A Stochastic SIS Epidemic Model with Ornstein-Uhlenbeck Process and Brown Motion

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A stochastic SIS epidemic model with Ornstein-Uhlenbeck process and Brown motion

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Abstract This paper studies a stochastic differential equation SIS epidemic model, disturbed randomly by the mean-reverting Ornstein-Uhlenbeck process and Brownian motion. We prove the existence and uniqueness of the positive global solutions of the model and obtain the controlling conditions for the extinction and persistence of the disease. The results show that when the basic reproduction number $\mathcal{R}_0^s < 1$, the disease will extinct, on the contrary, when the basic reproduction number $\mathcal{R}_0^s > 1$, the disease will persist. Furthermore, we can inhibit the outbreak of the disease by increasing the intensity of volatility ζ or decreasing the speed of reversion ϑ , respectively. Finally, we give some numerical examples to verify these results.

Keywords Epidemic model · Ornstein-Uhlenbeck process · Brown motion · Basic reproduction number · Extinction

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1 Introduction

As we all know, the epidemic infectious diseases often spread widely all over the world, posing a great threat to human life and property safety [1]. Since 1927, Kermack and McKendrick proposed the classical SIR model when studying the black death epidemic in London [2]. After nearly a hundred years of development, the epidemic compartmental model and its theoretical results have been widely expanded, and many excellent research findings have been obtained [3]. Many mathematicians have studied more epidemic infectious disease models based on the SIR model, such as the SEIR model [4, 5]; the $SEIR^{+CAQ}$ model [6]; the SEIRS model [7]; the SEIQR model [8, 9]; the SVEIR model [10]. In the field of public health, the SIR model and its extensions are widely used in the research and prediction of infectious diseases, such as the Ebola virus [11]; COVID-19 [12]; hand, foot, and mouth disease [13]; SARS [14].

Furthermore, many research findings indicate that various biological systems will be inevitably affected by white environmental noise for instance, natural disasters such as earthquakes, hurricanes, and floods [15]. Similarly, studying infectious disease models with stochastic disturbances has better practical significance [16]. In the past few decades, an increasing number of scholars have studied various forms of stochastic epidemic models and made many functional theoretical analyses and numerical discussions on the corresponding models. For instance, Krause [15] and Chen [16] have studied the nonlinear stochastic SIS epidemic system with white noise; Chang [17], Huang [18] and Gao [19] have proposed the nonlinear stochastic SIS epidemic system with multiple noises; Han et al. [20] have studied the stochastic SIS epidemic models with saturated incidence rate; Wang et al. [21] have proposed the stochastic SIS model with the mean regression Ornstein-Uhlenbeck process. In addition, one proved the stochastic persistence and the extinction of the disease with different conditions on the intensities of noises and different parameters of the model [24–31], and showed the stationary distribution of the existence of the disease and the expectation and the variance of the system [32–34].

In [35], Gray et al. have proposed the following stochastic SIS epidemic model with Brownian motion

$$\begin{cases} dS(t) = [uN - \bar{\lambda}S(t)I(t) + rI(t) - uS(t)]dt - \alpha_1 S(t)I(t)dW_1(t), \\ dI(t) = [\bar{\lambda}S(t)I(t) - (r + u)I(t)]dt + \alpha_1 S(t)I(t)dW_1(t). \end{cases} \quad (1)$$

Where u is the rate of birth, r is the rate of cure recovery, λ is the coefficient of the spread of the disease, α_1 is the white noise, $W_1(t)$ standard Brownian motion.

The corresponding ODE system of the SDE system (1) is

$$\begin{cases} dS(t) = [uN - \bar{\lambda}S(t)I(t) + rI(t) - uS(t)]dt, \\ dI(t) = [\bar{\lambda}S(t)I(t) - (r + u)I(t)]dt. \end{cases} \quad (2)$$

Here $S(t)$ is the number of the susceptible at the time t , $I(t)$ is the number of the infectious at time t . And $S(0) > 0$, $I(0) > 0$,

$$N = I(t) + S(t). \quad (3)$$

We can obtain the SDE system (1) from the system (2) by the following methods,

$$\bar{\lambda}dt \rightarrow \bar{\lambda}dt + \alpha_1 dW_1(t) \quad (4)$$

In view of (3), we can reshape the SDE system (1) into the following form

$$dI(t) = [\bar{\lambda}(N - I(t))I(t) - (r + u)I(t)]dt + \alpha_1(N - I(t))I(t)dW_1(t). \quad (5)$$

Recently, based on the work of Gray et al. [35], Cai et al. [36, 37] have studied a stochastic SIS epidemic model with two independent Brownian motions

$$\begin{cases} dS(t) = (uN - \bar{\lambda}S(t)I(t) + rI(t) - uS(t))dt \\ \quad - \alpha_1 S(t)I(t)dW_1(t) + \alpha_2 I(t)\sqrt{N - I(t)}dW_2(t), \\ dI(t) = (\bar{\lambda}S(t)I(t) - (r + u)I(t))dt \\ \quad + \alpha_1 S(t)I(t)dW_1(t) - \alpha_2 I(t)\sqrt{N - I(t)}dW_2(t). \end{cases} \quad (6)$$

The meaning of all parameters is consistent with the SDE system (1), and α_2 is the white noise, $W_2(t)$ standard Brownian motion.

Combine the work of Gray et al. [35], we can obtain the SDE model (6) from the SDE system (1) by disturbed randomly in the form as follows,

$$(r + u)dt = (r + u)dt + \alpha_2 \sqrt{N - I(t)}dW_2(t) \quad (7)$$

Then, in the light of (3), we can reshape the SDE system (6) into the following form

$$\begin{aligned} dI(t) = & [(\bar{\lambda}(N - I(t))I(t) - (r + u)I(t))]dt \\ & + \alpha_1(N - I(t))I(t)dW_1(t) - \alpha_2 I(t)\sqrt{N - I(t)}dW_2(t). \end{aligned} \quad (8)$$

On the basis of Gray et al. [35] and Cai et al. [36, 37], if we consider that the parameter $\bar{\lambda}$ is the mean-reverting Ornstein-Uhlenbeck process in a randomly changing environment (see [21–23]), that is

$$d\bar{\lambda}(t) = \vartheta(\lambda_e - \bar{\lambda}(t))dt + \zeta dW_i(t), \quad (i = 1, 2). \quad (9)$$

ϑ is the speed of reversion, ζ is the intensity of volatility, and they are all positive constants.

Integrating both ends of the equation (9) in the stochastic integral format, we obtain

$$\bar{\lambda}(t) = \lambda_e + (\lambda_0 - \lambda_e)e^{-\vartheta t} + \zeta \int_0^t e^{-\vartheta(t-s)} dW_i(s), \quad (i = 1, 2). \quad (10)$$

and $\lambda_0 := \bar{\lambda}(0)$.

We can get the expectation and the variance of $\bar{\lambda}(t)$ is, respectively

$$\mathbf{E}[\bar{\lambda}(t)] = \lambda_e + (\lambda_0 - \lambda_e)e^{-\vartheta t}, \quad (11)$$

$$\text{Var}[\bar{\lambda}(t)] = \frac{\zeta^2}{2\vartheta} (1 - e^{-2\vartheta t}). \quad (12)$$

Therefore, we can find out that $\zeta \int_0^t e^{-\vartheta(t-s)} dW_i(s) \sim N\left(0, \frac{\zeta^2}{2\vartheta} (1 - e^{-2\vartheta t})\right)$, that is

$$\zeta \int_0^t e^{-\vartheta(t-s)} = \frac{\zeta}{\sqrt{2\vartheta}} \sqrt{1 - e^{-2\vartheta t}} \frac{dW_i(t)}{dt}, \quad (i = 1, 2.) \quad (13)$$

and $W_i(t)$ is a Brownian motion.

Thus, the equation (10) can be written as

$$\bar{\lambda}(t)dt = \lambda_e dt + (\lambda_0 - \lambda_e) e^{-\vartheta t} dt + \alpha_1(t) dW_1(t), \quad (14)$$

where

$$\alpha_1(t) = \frac{\zeta}{\sqrt{2\vartheta}} \sqrt{1 - e^{-2\vartheta t}}. \quad (15)$$

Further, we can rewrite the SDE model (6) almost surely as

$$\begin{cases} dS(t) = \left[uN - (\lambda_e + (\lambda_0 - \lambda_e)e^{-\vartheta t})S(t)I(t) + rI(t) - uS(t) \right] dt \\ \quad - \alpha_1(t)S(t)I(t)dW_1(t) + \alpha_2 I(t) \sqrt{N - I(t)} dW_2(t), \\ dI(t) = \left[(\lambda_e + (\lambda_0 - \lambda_e)e^{-\vartheta t})S(t)I(t) - (r + u)I(t) \right] dt \\ \quad + \alpha_1(t)S(t)I(t)dW_1(t) - \alpha_2 I(t) \sqrt{N - I(t)} dW_2(t). \end{cases} \quad (16)$$

In view of (3), we can reshape the SDE (16) into the following form,

$$\begin{aligned} dI(t) = & [(\lambda_e + (\lambda_0 - \lambda_e)e^{-\vartheta t})(N - I(t))I(t) - (r + u)I(t)]dt \\ & + \alpha_1(t)(N - I(t))I(t)dW_1(t) - \alpha_2 I(t) \sqrt{N - I(t)} dW_2(t), \end{aligned} \quad (17)$$

satisfy the initial condition $I(0) = I_0 \in (0, N)$.

Remark I. When $\vartheta \rightarrow \infty$ and $\alpha_2 \rightarrow 0$, the SDE system (17) will tend to the system (2), when $\alpha_2 \rightarrow 0$, the SDE system (17) will tend to the SDE system (1.13) in [21].

Based on the thoughts and methods of Gray et al. [35] and Cai et al. [36, 37], we conduct research and discussion on a stochastic SIS epidemic model with the Ornstein-Uhlenbeck process and Brown motion. We make full use of the relevant knowledge of stochastic differential equations and some classic mathematical methods to discuss the existence and uniqueness of the positive global solution for the SDE system (17) and the conditions for persistence and extinction of the disease. Furthermore, focus on the effect of the intensity of volatility and the speed of reversion on the disease dynamics of the SDE system (17). Finally, we provide some numerical examples to verify the correctness of the theoretical analysis and discuss and summarize the results of the numerical experiments.

2 Existence and Uniqueness of the Global Positive Solution

Theorem 1. For any initial conditions $I_0 \in (0, N)$, the SDE system (17) exists a unique positive global solution $I(t) \in (0, N)$ for all $t \geq 0$.

Proof. Consider the following stochastic differential equation

$$\begin{aligned} dU(t) = & \left[(\lambda_e + (\lambda_0 - \lambda_e) e^{-\theta t}) (N - e^{U(t)}) - (r + u) \right] \\ & - \frac{\xi^2}{4\theta} (1 - e^{-2\theta t}) (N - e^{U(t)})^2 - \frac{\alpha_2^2}{2} (N - e^{U(t)}) \Big\} dt \quad (18) \\ & + \alpha_1(t) (N - e^{U(t)}) dW_1(t) - \alpha_2 \sqrt{(N - e^{U(t)})} dW_2(t). \end{aligned}$$

Obviously, for the initial conditions $U(0) = \ln I_0$, the coefficients of the system (18) obey the local Lipschitz condition. Thus, the system (18) has a local solution $V(t), t \in (0, \tau_e)$, where τ_e is the explosion time.

Therefore, we can find out that $I(t) = e^{U(t)}, t \in [0, \tau_e)$ is the local positive solution of the SDE system (17) with the initial conditions I_0 by the $It\hat{o}'s$ formula.

Next, we will prove the solution of the SDE system (17) is global, namely prove $\tau_e = \infty$.

Let $k_0 > 0$ be sufficiently large so that $I_0 \in [\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the explosion time

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : I(t) \notin \left(\frac{1}{K}, (1 - \frac{1}{K})N \right) \right\}.$$

And we set $\inf \Phi = \infty$, because when $k \rightarrow \infty$, the sequence τ_k is monotonically increasing and has an upper bound. Let $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$, then $\tau_\infty < \tau_e$.

Next, we assert $\tau_\infty = \infty$.

If the above conclusion is not true, for $\forall \varepsilon \in (0, 1), \exists T > 0$, so that when $t > T$, there is $P\{\tau_\infty \leq T\} > \varepsilon$.

Therefore, for $\mathcal{Q}_k = \{\tau_k \leq T\}$, there is $k_1 \geq k_0$, for all $k \geq k_1$ has $P(\mathcal{Q}_k) \geq \varepsilon$.

Next, we consider the following Lyapunov function

$$V(I) = \frac{1}{I} + \frac{1}{N - I}$$

then,

$$\begin{aligned}
dV(I(t)) &= \left\{ I(t) \left(-\frac{1}{I(t)^2} + \frac{1}{(N-I(t))^2} \right) \left[(\lambda_e + (\lambda_0 - \lambda_e)e^{-\theta t})(N-I(t)) - u - r \right] \right\} dt \\
&\quad + \left\{ \alpha_1^2 I(t)^2 (N-I(t))^2 \left(\frac{1}{I(t)^3} + \frac{1}{(N-I(t))^3} \right) \right. \\
&\quad \left. - \alpha_2^2 I(t)^2 (N-I(t)) \left(\frac{1}{I(t)^3} + \frac{1}{(N-I(t))^3} \right) \right\} dt \\
&\quad + \left\{ \left[-\frac{1}{I(t)^2} + \frac{1}{(N-I(t))^2} \right] \alpha_1 I(t)(N-I(t)) \right\} dW_1(t) \\
&\quad - \left\{ \left[-\frac{1}{I(t)^2} + \frac{1}{(N-I(t))^2} \right] \alpha_2 I(t) \sqrt{(N-I(t))} \right\} dW_2(t) \\
&= LV(I(t))dt + \left\{ \left[-\frac{1}{I(t)^2} + \frac{1}{(N-I(t))^2} \right] \alpha_1 I(t)(N-I(t)) \right\} dW_1(t) \\
&\quad - \left\{ \left[-\frac{1}{I(t)^2} + \frac{1}{(N-I(t))^2} \right] \alpha_2 I(t) \sqrt{(N-I(t))} \right\} dW_2(t),
\end{aligned} \tag{19}$$

where,

$$\begin{aligned}
LV(I(t)) &= I(t) \left(-\frac{1}{I(t)^2} + \frac{1}{(N-I(t))^2} \right) \left[(\lambda_e + (\lambda_0 - \lambda_e)e^{-\theta t})(N-I(t)) - u - r \right] \\
&\quad + \alpha_1^2 I(t)^2 (N-I(t))^2 \left(\frac{1}{I(t)^3} + \frac{1}{(N-I(t))^3} \right) \\
&\quad - \alpha_2^2 I(t)^2 (N-I(t)) \left(\frac{1}{I(t)^3} + \frac{1}{(N-I(t))^3} \right) \\
&\leq \frac{r+u}{I(t)} + \frac{(\lambda_e + (\lambda_0 - \lambda_e)e^{-\theta t})N}{N-I(t)} + \alpha_1^2 N^2 \left(\frac{1}{I(t)} + \frac{1}{N-I(t)} \right) \\
&\leq CV(I(t)).
\end{aligned} \tag{20}$$

when $\lambda_0 < \lambda_e$, then $\lambda_e + (\lambda_0 - \lambda_e)e^{-\theta t} \leq \lambda_e$; when $\lambda_0 > \lambda_e$, then $\lambda_e + (\lambda_0 - \lambda_e)e^{-\theta t} \leq \lambda_0$.

Thus, we obtain $LV \leq DV$, where $D = (r+u) \vee ((\lambda_0 \vee \lambda_e)N) + \alpha_1^2 N^2$.

Next, taking an integral on both sides of (20) on $[0, T \wedge \tau_k]$ and taking the expectation

$$EV(I(T \wedge \tau_k)) \leq V(I_0) + \int_0^{T \wedge \tau_k} D dt \leq V(I_0) + DT. \tag{21}$$

For $\forall w \in \Omega_k, \exists I(\tau_k, w) = k$ or $\frac{1}{k}$, so that

$$V(I(T \wedge \tau_k, \omega)) \geq \min \left\{ \left(k + \frac{1}{N - \frac{1}{k}} \right), \left(\frac{1}{k} + \frac{1}{N - k} \right) \right\}. \tag{22}$$

Therefore,

$$\begin{aligned} EV(I(T \wedge \tau_k)) &\geq E[1_{\Omega_k} V(I(t \wedge \tau_k))] \\ &\geq \varepsilon \min \left\{ \left(k + \frac{1}{N - \frac{1}{k}}\right), \left(\frac{1}{k} + \frac{1}{N - k}\right) \right\}. \end{aligned} \quad (23)$$

Where, 1_{Ω_k} is an indicator function defined in Ω_k .

$$V(I_0) + DT \geq \varepsilon \min \left\{ \left(k + \frac{1}{N - \frac{1}{k}}\right), \left(\frac{1}{k} + \frac{1}{N - k}\right) \right\}. \quad (24)$$

Therefore, when $k \rightarrow \infty$, then $\infty > V(I_0) + DT \geq \infty$, this is contradiction.

Therefore, $\tau_\infty = \infty$, it means that the SDE system (17) exists a unique global positive solution. $\square$

3 Extinction and stochastic persistence of the diseases

3.1 Extinction

With the findings of [21], [35] and [36], we define the basic reproduction number as follows, respectively,

$$\mathcal{R} = \frac{\lambda_e N}{r + u}, \quad \mathcal{R}_1 = \frac{\lambda_e N}{r + u} - \frac{\alpha_1^2 N^2}{2(r + u)}, \quad \mathcal{R}_2 = \frac{\lambda_e N}{r + u} - \frac{\zeta^2 N^2}{4\vartheta(r + u)}, \quad \mathcal{R}_0 = \frac{\lambda_e N}{r + u} - \frac{\alpha_2^2 N}{2(r + u)}.$$

The $\mathcal{R}$ corresponds to the system (2), the $\mathcal{R}_1$ corresponds to the SDE system (1), the $\mathcal{R}_2$ corresponds to the SDE system (1.13) in [21]. And the corresponding system of the $\mathcal{R}_0$ is

$$dI(t) = \left[((\lambda_e + (\lambda_0 - \lambda_e) e^{-\vartheta t})(N - I(t))I(t) - (r + u)I(t)) \right] dt - \alpha_2 I(t) \sqrt{N - I(t)} dW_2(t). \quad (25)$$

In [36], Cai et al. define the basic reproduction number of the SDE system (1.5) (it corresponds to the SDE system (8) in this paper)

$$\mathcal{R}^s = \frac{\lambda_e N}{r + u} - \frac{\alpha_1^2 N^2 + \alpha_2^2 N}{2(r + u)}. \quad (26)$$

Similarly, we also define the basic reproduction number of the SDE system (17) as follows

$$\mathcal{R}_0^s = \frac{\lambda_e N}{r + u} - \frac{\alpha_2^2 N}{2(r + u)} - \frac{\zeta^2 N^2}{4\vartheta(r + u)}. \quad (27)$$

Remark II. When $\alpha_2 = 0$ and $\vartheta \rightarrow \infty$, the $\mathcal{R}_0^s$ will degenerate into the $\mathcal{R}$, the corresponding SDE system (17) will tend to the system (2). When $\alpha_2 = 0$, $\vartheta = 0.5$ and $\zeta = \alpha_1$, the $\mathcal{R}_0^s$ will degenerate into the $\mathcal{R}_1$, the corresponding SDE system (17) will tend to the system (1). When $\alpha_2 = 0$, the $\mathcal{R}_0^s$ will degenerate into the $\mathcal{R}_2$, the corresponding SDE system (17) will tend to the system (1.13) in [21]. When $\zeta = 0$ or $\vartheta \rightarrow \infty$, the $\mathcal{R}_0^s$ will degenerate into the $\mathcal{R}_0$, the corresponding SDE system (17) will

tend to the system (25). When $\zeta = \alpha_1$ and $\vartheta = 0.5$, the $\mathcal{R}_0^s$ will degenerate into the $\mathcal{R}^s$, the corresponding SDE system (17) will tend to the system (8). Thus, the $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_0$ and $\mathcal{R}^s$ can be regarded as the promotion and expansion of the $\mathcal{R}_0^s$. Furthermore, the discussion about the extinction and the stochastic persistence of the diseases in this paper will have more extensive and practical significance.

So we have the following conclusion.

Theorem 2. When

$$\mathcal{R}_0^s < 1 \text{ and } \zeta^2 \leq \frac{2\lambda_e\vartheta - \vartheta\alpha_2^2}{N}, \quad (28)$$

or

$$\zeta^2 > \max \left\{ \frac{2\lambda_e\vartheta - \vartheta\alpha_2^2}{N}, \frac{\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2}{r+u} \right\}, \quad (29)$$

For $\forall I_0 \in (0, N)$, all solutions of the SDE system (17) satisfy $\lim_{t \rightarrow \infty} I(t) = 0$, that is $I(t)$ will go to die out.

Proof. We obtain by the $It\delta's$ formula,

$$\begin{aligned} \ln(I(t)) &= \ln I_0 + \int_0^t G(I(s))ds + \Phi(t) + \int_0^t \alpha_1(s)(N - I(s))dW_1(s) \\ &\quad - \int_0^t \alpha_2(s)\sqrt{N - I(s)}dW_2(s). \end{aligned} \quad (30)$$

Where $G : \mathbb{R} \rightarrow \mathbb{R}$ as

$$G(x) = \lambda_e(N - x) - (r + u) - \frac{\zeta^2}{4\vartheta}(N - x)^2 - \frac{\alpha_2^2}{2}(N - x), \quad (31)$$

$$\Phi(t) = \int_0^t \left((\lambda_0 - \lambda_e)e^{-\vartheta s}(N - I(s)) + \frac{\zeta^2 e^{-2\vartheta s}(N - I(s))^2}{4\vartheta} \right) ds. \quad (32)$$

We scale $\Phi(t)$ to get

$$\begin{aligned} \Phi(t) &\leq |\lambda_0 - \lambda_e|N \int_0^t e^{-\vartheta s} ds + \frac{\zeta^2 N^2}{4\vartheta} \int_0^t e^{-2\vartheta s} ds \\ &= \frac{|\lambda_0 - \lambda_e|N}{\vartheta} (1 - e^{-\vartheta t}) + \frac{\zeta^2 N^2}{8\vartheta^2} (1 - e^{-2\vartheta t}), \end{aligned} \quad (33)$$

$$\begin{aligned} G(I(s)) &= -\frac{\zeta^2}{4\vartheta}I(s)^2 - \left(\lambda_e - \frac{\zeta^2 N}{2\vartheta} - \frac{\alpha_2^2}{2} \right) I(s) + \lambda_e N - (r + u) - \frac{\zeta^2 N^2}{4\vartheta} - \frac{\alpha_2^2 N}{2} \\ &\leq \lambda_e N - (r + u) - \frac{\zeta^2 N^2}{4\vartheta} - \frac{\alpha_2^2 N}{2}. \end{aligned} \quad (34)$$

Next, we will discuss the two different situations for Theorem 2.

(i). With the condition (28) in Theorem 2.

When $\zeta^2 \leq \frac{2\lambda_e\vartheta - \vartheta\alpha_2^2}{N}$, for $I(s) \in (0, N)$, we have

$$\begin{aligned} G(I(s)) &\leq \lambda_e N - (r + u) - \frac{\zeta^2 N^2}{4\vartheta} - \frac{\alpha_2^2 N}{2} = (r + u) \left(\frac{\lambda_e N}{r + u} - \frac{\zeta^2 N^2}{4\vartheta(r + u)} - \frac{\alpha_2^2 N}{2(r + u)} - 1 \right) \\ &= (r + u)(\mathcal{R}_0^s - 1), \end{aligned} \quad (35)$$

We scale (30) to obtain

$$\begin{aligned} \frac{1}{t} \ln(I(t)) &\leq \frac{1}{t} \ln I_0 + \frac{\Phi(t)}{t} + \frac{1}{t} \int_0^t \alpha_1(s)(N - I(s))dW_1(s) \\ &\quad - \frac{1}{t} \int_0^t \alpha_2(s)\sqrt{N - I(s)}dW_2(s) + \frac{1}{t} \int_0^t G(I(s))ds. \end{aligned} \quad (36)$$

Obviously, $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = 0$, then

$$\int_0^t \alpha_1(s)(N - I(s))dW_1(s) = \frac{\zeta}{\sqrt{2\vartheta}} \int_0^t \sqrt{1 - e^{-2\vartheta s}}(N - I(s))dW_1(s),$$

and $\int_0^t \alpha_2(s)\sqrt{N - I(s)}dW_2(s)$ is a local martingale. With the large number theorem of the local martingale, we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \alpha_1(s)(N - I(s))dW_1(s) &= 0, \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \alpha_2(s)\sqrt{N - I(s)}dW_2(s) &= 0. \end{aligned}$$

When $\mathcal{R}_0^s < 1$, taking upper limit on both sides of (36), we obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq (r + u)(\mathcal{R}_0^s - 1) < 0. \quad (37)$$

(ii). With the condition (29) in Theorem 2,

$$\begin{aligned} G(I(s)) &= -\frac{\zeta^2}{4\vartheta} \left(I(s) - \frac{\zeta^2 N - 2\lambda_e\vartheta + \alpha_2^2\vartheta}{\zeta^2} \right)^2 + \frac{\lambda_e^2\vartheta}{\zeta^2} + \frac{\alpha_2^4\vartheta}{4\zeta^2} - \frac{\lambda_e\vartheta\alpha_2^2}{\zeta^2} - (r + u) \\ &\leq \frac{\lambda_e^2\vartheta}{\zeta^2} + \frac{\alpha_2^4\vartheta}{4\zeta^2} - \frac{\lambda_e\vartheta\alpha_2^2}{\zeta^2} - (r + u), \end{aligned} \quad (38)$$

When the condition (2.9) is established, as the same as the above discussion, we can obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq \frac{\lambda_e^2\vartheta}{\zeta^2} + \frac{\alpha_2^4\vartheta}{4\zeta^2} - \frac{\lambda_e\vartheta\alpha_2^2}{\zeta^2} - (r + u) < 0. \quad (39)$$

From (2.37) and (2.39), according to the operational nature of the limit, we can obtain

$$\lim_{t \rightarrow \infty} I(t) = 0.$$

Thus, we have completed the proof of Theorem 2. $\square$

3.2 Stochastic Persistence

Theorem 3. When $\mathcal{R}_0^s > 1$, the solutions of the SDE system (17) which satisfy the initial value $I_0 \in (0, N)$ satisfy

$$\limsup_{t \rightarrow \infty} I(t) \geq \psi, \quad (40)$$

and

$$\liminf_{t \rightarrow \infty} I(t) \leq \psi. \quad (41)$$

Where ψ is a positive root of the following equation in $(0, N)$

$$G(\psi) = \lambda_e(N - \psi) - (r + u) - \frac{\zeta^2}{4\vartheta}(N - \psi)^2 - \frac{\alpha_2^2}{2}(N - \psi) = 0, \quad (42)$$

i.e.

$$\psi = \frac{\vartheta\alpha_2^2 + N\zeta^2 - 2\vartheta\lambda_e + 2\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r + u)\right)}}{\zeta^2}. \quad (43)$$

In other words, $I(t)$ will rise to or above the level ψ endlessly and frequently. That is, this disease will persist.

Proof. When $\mathcal{R}_0^s > 1$, we can find that the $G(x) = 0$ has a positive root ψ and a negative root, then

$$G(0) = (r + u)(\mathcal{R}_0^s - 1) > 0,$$

$$G(N) = -(r + u) < 0.$$

Owing to

$$G(I(s_0)) \geq G(I(s)), I(s_0) = \frac{N\zeta^2 - 2\lambda_e\vartheta + \alpha_2^2\vartheta}{\zeta^2}.$$

Thus, when $\zeta \in (0, N)$,

$$G(I(t)) > 0 \text{ is increasing strictly at } (0, 0 \vee \frac{N\zeta^2 - 2\lambda_e\vartheta + \alpha_2^2\vartheta}{\zeta^2}), \quad (44)$$

$$G(I(t)) > 0 \text{ is decreasing strictly at } (0 \vee \frac{N\zeta^2 - 2\lambda_e\vartheta + \alpha_2^2\vartheta}{\zeta^2}, \psi), \quad (45)$$

$$G(I(t)) < 0 \text{ is decreasing strictly at } (\psi, N). \quad (46)$$

Next, we will prove $\limsup_{t \rightarrow \infty} I(t) \geq \psi$. When it is not true, then we assume that $\delta \in (0, 1)$ is sufficiently small satisfy,

$$P(\mathcal{Q}_1) > \delta,$$

and $\mathcal{Q}_1 = \left\{ \limsup_{t \rightarrow \infty} I(t) \leq \psi - 2\delta \right\}$. Hence, for $\delta \in \mathcal{Q}_1$, we will find that there is a $T = T(\delta) > 0$ satisfy

$$I(t, \delta) \leq \psi - \delta, \quad t \geq T(\vartheta). \quad (47)$$

Thus, we can choose a δ , which is sufficiently small satisfy $G(0) > G(\psi - \delta)$.

From (44), (45), and (47), we obtain

$$G(I) \geq G(\psi - \delta), \quad t \geq T(\delta). \quad (48)$$

On the other hand, with the law of large number for local martingales, $\exists \Omega_2 \in \mathcal{Q}$ so that $\forall \delta \in \Omega_2$, we have

$$\mathbb{P}(\Omega_2) = 1,$$

that is

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \alpha_1(s)(N - I(s, \delta))dW_1(s, \delta) = 0, \quad (49)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \alpha_2(s) \sqrt{(N - I(s, \delta))}dW_2(s, \delta) = 0. \quad (50)$$

(i) If $\lambda_0 > \lambda_e$, for $I(s) \in (0, N)$,

$$\Phi(t) = \int_0^t \left((\lambda_0 - \lambda_e) e^{-\vartheta s} (N - I(s)) + \frac{\zeta^2 e^{-2\vartheta s} (N - I(s))^2}{4\vartheta} \right) ds \geq 0. \quad (51)$$

Set any $\delta \in \Omega_1 \cap \Omega_2$, for $t \geq T(\delta)$, from (48), (49) and (50), we obtain

$$\begin{aligned} \ln(I(t)) &\geq \ln I_0 + \int_0^T G(I(s))ds + G(\psi - \delta)(t - T) + \int_0^t \alpha_1(s)(N - I(s))dW_1(s) \\ &\quad - \int_0^t \alpha_2(s) \sqrt{(N - I(s))}dW_2(s), \end{aligned} \quad (52)$$

then

$$\liminf_{t \rightarrow \infty} \frac{\ln I(t)}{t} \geq G(\psi - \delta) > 0, \quad (53)$$

as a result,

$$\lim_{t \rightarrow \infty} I(t, \delta) = 0. \quad (54)$$

However this contradicts with (47), namely $\limsup_{t \rightarrow \infty} I(t) \geq \psi$.

(ii) If $\lambda_0 < \lambda_e$,

$$\begin{aligned} \Phi(t) &= \int_0^t \left((\lambda_0 - \lambda_e) e^{-\vartheta s} (N - I(s)) + \frac{\zeta^2 e^{-2\vartheta s} (N - I(s))^2}{4\vartheta} \right) ds \\ &\geq \int_0^t (\lambda_0 - \lambda_e) N e^{-\vartheta s} ds \\ &= \frac{(\lambda_e - \lambda_0)N}{\vartheta} (e^{-\vartheta t} - 1). \end{aligned} \quad (55)$$

then, for $\forall \delta \in \Omega_3$, $\exists \Omega_3 \subset \mathcal{Q}$ such that $\mathbb{P}(\Omega_3) = 1$ satisfy

$$\limsup_{t \rightarrow \infty} \frac{\Phi}{t} \geq 0. \quad (56)$$

Thus, for $\delta \in \Omega_1 \cap \Omega_2 \cap \Omega_3$, we also obtain

$$\lim_{t \rightarrow \infty} I(t, \delta) = \infty.$$

It also contradicts the equation (47).

Next, we will prove $\limsup_{t \rightarrow \infty} I(t) \leq \psi$, when it is not true, then $\exists \varepsilon \in (0, 1)$ is sufficiently small so that

$$P(\Omega_4) > \varepsilon,$$

$$\text{where } \Omega_4 = \left\{ \limsup_{t \rightarrow \infty} I(t) \geq \psi + 2\varepsilon \right\}.$$

Thus, for $\omega \in \Omega_4$, $\exists T_1 = T_1(\omega) > 0$ so that

$$I(t, \omega) \geq \psi + \zeta, \quad t \geq T_1(\omega). \quad (57)$$

Hence,

$$G(I(t, \omega)) \leq G(\psi + \varepsilon), \quad t \geq T_1(\omega). \quad (58)$$

On the one hand,

$$\Phi(t) \leq \frac{|\lambda_0 - \lambda_e| N}{\vartheta} (1 - e^{-\vartheta t}) + \frac{\zeta^2 N^2}{8\vartheta^2} (1 - e^{-2\vartheta t}), \quad (59)$$

as a result,

$$\frac{1}{t} \Phi(t) \leq \frac{|\lambda_0 - \lambda_e| N}{\vartheta} \frac{(1 - e^{-\vartheta t})}{t} + \frac{\zeta^2 N^2}{8\vartheta^2} \frac{(1 - e^{-2\vartheta t})}{t}. \quad (60)$$

With taking upper limit on both sides of (60), we obtain

$$\limsup_{t \rightarrow \infty} \frac{\Phi}{t} \leq 0. \quad (61)$$

Simultaneously,

$$\Phi(t) \geq (\lambda_0 - \lambda_e) N \int_0^t e^{-\vartheta s} ds = \frac{(\lambda_0 - \lambda_e) N}{\vartheta} (1 - e^{-\vartheta t}), \quad (62)$$

then,

$$\frac{1}{t} \Phi(t) \geq \frac{(\lambda_0 - \lambda_e) N}{\vartheta} \frac{(1 - e^{-\vartheta t})}{t}. \quad (63)$$

With taking upper limit on both sides of (63), we obtain

$$\limsup_{t \rightarrow \infty} \frac{\Phi}{t} \geq 0. \quad (64)$$

Therefore, for $\forall \delta \in \Omega_5$, $\exists \Omega_5 \subset \Omega$ such that $\mathbb{P}(\Omega_5) = 1$, satisfy

$$\lim_{t \rightarrow \infty} \frac{\Phi(t, \delta)}{t} = 0. \quad (65)$$

At the moments, set any $\omega \in \Omega_2 \cap \Omega_4 \cap \Omega_5$. When $t \geq T_1(\omega)$, from (30), (49), (50) and (65), we obtain

$$\begin{aligned} \ln(I(t)) &\leq \ln I_0 + \int_0^{T_1} G(I(s))ds + G(\psi + \varepsilon)(t - T_1) + \Phi(t) \\ &\quad + \int_0^t \alpha_1(s)(N - I(s))dW_1(s) - \int_0^t \alpha_2(s)\sqrt{(N - I(s))}dW_2(s), \end{aligned} \quad (66)$$

which, combine (57) and (58), yields

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq G(\psi + \varepsilon) < 0, \quad (67)$$

hence,

$$\lim_{t \rightarrow \infty} I(t, \omega) = 0. \quad (68)$$

However, this is contradicts with (57). That is namely $\limsup_{t \rightarrow \infty} I(t) \leq \psi$.

Therefore, our assertion (57) holds. $\square$

Remark III. If the solutions of the SDE system (17) are converging to disease-free dynamics, then the solutions of the SDE system (17) tend to 0 from Theorem 2. On the other hand, if the solutions of the SDE system (17) are converging to the endemic dynamics, then the solutions of the SDE system (17) will rise to or above the ψ from Theorem 3. In other words, we can use the basic reproduction number $\mathcal{R}_0^s$ to give the extinction and the stochastic persistence of the diseases.

Remark IV. From Theorem 2 and Theorem 3, we obtain

$$\mathcal{R}_0^s = \frac{\lambda_e N}{r + u} - \frac{\alpha_2^2 N}{2(r + u)} - \frac{\zeta^2 N^2}{4\vartheta(r + u)} < \mathcal{R}_1 = \frac{\lambda_e N}{r + u} - \frac{\alpha_1^2 N}{2(r + u)} < \mathcal{R} = \frac{\lambda_e N}{r + u}$$

$$\mathcal{R}_0^s = \frac{\lambda_e N}{r + u} - \frac{\alpha_2^2 N}{2(r + u)} - \frac{\zeta^2 N^2}{4\vartheta(r + u)} < \mathcal{R}_2 = \frac{\lambda_e N}{r + u} - \frac{\zeta^2 N^2}{4\vartheta(r + u)} < \mathcal{R} = \frac{\lambda_e N}{r + u}$$

$$\mathcal{R}_0^s = \frac{\lambda_e N}{r + u} - \frac{\alpha_2^2 N}{2(r + u)} - \frac{\zeta^2 N^2}{4\vartheta(r + u)} < \mathcal{R}_0 = \frac{\lambda_e N}{r + u} - \frac{\alpha_2^2 N}{2(r + u)} < \mathcal{R} = \frac{\lambda_e N}{r + u}$$

Moreover, when $\mathcal{R}_1 > 1$ but $\mathcal{R}_0^s < 1$, then the disease of the SDE system (5) will persist; but the disease of the SDE system (17) will extinct. When $\mathcal{R}_2 > 1$ but $\mathcal{R}_0^s < 1$, then the disease of the SDE system (1.13) in [21] will persist, that is $\alpha_2 = 0$ in the SDE system (17); but when $\alpha_2 > 0$ in the SDE system (17), the disease of the SDE system (17) will extinct. When $\mathcal{R}_0 > 1$ but $\mathcal{R}_0^s < 1$, then the disease of the SDE system (25) will persist, that is $\zeta = 0$ in the SDE system (17); but when $\zeta > 0$ in the SDE system (17), the disease of the SDE system (17) will extinct. When $\mathcal{R} > 1$ but $\mathcal{R}_0^s < 1$, then the disease of the deterministic system (2) will persist, that is $\zeta = 0$, $\alpha_2 = 0$ in the system (17); but when $\zeta > 0$, $\alpha_2 > 0$ in the SDE system (17), the disease of the SDE system (17) will be extinct. We can conclude that the more significant environmental fluctuations will suppress the disease outbreak in a randomly varying environment.

Next, let us further discuss the impact of the ζ and the ϑ on the ψ . We will focus on the limit dynamics of the SDE model (17) concerning the ζ and the ϑ , respectively.

Theorem 4. If $\mathcal{R}_0^s > 1$, and define ψ by (42) as function of the ζ for

$$0 < \zeta < \frac{2\sqrt{\vartheta\left(\lambda_e N - \frac{\alpha_2^2 N}{2} - (r+u)\right)}}{N} := \zeta^*.$$

Then ψ is decreasing strictly, and we obtain

$$\lim_{\zeta \rightarrow 0^+} \psi = N \left(1 - \frac{1}{\mathcal{R}_0}\right),$$

$$\lim_{\zeta \rightarrow \zeta^*} \psi = \begin{cases} 0, & \text{if } 1 < \mathcal{R}_0 \leq 2 \\ N \left(1 - \frac{1}{\mathcal{R}_0 - 1}\right), & \text{if } \mathcal{R}_0 > 2. \end{cases}$$

Proof. Considering

$$\begin{aligned} \frac{d\psi}{d\zeta} &= \frac{2N - \frac{4\vartheta(r+u)}{\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}}{\zeta} \\ &\quad - \frac{2\left(N\zeta^2 - 2\vartheta\lambda_e + \alpha_2^2\vartheta\right)}{\zeta^3} + \frac{4\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}{\zeta^3} \\ &= \frac{2\zeta^2\vartheta(r+u) + 4\left(\frac{\alpha_2^2\vartheta}{2} - \lambda_e\vartheta\right)\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}{\zeta^3\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}} \\ &\quad + \frac{4\left(\vartheta\left(\vartheta\lambda_e^2 + \frac{\alpha_2^4\vartheta}{4} - \lambda_e\alpha_2^2\vartheta - \zeta^2(r+u)\right)\right)}{\zeta^3\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}} \\ &= -\frac{2\left(\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)} - \left(\lambda_e\vartheta - \frac{\alpha_2^2\vartheta}{2}\right)\right)^2}{\zeta^3\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}. \end{aligned}$$

Due to $\zeta \neq 0$, hence $\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)} \neq 0$.

Then $\frac{d\psi}{d\zeta} < 0$, that is when ζ increases, the ψ is decreasing strictly.

In addition, making use of the L'Hospital Law, we obtain

$$\begin{aligned}
\lim_{\zeta \rightarrow 0^+} \psi &= \lim_{\zeta \rightarrow 0^+} \frac{2N\zeta - \frac{2\zeta\vartheta(r+u)}{\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}}{2\zeta} \\
&= \lim_{\zeta \rightarrow 0^+} \left(\frac{2N - \frac{2N\vartheta(r+u)}{\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}}{2} \right. \\
&\quad \left. - \frac{2\zeta^2\vartheta^2(r+u)^2\left(\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)\right)^{-\frac{3}{2}}}{2} \right) \\
&= \lim_{\zeta \rightarrow 0^+} \left(\frac{2N - \frac{2N(r+u)}{\lambda_e - \frac{\alpha_2^2}{2}}}{2} \right) = N \left(1 - \frac{r+u}{\lambda_e - \frac{\alpha_2^2}{2}} \right) = N \left(1 - \frac{1}{\mathcal{R}_0} \right).
\end{aligned}$$

What's more, it is easy to find out that

$$\begin{aligned}
\lim_{\zeta \rightarrow \zeta^{*-}} \psi &= \lim_{\zeta \rightarrow \zeta^{*-}} \frac{N\zeta^2 + \alpha_2^2\vartheta - 2\vartheta\lambda_e + 2\lambda_e - \frac{\alpha_2^2}{2}}{\zeta^2} \\
&= \frac{N^2 \left(2\lambda_e\vartheta - \alpha_2^2\vartheta - \frac{4\vartheta(r+u)}{N} + 2\sqrt{\left(\lambda_e\vartheta - \frac{\alpha_2^2\vartheta}{2} - \frac{2\vartheta(r+u)}{N}\right)^2} \right)}{4\vartheta\left(\lambda_e N - \frac{\alpha_2^2 N}{2} - (r+u)\right)} \\
&= \frac{N \left(\lambda_e N - \frac{\alpha_2^2 N}{2} - 2(r+u) + \left| \lambda_e N - \frac{\alpha_2^2 N}{2} - 2(r+u) \right| \right)}{2\left(\lambda_e N - \frac{\alpha_2^2 N}{2} - 2(r+u)\right)} \\
&= \frac{N(|\mathcal{R}_0 - 2| + \mathcal{R}_0 - 2)}{2(\mathcal{R}_0 - 1)}.
\end{aligned} \tag{69}$$

Hence, if $1 < \mathcal{R}_0 < 2$, then $\lim_{\zeta \rightarrow \zeta^{*-}} \psi = 0$. If $\mathcal{R}_0 > 2$, then $\lim_{\zeta \rightarrow \zeta^{*-}} \psi = N(1 - \frac{1}{\mathcal{R}_0 - 1})$.

This completes the proof of Theorem 4. $\square$

Theorem 5. Assume that $\mathcal{R}_0^s > 1$, and define ψ by (42) as function of the ϑ for

$$\vartheta > \frac{\zeta^2 N^2}{4\left[\lambda_e N - \frac{\alpha_2^2 N}{2} - (r+u)\right]} := \vartheta^*.$$

Then ψ is increasing strictly, and we obtain

$$\begin{aligned}
\lim_{\vartheta \rightarrow \infty} \psi &= N \left(1 - \frac{1}{\mathcal{R}_0} \right), \\
\lim_{\vartheta \rightarrow \vartheta^{*+}} \psi &= \begin{cases} 0, & \text{if } 1 < \mathcal{R}_0 \leq 2 \\ N \left(1 - \frac{1}{\mathcal{R}_0 - 1} \right), & \text{if } \mathcal{R}_0 > 2. \end{cases}
\end{aligned}$$

Proof. Considering

$$\begin{aligned}
\frac{d\psi}{d\vartheta} &= \frac{\alpha_2^2 - 2\lambda_e + \frac{2\vartheta\lambda_e^2 + \frac{\alpha_2^4\vartheta}{2} - 2\lambda_e\alpha_2^2\vartheta - \zeta^2(r+u)}{\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}}{\zeta^2} \\
&= \frac{2\vartheta\lambda_e^2 + \frac{\alpha_2^4\vartheta}{2} - 2\lambda_e\alpha_2^2\vartheta - \zeta^2(r+u)}{\zeta^2 \sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}} \\
&= \frac{(\alpha_2^2 - 2\lambda_e) \sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}{\zeta^2 \sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}} \\
&= \frac{\left(\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)} - \vartheta\left(\lambda_e - \frac{\alpha_2^2}{2}\right)\right)^2}{\vartheta\zeta^2 \sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}, \tag{70}
\end{aligned}$$

Due to $\vartheta \neq 0$,

$$\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)} \neq 0. \tag{71}$$

Thus, we can easily calculate that $\frac{d\psi}{d\vartheta} > 0$, that is when ϑ increases, the ψ is increasing strictly.

Next, following (42), we obtain

$$\begin{aligned}
\psi &= \frac{\vartheta\alpha_2^2 + N\zeta^2 - 2\vartheta\lambda_e + 2\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}{\zeta^2} \\
&= \frac{\alpha_2^2\vartheta + N\zeta^2 - 2\vartheta\lambda_e + 2\sqrt{(\lambda_e - \frac{\alpha_2^2}{2})^2(\vartheta - \frac{\zeta^2(r+u)}{2(\lambda_e - \frac{\alpha_2^2}{2})})^2 - \frac{\zeta^4(r+u)^2}{4(\lambda_e - \frac{\alpha_2^2}{2})^4}}}{\zeta^2} \\
&\leq \frac{\alpha_2^2\vartheta + N\zeta^2 - 2\lambda_e\vartheta + 2(\lambda_e - \frac{\alpha_2^2}{2})\left(\vartheta - \frac{\zeta^2(r+u)}{2(\lambda_e - \frac{\alpha_2^2}{2})}\right)}{\zeta^2} \\
&= N - \frac{r+u}{(\lambda_e - \frac{\alpha_2^2}{2})} = N\left(1 - \frac{1}{\mathcal{R}_0}\right).
\end{aligned}$$

Thus, ψ is bounded and monotonically increasing strictly with ϑ , then

$$\psi^* \triangleq \lim_{\vartheta \rightarrow \infty} \psi \leq N\left(1 - \frac{1}{\mathcal{R}_0}\right) \tag{72}$$

and

$$G(\psi) = \lambda_e(N - \psi) - (r + u) - \frac{\zeta^2}{4\vartheta}(N - \psi)^2 - \frac{\alpha_2^2}{2}(N - \psi) = 0, \quad (73)$$

Taking upper limit on both sides of (73), we obtain

$$\lambda_e N - (r + u) - \frac{\alpha_2^2}{2}N - \lambda_e \psi^* - \frac{\alpha_2^2}{2}\psi^* = 0, \quad (74)$$

then

$$\psi^* = N \left(1 - \frac{1}{\mathcal{R}_0} \right), \quad (75)$$

hence,

$$\lim_{\vartheta \rightarrow \infty} \psi = N \left(1 - \frac{1}{\mathcal{R}_0} \right), \quad (76)$$

$$\begin{aligned} \lim_{\vartheta \rightarrow \vartheta^{**}} \psi &= \lim_{\vartheta \rightarrow \vartheta^{**}} \frac{\vartheta \alpha_2^2 + N \zeta^2 - 2\vartheta \lambda_e + 2 \sqrt{\vartheta \left(\vartheta \lambda_e^2 + \frac{\vartheta \alpha_2^4}{4} - \vartheta \lambda_e \alpha_2^2 - \zeta^2(r + u) \right)}}{\zeta^2} \\ &= \frac{4N(\lambda_e N - (r + u) - \frac{\alpha_2^2 N}{2}) + \alpha_2^2 N^2 - 2\lambda_e N^2}{4(\lambda_e N - (r + u) - \frac{\alpha_2^2 N}{2})} \\ &\quad + 2 \sqrt{\frac{N^2(\lambda_e^2 N^2 + \frac{\alpha_2^4 N^2}{4} - \lambda_e \alpha_2^2 N^2 - 4\lambda_e(r + u)N + 2\alpha_2^2(r + u)N + 4(r + u)^2)}{16(\lambda_e N - (r + u) - \frac{\alpha_2^2 N}{2})^2}} \\ &= \frac{N(\lambda_e N - 2(r + u) - \frac{\alpha_2^2 N}{2})}{2(\lambda_e N - (r + u) - \frac{\alpha_2^2 N}{2})} + \frac{N \left| \lambda_e N - 2(r + u) - \frac{\alpha_2^2 N}{2} \right|}{2(\lambda_e N - (r + u) - \frac{\alpha_2^2 N}{2})} \\ &= \frac{N(|\mathcal{R}_0 - 2| + \mathcal{R}_0 - 2)}{2(\mathcal{R}_0 - 1)}. \end{aligned} \quad (77)$$

Therefore, if $1 < \mathcal{R}_0 < 2$, then $\lim_{\vartheta \rightarrow \vartheta^{**}} \psi = 0$; if $\mathcal{R}_0 > 2$, then $\lim_{\vartheta \rightarrow \vartheta^{**}} \psi = N(1 - \frac{1}{\mathcal{R}_0 - 1})$.

Remark V. The way of definition of $\mathcal{R}_0^s$ of the SDE system (17) as the same as $\mathcal{R}_1$ of the system (5) from [35] $\mathcal{R}_2$ of the model (1.13) in [21] and $\mathcal{R}^s$ of the model (8) from [36]. In order to show the similarities and differences between them, we choose the following parameter values.

$$\lambda_e = 0.55; \lambda = 0.55; N = 200; u = 20; r = 25; \lambda_0 = 0.40; \alpha_2 = 0.03; \quad (78)$$

Example I. We choose $\zeta = \alpha_1 = \alpha_2 = 0.03$ to show the relations between the basic reproduction number $\mathcal{R}_0^s$ with ϑ in Fig 1. From Fig 1, when $\vartheta \leq \vartheta_0$, (ϑ_0 is the root of ϑ at $\mathcal{R}_0^s = 1$), then $\mathcal{R}_0^s \geq 1$, the disease of the SDE system (17) will die out. If $\vartheta > \vartheta_0$, then $\mathcal{R}_0^s < 1$, the $I(t)$ of the SDE system (17) will persist. That is, the lower the speed of reversion ϑ will suppress the disease outbreak.

Fig. 1. The relations $\mathcal{R}_0^s$ with ϑ

Example II. We choose $\zeta = \alpha_1, \alpha_2 = 0.03$, and use different values of ϑ to show the relations between the basic reproduction number $\mathcal{R}^s$ and $\mathcal{R}_0^s$ with ζ in Fig 2. In Fig 2 (a), $\vartheta = \frac{1}{2}$, if $\zeta > \zeta_0$, (ζ_0 is the root of ζ at $\mathcal{R}_0^s = 1$), then the $I(t)$ of the SDE system (17) will die out; if $\zeta > \zeta_0^*$ (ζ_0^* is the root of ζ at $\mathcal{R}^s = 1$), then the $I(t)$ of the model (8) will die out. If $\zeta_0 < \zeta_0^*$, then $\zeta_0 < \zeta < \zeta_0^*$, the $I(t)$ of the SDE system (17) will die out, but the $I(t)$ of the system (8) will persist. In Fig 2(b), if $\vartheta = \frac{1}{2}$, then $\mathcal{R}^s = \mathcal{R}_0^s$, the stochastic dynamics of the SDE system (17) as the same as the system (8). In Fig 2 (c), if $\vartheta = 1$, $\zeta_0 > \zeta$, then $\zeta_0^* < \zeta < \zeta_0$, the $I(t)$ of the SDE system (17) will persist, but the $I(t)$ of the system (8) will die out.

Fig. 2. The relations between $\mathcal{R}^s$ and $\mathcal{R}_0^s$ with ζ

Example III. We choose $\zeta = \alpha_1, \alpha_2 = 0$, and use different values of ϑ to show the relations between the basic reproduction number $\mathcal{R}_1$ and $\mathcal{R}_0^s$ with ζ in Fig 3. In Fig 3 (a), $\vartheta = \frac{1}{4}$, if $\zeta > \zeta_0$, the $I(t)$ of the SDE system (17) will die out; if $\zeta > \zeta_0^*$ (ζ_0^* is the root of ζ at $\mathcal{R}_1 = 1$), the $I(t)$ of the system (5) will die out. If $\zeta_0 < \zeta_0^*$, then $\zeta_0 < \zeta < \zeta_0^*$, the $I(t)$ of the SDE system (17) will die out, but the $I(t)$ of the system (5) will persist. In Fig 3(b), if $\vartheta = \frac{1}{2}$, then $\mathcal{R}_1 = \mathcal{R}_0^s$, the stochastic dynamics of the SDE system(17) as the same as the system (5). In Fig 3 (c), if $\vartheta = 1$, $\zeta_0 > \zeta$, then $\zeta_0^* < \zeta < \zeta_0$, the $I(t)$ of the SDE system (17) will persist, but the $I(t)$ of the system (5) will die out.

Remark VI. Compared with the $\mathcal{R}_1$ and the $\mathcal{R}^s$, the $\mathcal{R}_0^s$ is more flexible and more effective in controlling diseases. Thus, the SDE system (17) has better effect and versatility in suppressing the disease outbreak. The SDE system (5) and the SDE system (8) are the promotion and expansion of the SDE system (17). Furthermore,

the lower the speed of reversion ϑ , the higher intensity of volatility ζ will suppress the disease outbreak, respectively.

Fig. 3. The relations between $\mathcal{R}_1$ and $\mathcal{R}_0^s$ with ζ

4 Numerical results

Now, we will use Milstein's method from Tian et al. [22], which provides some numerical examples to explain the complex disease dynamic results of the SDE system (17).

$$\begin{aligned}
 I_{k+1} = & I_k + I_k[(\lambda_e + (\lambda_0 - \lambda_e)e^{-\vartheta k \Delta t})(N - I_k) - (r + u)]\Delta t \\
 & + \frac{\zeta \sqrt{1 - e^{-2\vartheta(k\Delta t)}}}{\sqrt{2\vartheta}}(N - I_k)I_k\eta_{1k} \sqrt{\Delta t} + \alpha_2 \sqrt{(N - I_k)I_k}\eta_{2k} \sqrt{\Delta t} \\
 & + \frac{\zeta^2(1 - e^{-2\vartheta(k\Delta t)})}{4\vartheta}(N - I_k)(N - 2I_k)I_k[(\eta_{1k})^2 - 1]\Delta t \\
 & + \frac{\alpha_2^2}{2}(N - \frac{3}{2}I_k)I_k[(\eta_{2k})^2 - 1]\Delta t.
 \end{aligned} \tag{79}$$

Where the meaning of η_{nk} , $k \in \mathbb{Z}^+$ as independent Gaussian random variables, Δt is the time step.

Next, we will further discuss the impacts of the intensity of volatility ζ and the speed of reversion ϑ on the disease dynamics of the SDE system (17). We choose the same parameters as (78) and set the initial value $I(0) = 10$.

4.1 Intensity of Volatility ζ

We choose $\vartheta = 0.8000$, $\zeta = \alpha_1 = 0.0650$. It is accessible to calculate that $\mathcal{R} = 2.4444 > 1$, $\mathcal{R}_1 = 0.5667 < 1$, $\mathcal{R}^s = 0.5647 < 1$, then the disease of the SDE system (5) and the SDE system (8) will die out. However, $\mathcal{R}_0^s = 1.2688 > 1$, we can know from Theorem 3, the disease of the SDE system (17) will persist, we obtain

$$\psi = \frac{\vartheta\alpha_2^2 + N\zeta^2 - 2\vartheta\lambda_e + 2\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r + u)\right)}}{\zeta^2} = 87.9511.$$

Fig. 4. The impact of ζ on the extinction and the stochastic persistence of the diseases of the SDE system (17), the SDE system (8), the SDE system (5), and the deterministic system (2).

Therefore, with Theorem 3, when the initial value $I(0) = 10$, we can conclude that the solutions of the SDE system (17) satisfy

$$\liminf_{t \rightarrow \infty} I(t) \leq 87.9511 \leq \limsup_{t \rightarrow \infty} I(t).$$

It can be seen $I(t)$ has been fluctuating around the level of $\psi = 87.9511$ (See Fig 4(a)).

If we choose $\zeta = \alpha_1 = 0.085$, It is accessible to calculate that $\mathcal{R} = 2.4444 > 1$, $\mathcal{R}_1 = -0.7667 < 1$, $\mathcal{R}^s = -0.7687 < 1$, and $\mathcal{R}_0^s = 0.4355 < 1$, we can find from Theorem 2 that the disease of the SDE system (17) will die out (See Fig 4(b)).

From Fig 4, comparing with the SDE system (5) and the SDE system (8), the SDE system (17) exhibits different dynamic properties. Under the same conditions, the intensity of volatility of the environment has a more significant impact on the SDE system (17). The response of the SDE system (17) to the fluctuation intensity of the environment is more natural and sensitive. It fully shows that the SDE system (17) proposed in this paper is more suitable for describing the corresponding epidemic infectious diseases. It is also closer to the natural situation in reality. The numerical fitting result supports this conclusion very well (see Fig 4). It also further verifies the correctness of the theoretical analysis in this paper. In addition, we also found that with the higher intensity of volatility ζ , the outbreak of epidemic infectious diseases will be suppressed. With the lower intensity of volatility ζ , the outbreak of epidemic infectious diseases will be promoted.

Besides, with Theorem 4, we can easily know that when the intensity of volatility ζ increases, the ψ is decreasing strictly. So we choose $\zeta = \alpha_1 = \alpha_2 = 0.001$, then $\mathcal{R} = 2.4444 > 1$, $\mathcal{R}_1 = 2.4440 > 1$, $\mathcal{R}^s = 2.443998 > 1$, $\mathcal{R}_0^s = 2.4442 > 1$, and calculate maximum value of ψ is approximately,

$$\psi = \frac{\vartheta \alpha_2^2 + N \zeta^2 - 2 \vartheta \lambda_e + 2 \sqrt{\vartheta \left(\vartheta \lambda_e^2 + \frac{\vartheta \alpha_2^4}{4} - \vartheta \lambda_e \alpha_2^2 - \zeta^2 (r + u) \right)}}{\zeta^2} = 118.1779.$$

Thus, with Theorem 3, for the initial value $I(0) = 10$, one can conclude that the solutions of the SDE system (17) satisfy

$$\liminf_{t \rightarrow \infty} I(t) \leq 118.1779 \leq \limsup_{t \rightarrow \infty} I(t).$$

Furthermore, we obtain $I^* := N(1 - \frac{1}{\mathcal{R}_0}) = 118.1779$ is the positive equilibrium point of the following equation

$$dI(t) = I(t) \left[\left(\lambda_e - \frac{\alpha_2^2}{2} \right) (N - I(t)) - (r + u) \right] dt, \quad (80)$$

Thus, when the intensity of volatility ζ tends to 0, then the solutions of the SDE system (17) will approach the limiting value of the model (80) (See Fig 4 (c)).

Next, when we take the intensity of volatility ζ sufficiently big, for example, we adopt $\zeta = 0.072 \rightarrow \zeta^* = 0.07211$, $\alpha_2 = 0.001$. one can conclude that $\mathcal{R}_0 = 2.4444 > 2$, and we get

$$\psi = \frac{\vartheta \alpha_2^2 + N \zeta^2 - 2 \vartheta \lambda_e + 2 \sqrt{\vartheta \left(\vartheta \lambda_e^2 + \frac{\vartheta \alpha_2^4}{4} - \vartheta \lambda_e \alpha_2^2 - \zeta^2 (r + u) \right)}}{\zeta^2} = 62.4694.$$

Therefore, with Theorem 3, for the initial value $I(0) = 10$, the solutions of the SDE system (17) satisfy

$$\liminf_{t \rightarrow \infty} I(t) \leq 62.4694 \leq \limsup_{t \rightarrow \infty} I(t),$$

(See Fig 4 (a)).

On the flip side, we get $I^* := N(1 - \frac{1}{\mathcal{R}_0-1}) = 62.4694$ is the positive equilibrium of the following equation

$$dI(t) = I(t) \left[\left(\lambda_e - \frac{N \alpha_2^2}{2} + (r + u) \right) (N - I(t)) - (r + u) \right] dt. \quad (81)$$

Therefore, when $\zeta \rightarrow \zeta^*$, the solutions of the SDE system (17) will approach the limiting value of the model (81).

Moreover, when we choose $N = 100$, and $2 > \mathcal{R}_0 = 1.2222 > 1$, then the disease will be extinct (See Fig 4 (b)).

At the same time, according to the above analysis, we obtain $I^* := N(1 - \frac{1}{\mathcal{R}}) = 118.1818$ is the positive equilibrium point of the following equation

$$dI(t) = I(t) [\lambda_e (N - I(t)) - (r + u)] dt, \quad (82)$$

and $I^* := N(1 - \frac{1}{\mathcal{R}-1}) = 61.5385$ is the positive equilibrium of the following equation

$$dI(t) = I(t) \left[\left(\lambda_e - \frac{(r + u)}{N} \right) (N - I(t)) - (r + u) \right] dt. \quad (83)$$

Similarly, when the intensity of volatility ζ tends to 0, and α_2 tends to 0, then the solutions of the SDE system (17) will approach the limiting value of the model (82). When $\zeta \rightarrow \zeta^*$, and α_2 tends to 0, then the solutions of the SDE system (17) will approach the limiting value of the model (83) (See Fig 4 (c)).

Fig. 5. The impact of ϑ on the extinction and the stochastic persistence of the diseases of the SDE system (17), the SDE system (8), the SDE system (5), and the deterministic system (2).

4.2 The Speed Reversion ϑ

Next, we will show the impact on the disease dynamics of the SDE system (17) with the speed reversion ϑ . If set $\zeta = \alpha_1 = \alpha_2 = 0.030, \vartheta = 0.10$; in this case, we can calculate that the basic reproduction number $\mathcal{R} = 2.4444 > 1, \mathcal{R}_1 = 2.0444 > 1, \mathcal{R}^s = 2.0424 > 1$. Thus, we can get the disease to persist for the SDE system (5) and the SDE system (8). We also get $\mathcal{R}_0^s = 0.4424 < 1$, with Theorem 3, then the disease of the SDE system (17) will die out (See Fig 5 (a)). When we choose $\vartheta = 0.25$, one can calculate $\mathcal{R} = 2.4444 > 1, \mathcal{R}_1 = 2.0444 > 1, \mathcal{R}^s = 2.0424 > 1, \mathcal{R}_0^s = 1.6424 > 1$. Thus, we can draw a conclusion that the persistence of the disease does not change for the SDE system (5) and the SDE system (8), the disease of the SDE system (17) will persist by Theorem 3, and we obtain

$$\psi = \frac{\vartheta\alpha_2^2 + N\zeta^2 - 2\vartheta\lambda_e + 2\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}{\zeta^2} = 102.5682.$$

Therefore, with Theorem 3, for the initial value $I(0) = 10$, one can conclude that the solutions of the SDE system (17) satisfy

$$\liminf_{t \rightarrow \infty} I(t) \leq 102.5682 \leq \limsup_{t \rightarrow \infty} I(t).$$

We can see that $I(t)$ has been fluctuating around the level $\psi = 102.5682$ (See Fig 5(b)).

From Fig 5, comparing with the SDE system (5) and the SDE system(8), the SDE system (17) exhibits different dynamic properties. With the same conditions, when we increase the speed reversion ϑ , the basic reproduction number of the SDE system (17) will change from $\mathcal{R}_0^s < 1$ to $\mathcal{R}_0^s > 1$, and the disease will also change from an extinct state to persist. When we choose the speed reversion ϑ to infinity, the disease dynamics of the SDE system (17) will degenerate into a corresponding deterministic system. Thus, we can conclude that one can control the basic reproduction number by controlling the speed reversion of the SDE system (17) to control the extinction and persistence of the disease. The numerical fitting result supports this conclusion very well (see Fig 5). It also further verifies the correctness of the theoretical analysis

in this paper. In addition, it is easy to find that the disease outbreak will be caused by the more significant speed reversion ϑ , and the disease outbreak will be suppressed by the lower speed reversion ϑ .

When we fix $\vartheta = 0.1387 := \vartheta^*$, we can get $\mathcal{R}_0 = 2.4424 > 2$, and,

$$\psi = \frac{\vartheta\alpha_2^2 + N\zeta^2 - 2\vartheta\lambda_e + 2\sqrt{\vartheta\left(\vartheta\lambda_e^2 + \frac{\vartheta\alpha_2^4}{4} - \vartheta\lambda_e\alpha_2^2 - \zeta^2(r+u)\right)}}{\zeta^2} = 61.3465.$$

Therefore, with Theorem 3, for the initial value $I(0) = 10$, one can conclude that the solutions of the SDE system (17) satisfy

$$\liminf_{t \rightarrow \infty} I(t) \leq 61.3465 \leq \limsup_{t \rightarrow \infty} I(t).$$

On the other hand, with Theorem 5, $I^* := N(1 - \frac{1}{\mathcal{R}_0 - 1}) = 61.3465$ is the positive equilibrium of the model (81), i.e., when $\vartheta \rightarrow \vartheta^*$, the solutions of the SDE system (17) will approach the limiting value of the model (81).

In addition, if set the parameters are same as (78) and choose $\zeta = 0.030, \alpha_1 = \alpha_2 = 0.001$, then $\lim_{\vartheta \rightarrow \infty} \psi = 118.1815$; with the discussion in Section 3.2 as above, we can get the positive equilibrium point of the model (80) which is $I^* = N(1 - \frac{1}{\mathcal{R}_0}) = 118.1815$. In other words, when $\vartheta \rightarrow \infty, \alpha_2 \rightarrow 0$, the solutions of the SDE system (17) will approach the limiting of the model (80).

When we choose $\vartheta = 10000$, the numerical fitting result is shown in Fig 5 (c). As proved in Section 3.2, if $\vartheta \rightarrow \infty, \alpha_2 \rightarrow 0$, the SDE system (17) will tend to the deterministic model (2).

Fig. 6. Frequency distribution of $I(t)$ at $t = 1000$ of the SDE system (17) when ζ is 0.01, 0.03, 0.05, respectively

4.3 Discussion on the stationary distribution

In this section, we will further study the impact of the dynamics of the SDE model (17) with the intensity of volatility ζ and the intensity of volatility ϑ . We choose all parameters such as (78) and conducted more than 100000 simulation experiments.

At $t = 1000$, we have never seen any extinction of disease. Fig 6 and Fig 7 show the frequency distribution histogram of $I(t)$ in the SDE system (17) at $t = 1000$.

For the sake of studying the impact on the stationary distributions of $I(t)$ of the SDE model (17) with the intensity of volatility ζ . We choose $\vartheta = 1.0000$ and different values of the intensity of volatility ζ , such as 0.01, 0.03 and 0.05, and choose other all parameters as (78). Then we can obtain that the corresponding $\mathcal{R}_0^s$ are 2.4202, 2.2424 and 1.8869, respectively. With Theorem 3, the disease of the SDE system (17) will persist with probability 1 in the above situations. That is, if $\zeta = 0.01$, the distribution looks closer to a normal (See Fig 6(a)). If $\zeta = 0.03$, the distribution looks closer to the normal (See Fig 6 (b)), while $\zeta = 0.05$, the distribution is positively skewed (See Fig 6(c)). To this end, when the intensity of volatility ζ is lower, with less amplitude of fluctuation and more symmetrical oscillation distribution. The stationary distribution will tend to be normal. On the contrary, when the intensity of volatility ζ is higher, more fluctuation and oscillation distribution are skewed. The stationary distribution will tend to be left. The numerical simulation supports this result (see Fig 6). It also further verifies the correctness of Theorem 4.

Fig. 7. Frequency distribution of $I(t)$ at $t = 1000$ of the SDE system (17) when $\zeta = \alpha_1 = \alpha_2 = 0.03$, and ϑ is 0.25, 0.50, 1.00, respectively.

Similarly, for the sake of discussing the impact on the stationary distributions of $I(t)$ with the intensity of volatility ϑ . We choose $\zeta = 0.03$ and different values of the intensity of volatility ϑ , which are 0.25, 0.5, 1, respectively, set other parameters as (78). Then we can obtain the corresponding $\mathcal{R}_0^s$ are 1.6424, 2.0424 and 2.2424, respectively. With Theorem 3, the disease of the SDE model (17) will persist with probability 1 in the above situations. In Fig 7, if $\vartheta = 1$, the distribution looks closer to be normal (See Fig 7 (c)), while if $\vartheta = 0.25$, the distribution is positively skewed (See Fig 7(a)). Furthermore, if $\vartheta = 0.5$, the result is the same as the SDE model (1) (See Fig 7(b)). For this reason, when the intensity of volatility ϑ is higher, with less amplitude of fluctuation and more symmetrical oscillation distribution. The stationary distribution will tend to be normal. On the contrary, when the intensity of volatility ϑ is lower, more fluctuation and oscillation distribution are skewed. The stationary distribution will tend to be left. The numerical simulation supports this result (see Fig 7). It also further verifies the correctness of Theorem 5.

5 Conclusion

Since the mean-reverting Ornstein-Uhlenbeck process has the characteristics of continuity, non-negativity, practicality and asymptotic distribution, and it has the significant feature better than traditional Gaussian white noise on characterizing the variability of the environment in biological systems [21]. In this paper, we introduced the mean-reverting Ornstein-Uhlenbeck process to a new stochastic SIS model. Moreover, it proved the impact of the spread of disease in the SDE system (17) by environmental fluctuations. One can determine the stochastic persistence and the stochastic extinction of the SDE system (17) by the stochastic basic reproduction number $\mathcal{R}_0^s$. That is when the stochastic basic reproduction number $\mathcal{R}_0^s < 1$, the disease will go extinct, while $\mathcal{R}_0^s > 1$, the disease will go to persist.

Furthermore, due to $\mathcal{R}_0^s > 1$ means $\vartheta < \vartheta^*$ or $\zeta > \zeta^*$, and $\mathcal{R}_0^s < 1$ means $\vartheta > \vartheta^*$ or $\zeta < \zeta^*$. Thus, we can control the disease dynamic of the SDE model (17) by ϑ^* and ζ^* . That is to say, when $\vartheta < \vartheta^*$ or $\zeta > \zeta^*$, the disease of the SDE system (17) will die out. Furthermore, when $\vartheta > \vartheta^*$ or $\zeta < \zeta^*$, the disease of the SDE system (17) will persist. It demonstrated that higher intensity of volatility ζ or lower speed reversion ϑ will suppress the outbreak of the disease. Oppositely, lower intensity of volatility ζ or higher speed reversion ϑ will help the outbreak of the disease, respectively.

We provide the limit dynamics of the SDE system (17) relative to the speed reversion ϑ and the intensity of volatility ζ in Theorem 4 and Theorem 5. That is, the solutions of the SDE system (17) in the different limit processes will approach the limiting value $N(1 - \frac{1}{\mathcal{R}_0})$ corresponding to the system (80). Similarly, the solutions of the SDE system (17) in the different limit processes will approach the limiting value $N(1 - \frac{1}{\mathcal{R}_0-1})$ corresponding to the system (81) too.

On the other hand, the disease infection rate of the system (80) is λ_e , the disease infection rate of the system (81) is $\lambda_e - \frac{r+u}{N}$. That is say, if $\zeta \rightarrow \zeta^*$ or $\vartheta \rightarrow \vartheta^*$, the disease infection rate will decrease from λ_e to $\lambda_e - \frac{u+r}{N}$. Therefore, we can decrease the disease infection rate to control the disease spread by decreasing the speed reversion ϑ or increasing the intensity of volatility ζ . In order to control the disease spread, we should decrease the speed reversion ϑ or increase the intensity of volatility ζ , respectively.

Finally, in case $\zeta = \alpha_1$, if $\vartheta = 0.5$ and $\mathcal{R} = \mathcal{R}_0^s$, the stochastic disease dynamics of the SDE system (17) is the same as that of the system (1). If $\vartheta < 0.5$, and $\zeta^* < \zeta < \zeta_*$, the disease of the SDE system (17) will die out, and the disease of the system (1) will persist. If $\vartheta > 0.5$, and $\zeta^* < \zeta < \zeta_*$, the disease of the SDE system (17) will persist; simultaneously, the disease of the SDE system (1) will die out. Therefore, we can get different stochastic dynamics of the system perturbed with the mean-reverting Ornstein-Uhlenbeck process. Compared with the SDE systems proposed in [21], [35] and [36], the SDE system (17) proposed in this paper can better characterize stochastic dynamics and better control the outbreak and extinction of the disease. The SDE system (17) proposed in this paper can be regarded as the extension and promotion of the systems proposed in [21], [35] and [36].

Remark VII. Based on the SDE system (17), much exciting work is still worth further studying. One of the most compelling questions is whether there is a relationship between these two disturbances? For example, Cai et al. [37] considered two

correlated Brownian motions in the SIS system. The findings show that the correlation stochastic dynamics are different from the general stochastic SIS system. Furthermore, it will be fascinating to propose a more realistic and more complex system. Thus, we will further consider the following stochastic differential system.

$$\begin{aligned} dI(t) = & [((\lambda_e + (\lambda_0 - \lambda_e) e^{-\theta t})(N - I(t))I(t) - (r + u)I(t))]dt \\ & + [m_1\alpha_1(t)(N - I(t))I(t) - m_2\alpha_2I(t)\sqrt{N - I(t)}]dW_1(t) \\ & - m_3\alpha_2I(t)\sqrt{N - I(t)}dW_2(t), \end{aligned} \quad (84)$$

where m_1, m_2, m_3 are constants. The SDE system (84) can be regarded as the extension and promotion of the systems proposed in [21], [35], [36] and [37].

Author contributions

All authors contributed to the study's conception and design. Material preparation, data collection, and analysis were performed by Xingzhi Chen, Baodan Tian, Xin Xu, Ruoxi Yang, and Shouming Zhong. The first draft of the manuscript was written by Xingzhi Chen, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Conflict of interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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