$ are independent random variables and $\tau_1, \tau_2, \dots, \tau_m$ are uncertain variables, also $\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_m)$ is an uncertain random variable. Partial Tsallis entropy of uncertain random variable ξ is defined as following

$$PTE_\beta[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} T_\beta(F(x, y_1, \dots, y_m))dx d\Psi_1(y_1) \cdots d\Psi_m(y_m),$$

where $T_\beta(s) = \frac{1-s^\beta-(1-s)^\beta}{\beta-1}$ and $F(x, y_1, \dots, y_m)$ is the uncertainty distribution of uncertain variable $f(y_1, \dots, y_m, \tau_1, \dots, \tau_m)$ for any real numbers $y_1, \dots, y_m$.

For $\beta = 1$,

$$\begin{aligned} \lim_{\beta \rightarrow 1} T_\beta(F(x, y_1, \dots, y_m)) &= \lim_{\beta \rightarrow 1} \frac{1 - (F(x, y_1, \dots, y_m))^\beta - (1 - F(x, y_1, \dots, y_m))^\beta}{\beta - 1} \\ &= -F(x, y_1, \dots, y_m) \ln F(x, y_1, \dots, y_m) \\ &\quad - (1 - F(x, y_1, \dots, y_m)) \ln(1 - F(x, y_1, \dots, y_m)) \\ &= T(F(x, y_1, \dots, y_m)). \end{aligned}$$

This means, for $\beta = 1$ the partial Tsallis entropy becomes the partial entropy in [Definition 4](#).

Remark 1 If uncertain random variables reduce to uncertain ones, we can use above definition for uncertain random variables.

Example 1 Suppose τ is an uncertain variable with linear uncertainty distribution $\mathcal{L}(0, 1)$. Tsallis entropy of order β for τ is

$$\begin{aligned} TE_\beta(\tau) &= \frac{1}{\beta-1} \int_0^1 (1-x^\beta - (1-x)^\beta)dx \\ &= \frac{1}{\beta+1}. \end{aligned}$$

Besides, Liu [12] showed Shanon entropy for τ is equal to 0.5. We can see

$$\begin{aligned}\lim_{\beta \rightarrow 1} TE_{\beta}(\tau) &= \lim_{\beta \rightarrow 1} \frac{1}{\beta + 1} \\ &= \frac{1}{2} = H[\tau],\end{aligned}$$

which says when β tends to one, Tsallis entropy converges to Shanon (logarithm) entropy.

Theorem 3 Let $\eta_1, \eta_2, \dots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \dots, \Psi_m$, and $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_n$, respectively, and let f be a measurable function. Then

$$\xi = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$$

has partial Tsallis entropy

$$PTE_{\beta}[\xi] = - \int_{\mathbb{R}^m} \int_0^1 F^{-1}(\alpha, y_1, \dots, y_m) T'_{\beta}(\alpha) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m),$$

where, $T'_{\beta}(\alpha) = \frac{\beta((1-s)^{\beta-1} - s^{\beta-1})}{\beta-1}$.

Proof: It is clear that

$$T_{\beta}(F(x, y_1, \dots, y_m)) = \int_0^{F(x, y_1, \dots, y_m)} T'_{\beta}(\alpha) d\alpha = \int_{F(x, y_1, \dots, y_m)}^1 T'_{\beta}(\alpha) d\alpha.$$

By definition of partial Tsallis entropy of uncertain random variable, we have

$$\begin{aligned}PTE_{\beta}[\xi] &= \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} T_{\beta}(F(x, y_1, \dots, y_m)) dx d\Psi_1(y_1) \cdots d\Psi_m(y_m) \\ &= \int_{\mathbb{R}^m} \int_{-\infty}^0 \int_0^{F(x, y_1, \dots, y_m)} T'_{\beta}(\alpha) d\alpha dx d\Psi_1(y_1) \cdots d\Psi_m(y_m) \\ &\quad - \int_{\mathbb{R}^m} \int_0^{\infty} \int_{F(x, y_1, \dots, y_m)}^1 T'_{\beta}(\alpha) d\alpha dx d\Psi_1(y_1) \cdots d\Psi_m(y_m).\end{aligned}$$

It follows from Fubini's theorem that

$$\begin{aligned}
PH_q[\xi] &= \int_{\mathbb{R}^m} \int_0^{F(0,y_1,\dots,y_m)} \int_{F^{-1}(\alpha,y_1,\dots,y_m)}^0 T'_\beta(\alpha) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m) \\
&\quad - \int_{\mathbb{R}^m} \int_{F(0,y_1,\dots,y_m)}^1 \int_0^{F^{-1}(\alpha,y_1,\dots,y_m)} T'_\beta(\alpha) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m) \\
&= - \int_{\mathbb{R}^m} \int_0^{F(0,y_1,\dots,y_m)} F^{-1}(\alpha,y_1,\dots,y_m) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m) \\
&\quad - \int_{\mathbb{R}^m} \int_{F(0,y_1,\dots,y_m)}^1 F^{-1}(\alpha,y_1,\dots,y_m) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m) \\
&= - \int_{\mathbb{R}^m} \int_0^1 F^{-1}(\alpha,y_1,\dots,y_m) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) \cdots d\Psi_m(y_m).
\end{aligned}$$

Theorem 4 Let τ be an uncertain variable with uncertainty distribution function Φ and η be a random variable with probability distribution function Ψ . If $\xi = \gamma + \tau$, then

$$PTE_\beta[\xi] = TE_\beta[\tau].$$

Proof: It is obvious that $F^{-1}(\alpha, y) = \Phi^{-1}(\alpha) + y$, therefore by using Theorem 3, we obtain

$$\begin{aligned}
PTE_\beta[\xi] &= - \int_{\mathbb{R}} \int_0^1 F^{-1}(\alpha, y) T'_\beta(\alpha) d\alpha d\Psi(y) = - \int_{\mathbb{R}} \int_0^1 (\Phi^{-1}(\alpha) + y) T'_\beta(\alpha) d\alpha d\Psi(y) \\
&= - \int_{\mathbb{R}} \int_0^1 \Phi^{-1}(\alpha) T'_\beta(\alpha) d\alpha d\Psi(y) - \int_{\mathbb{R}} \int_0^1 y T'_\beta(\alpha) d\alpha d\Psi(y) \\
&= TE_\beta[\tau].
\end{aligned}$$

Theorem 5 Let τ be an uncertain variable with uncertainty distribution function Φ and η be a random variable with probability distribution function Ψ . If $\xi = \gamma\tau$, then $PTE_\beta[\xi] = TE_\beta[\tau]E[\gamma]$.

Proof: It is obvious that $F^{-1}(\alpha, y) = \Phi^{-1}(\alpha)y$, therefore by using Theorem 3, we obtain

$$\begin{aligned} PTE_{\beta}[\xi] &= - \int_{\mathbb{R}} \int_0^1 F^{-1}(\alpha, y) T'_{\beta}(\alpha) d\alpha d\Psi(y) = - \int_{\mathbb{R}} \int_0^1 y \Phi^{-1} T'_{\beta}(\alpha) d\alpha d\Psi(y) \\ &= - \int_0^1 \Phi^{-1}(\alpha) T'_{\beta}(\alpha) d\alpha \int_{\mathbb{R}} y d\Psi(y) \\ &= TE_{\beta}[\tau] E[\gamma]. \end{aligned}$$

Theorem 6 Let $\eta_1, \eta_2, \dots, \eta_n$ be independent random variables and $\tau_1, \tau_2, \dots, \tau_n$ be independent uncertain variables. Also, suppose that

$$\xi_1 = f_1(\eta_1, \tau_1), \xi_2 = f_2(\eta_2, \tau_2), \dots, \xi_n = f_n(\eta_n, \tau_n)$$

If $f(x_1, x_2, \dots, x_n)$ is strictly increasing with respect to $x_1, x_2, \dots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \dots, x_n$, then $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ has partial Tsallis entropy

$$PTE_{\beta}[\xi] = - \int_{\mathbb{R}^n} \int_0^1 g(\alpha, y_1, \dots, y_m) T'_{\beta}(\alpha) d\alpha d\Psi_1(y_1) \cdots d\Psi_n(y_n).$$

where $g(\alpha, y_1, \dots, y_m)$ is equal to

$$f(F_1^{-1}(\alpha, y_1), F_2^{-1}(\alpha, y_2), \dots, F_m^{-1}(\alpha, y_m), F_{m+1}^{-1}(1 - \alpha, y_{m+1}), \dots, F_n^{-1}(1 - \alpha, y_n)),$$

and $F_i^{-1}(\alpha, y_i)$ is the inverse uncertainty distribution of uncertain variable $f_i(y_i, \tau_i)$ for any real number $y_i, i = 1, 2, \dots, n$

Proof: Since $f(x_1, x_2, \dots, x_n)$ is strictly increasing with respect to $x_1, x_2, \dots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \dots, x_n$, **operational law of inverse uncertainty distribution implies that**

$$F^{-1}(\alpha, y_1, \dots, y_n) = f(F_1^{-1}(\alpha, y_1), \dots, F_m^{-1}(\alpha, y_m), F_{m+1}^{-1}(1 - \alpha, y_{m+1}), \dots, F_n^{-1}(1 - \alpha, y_n)).$$

By invoking Theorem 3, the proof is complete.

Theorem 7 Let η_1 and η_2 be random variables with probability distribution functions Ψ_1 and Ψ_2 respectively, and τ_1 and τ_2 be uncertain variables with uncertainty distribution functions Φ_1 and Φ_2 , respectively. If $\xi_1 = \eta_1 + \tau_1$ and $\xi_2 = \eta_2 + \tau_2$, then

$$PTE_{\beta}[\xi_1 \xi_2] = TE_{\beta}[\tau_1 \tau_2] + E[\eta_1] TE_{\beta}[\tau_2] + E[\eta_2] TE_{\beta}[\tau_1].$$

Proof: It is clear that $F_1^{-1}(\alpha, y_1) = y_1 + \Phi_1^{-1}(\alpha)$ and $F_2^{-1}(\alpha, y_2) = y_2 + \Phi_2^{-1}(\alpha)$. By using Theorem 6, we have

$$\begin{aligned}
PTE_\beta[\xi_1\xi_2] &= - \int_{\mathbb{R}^2} \int_0^1 F^{-1}(\alpha, y_1, y_2) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) d\Psi_2(y_2) \\
&= - \int_{\mathbb{R}^2} \int_0^1 F_1^{-1}(\alpha, y_1) F_2^{-1}(\alpha, y_2) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) d\Psi_2(y_2) \\
&= - \int_{\mathbb{R}^2} \int_0^1 \{y_1 + \Phi_1^{-1}(\alpha)\} \{y_2 + \Phi_2^{-1}(\alpha)\} T'_\beta(\alpha) d\alpha d\Psi_1(y_1) d\Psi_2(y_2) \\
&= - \int_0^1 \Phi_1^{-1}(\alpha) \Phi_2^{-1}(\alpha) T'_\beta(\alpha) d\alpha - \int_{\mathbb{R}} \int_0^1 y_1 \Phi_2^{-1}(\alpha) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) \\
&\quad - \int_{\mathbb{R}} \int_0^1 y_2 \Phi_1^{-1}(\alpha) T'_\beta(\alpha) d\alpha d\Psi_2(y_2) \\
&= TE_\beta[\tau_1\tau_2] + E[\eta_1]TE_\beta[\tau_2] + E[\eta_2]TE_\beta[\tau_1].
\end{aligned}$$

Theorem 8 Let η_1 and η_2 be random variables with probability distribution functions Ψ_1 and Ψ_2 respectively, and τ_1 and τ_2 be uncertain variables with uncertainty distribution functions Φ_1 and Φ_2 , respectively. If $\xi_1 = \eta_1\tau_1$ and $\xi_2 = \eta_2\tau_2$, then

$$PTE_\beta\left[\frac{\xi_1}{\xi_2}\right] = TE_\beta\left[\frac{\tau_1}{\tau_2}\right]E[\eta_1]E\left[\frac{1}{\eta_2}\right].$$

Proof: By using a similar method of Theorem 7 and independence of random variables, the proof is straightforward.

Theorem 9 Let η_1 and η_2 be independent random variables and τ_1 and τ_2 be independent uncertain variables. Also suppose that $\xi_1 = f_1(\eta_1, \tau_1)$ and $\xi_2 = f_2(\eta_2, \tau_2)$. Then for any real numbers a and b , we have

$$PTE_\beta[a\xi_1 + b\xi_2] = |a|PTE_\beta[\xi_1] + |b|PTE_\beta[\xi_2].$$

Proof: We should consider four states of a and b . i) $a > 0, b > 0$, ii) $a > 0, b < 0$, iii) $a < 0, b > 0$ and iv) $a < 0, b < 0$. without loss generality, consider $a > 0$ and $b < 0$. In this case, it is easy to

see that the inverse of uncertainty distribution of $a\xi_1 + b\xi_2$ is equal to $aF_1^{-1}(\alpha, y_1) + bF_2^{-1}(1 - \alpha, y_2)$. Thus, implies that

$$\begin{aligned}
PTE_\beta(a\xi_1 + b\xi_2) &= - \int_{\mathbb{R}^2} \int_0^1 (aF_1^{-1}(\alpha, y_1) + bF_2^{-1}(1 - \alpha, y_2)) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) d\Psi_2(y_2) \\
&= - \int_{\mathbb{R}} \int_0^1 aF_1^{-1}(\alpha, y_1) T'_\beta(\alpha) d\alpha d\Psi_1(y_1) \\
&\quad - \int_{\mathbb{R}} \int_0^1 bF_2^{-1}(1 - \alpha, y_2) T'_\beta(\alpha) d\alpha d\Psi_2(y_2) \\
&= aPTE_\beta(\xi_1) - b \int_{\mathbb{R}} \int_0^1 F_2^{-1}(1 - \alpha, y_2) T'_\beta(\alpha) d\alpha d\Psi_2(y_2)
\end{aligned}$$

Since $T'_\beta(\alpha) = -T'_\beta(1 - \alpha)$, by putting $u = 1 - \alpha$, we have

$$\begin{aligned}
-b \int_{\mathbb{R}} \int_0^1 F_2^{-1}(1 - \alpha, y_2) T'_\beta(\alpha) d\alpha d\Psi_2(y_2) &= b \int_{\mathbb{R}} \int_0^1 F_2^{-1}(u, y_2) T'_\beta(u) du d\Psi_2(y_2) \\
&= -bPTE_\beta(\xi_2) = |b|PTE_\beta(\xi_2).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
PTE_\beta(a\xi_1 + b\xi_2) &= aPTE_\beta(\xi_1) + |b|PTE_\beta(\xi_2) \\
&= |a|PTE_\beta(\xi_1) + |b|PTE_\beta(\xi_2).
\end{aligned}$$

Similarly, for other cases of a and b , we can prove linearity for partial Tsallis entropy.

4 Monte Carlo Simulation for Partial Tsallis Entropy

By invoking Theorem 3, we can write partial Tsallis entropy of an uncertain random variable via expectation of a function of random variables as follows:

$$PTE_\beta[\xi] = -E[F^{-1}(U, \eta_1, \dots, \eta_m) T'_\beta(U)],$$

where, $U, \eta_1, \dots, \eta_m$ are random variables with standard uniform distribution, $\Psi_1(y_1), \dots, \Psi_m(y_m)$, respectively. Thus, we can use Monte Carlo simulation for partial Tsallis entropy via the following steps. Consider $\xi = f(\eta, \tau)$ as an uncertain random variable.

Step 1. Randomly generate $u_1, u_2, \dots, u_N$ and $y_1, y_2, \dots, y_M$ from uniform distribution and

probability distribution $\Psi(y)$, respectively.

Step 2. Compute $F^{-1}(u_i, y_j)T'_\beta(u_j)$ for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$ and fixed β .

Step 3. Consider $\frac{1}{NM} \sum_{i=1}^N \sum_{j=1}^M F^{-1}(u_i, y_j)T'_\beta(u_j)$ as an approximation for partial Tsallis entropy PTE_β .

Example 2 Let η be a random variable such that $\eta \sim \text{Exp}(5)$. Also, suppose that τ is an uncertain variable with uncertainty distribution $\mathcal{N}(2, 3)$. Consider $\xi = \eta^3\tau$ as an uncertain random variable. Partial Tsallis entropy of ξ is

$$\begin{aligned} PTE_\beta(\xi) &= - \int_0^\infty \int_0^1 F^{-1}(\alpha, y)T'_\beta(\alpha)d\alpha d\Psi(y) \\ &= - \int_0^\infty \int_0^1 y^3(2 + \frac{3\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}) \frac{\beta}{\beta-1} [(1-\alpha)^{\beta-1} - \alpha^{\beta-1}] 5e^{-5y} d\alpha dy. \end{aligned}$$

By using Monte-Carlo simulation, we have $PTE_{0.5}(\xi) = 0.703$ and $PTE_{1.5}(\xi) = 0.323$.

Example 3 Suppose that τ is an uncertain variable with uncertainty distribution $\mathcal{N}(2, 3)$. Tsallis entropy of τ is

$$\begin{aligned} PTE_\beta(\xi) &= - \int_0^1 \Phi^{-1}(\alpha, y)T'_\beta(\alpha)d\alpha d\Psi(y) \\ &= - \int_0^1 (2 + \frac{3\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}) \frac{\beta}{\beta-1} [(1-\alpha)^{\beta-1} - \alpha^{\beta-1}] d\alpha. \end{aligned}$$

Liu [12] showed the Shanon (logarithm) entropy for $\mathcal{N}(e, \sigma)$ is equal to $\frac{\pi\sigma}{\sqrt{3}}$. Thus,

$$H[\tau] = \frac{3\pi}{\sqrt{3}} = \pi\sqrt{3} = 5.43.$$

By using Monte-Carlo simulation, we calculate Tsallis entropy for different values of β in Figure 1. Also, Figure 1 displays when β tends to one, Tsallis entropy converges to logarithm entropy.

5 Portfolio Optimization of Uncertain Random Returns

In this section, in order to solve the portfolio selection problems with uncertain random returns, we propose two mean-variance-entropy models via partial Tsallis entropy.

Figure 1: Tsallis Entropy for Uncertain Normal Variable

Assume we have n securities with uncertain random returns $\xi_1, \xi_2, \dots, \xi_n$, respectively. Also, consider x_i 's as investment proportions in security $i, i = 1, 2, \dots, n$. In order to optimize the portfolio selection problems, we present two following mean-variance-entropy models.

$$\left\{ \begin{array}{l} \max_{x_i} E[x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n] \\ \text{subject to :} \\ V[x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n] \leq \delta \\ PTE_\beta[x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n] \geq \lambda \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n, \end{array} \right.$$

where, δ and λ are predetermined parameters. Another model is multi-objective mean-variance-

entropy model as follows:

$$\left\{ \begin{array}{l} \max_{x_i} E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \min_{x_i} V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \min_{x_i} PTE_\beta[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to :} \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n, \end{array} \right.$$

We want to convert the above models to crisp models. Thus, by using Theorem 1, we can obtain

$$E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] = \sum_{i=1}^n x_i E[\xi_i].$$

Furthermore, Theorem 2 concludes that

$$V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] = \sum_{i=1}^n x_i^2 V[\xi_i] + 2 \sum_{i=1}^n \sum_{j=i+1}^n x_i x_j Cov(\xi_i, \xi_j).$$

Finally, by using Theorem 6, we have

$$PTE_\beta[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] = \sum_{i=1}^n x_i PTE_\beta[\xi_i].$$

Therefor, the above mean-variance-entropy problems are interchanged to ordinary(crisp) optimization problems as follows:

$$\left\{ \begin{array}{l} \max_{x_i} \sum_{i=1}^n x_i E[\xi_i] \\ \text{subject to :} \\ \sum_{i=1}^n x_i^2 V[\xi_i] + 2 \sum_{i=1}^n \sum_{j=i+1}^n x_i x_j Cov(\xi_i, \xi_j) \leq \delta \\ \sum_{i=1}^n x_i PTE_\beta[\xi_i] \geq \lambda \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n, \end{array} \right.$$

Also, the multi-objective problem is reduced as follows:

$$\left\{ \begin{array}{l} \max_{x_i} \sum_{i=1}^n x_i E[\xi_i] \\ \min_{x_i} \sum_{i=1}^n x_i^2 V[\xi_i] + 2 \sum_{i=1}^n \sum_{j=i+1}^n x_i x_j Cov(\xi_i, \xi_j) \\ \min_{x_i} \sum_{i=1}^n x_i PTE_{\beta}[\xi_i] \\ \text{subject to :} \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n, \end{array} \right.$$

Example 4 Consider we have five securities with uncertain random returns shown in Table 1, with $\xi_i = \tau_i + \eta_i$, $i = 1, 2, \dots, n$.

Table 1 **Securities**

No	Uncertain Term	Random Term
1	$\tau_1 \sim \mathcal{N}(0.032, 0.18)$	$\eta_1 \sim \mathcal{N}(0.3, 0.09)$
2	$\tau_2 \sim \mathcal{N}(0.039, 0.21)$	$\eta_2 \sim \mathcal{N}(0.1, 0.16)$
3	$\tau_3 \sim \mathcal{N}(-0.031, 0.16)$	$\eta_3 \sim \mathcal{N}(-0.01, 0.05)$
4	$\tau_4 \sim \mathcal{N}(-0.2, 0.26)$	$\eta_4 \sim \mathcal{N}(-0.02, 0.06)$
5	$\tau_5 \sim \mathcal{N}(-0.2, 0.3)$	$\eta_5 \sim \mathcal{N}(0.02, 0.16)$

After calculating expectations, variances and partial Tsallis entropy of order $\beta = 1.5$ via Monte Carlo simulation, we want to optimize the mean-variance-entropy with constrained variance and

entropy.

$$\left\{ \begin{array}{l} \max_{x_i} 0.332x_1 + 0.139x_2 - 0.041x_3 - 0.22x_4 - 0.18x_5 \\ \text{subject to :} \\ 0.1224x_1^2 + 0.2041x_2^2 + 0.0756x_3^2 + 0.1276x_4^2 + 0.25x_5^2 + 0.0756x_1x_2 \\ +0.0576x_1x_3 + 0.093x_1x_4 + 0.185x_1x_5 + .0672x_2x_3 + 0.1144x_2x_4 \\ +0.126x_2x_5 + 0.0832x_3x_4 + 0.096x_3x_5 + 0.156x_4x_5 \leq 2 \\ 0.239x_1 + 0.286x_2 + 0.22x_3 + 0.344x_4 + 0.406x_5 \geq 0.35 \\ x_1 + x_2 + \dots + x_5 = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, 5, \end{array} \right. \quad (1)$$

Now, by solving the crisp optimization problem, we obtain the optimal solutions as Table 2. Also, the expected value of the total returns is -0.007591 . The convergence curve of the expected value is displayed in Figure 2.

Table 2 Portfolio Proportion in Model (1)

No	1	2	3	4	5
Proportion of Portfolio	0.335	0.003	0	0	0.662

Now, we want to optimize the portfolio selection problem via multi-objective model.

$$\left\{ \begin{array}{l} \max_{x_i} 0.332x_1 + 0.139x_2 - 0.041x_3 - 0.22x_4 - 0.18x_5 \\ \min_{x_i} 0.1224x_1^2 + 0.2041x_2^2 + 0.0756x_3^2 + 0.1276x_4^2 + 0.25x_5^2 + 0.0756x_1x_2 \\ +0.0576x_1x_3 + 0.093x_1x_4 + 0.185x_1x_5 + .0672x_2x_3 + 0.1144x_2x_4 \\ +0.126x_2x_5 + 0.0832x_3x_4 + 0.096x_3x_5 + 0.156x_4x_5 \\ \min_{x_i} 0.239x_1 + 0.286x_2 + 0.22x_3 + 0.344x_4 + 0.406x_5 \\ \text{subject to :} \\ x_1 + x_2 + \dots + x_5 = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, 5, \end{array} \right. \quad (2)$$

Figure 2: Convergence Curve in Model (1)

Here, by solving the ordinary optimization problem, we get the optimal proportions of five securities as Table 3. Also, the optimal expected value of the total return is 0.083696. Furthermore, optimal variance and entropy are equal to 0.056241 and 0.237339, respectively.

Table 3		Portfolio Proportion in Model (2)				
No		1	2	3	4	5
Proportion of Portfolio		0.29	0.12	0.56	0.03	0

6 Conclusions

This paper proposed the concept of partial Tsallis entropy of uncertain random variables as an extension Shanon entropy, and studied some properties of this concept. Also, in order to calculate partial Tsallis entropy for uncertain random variables, Monte-Carlo simulation was proposed. Furthermore, we showed how portfolio selection problems resolution can be optimize by using mean-variance-Tsallis entropy, and provided several examples for better illustration of main results.

Author Contributions

Conceptualization, Zhenhua He; Formal analysis, Hamed Ahmadzade; Methodology and Editing, Habib Naderi; Resources and Editing, Hassan Rezaei; Software, Kamran Rezaei; Writing original draft and Revising, Hamed Ahmadzade; Funding acquisition, Zhenhua He. All authors have read and agreed to the published version of the manuscript.

Compliance with ethical standards

Conflict of interest.

The authors declare that they have no conflict of interest.

Ethics statement.

This work did not involve any active collection of human data.

Competing interests statement.

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