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From Generalization of Bacon-Shor Codes to High Performance Quantum LDPC Codes

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Abstract

We utilize a concatenation scheme to construct new families of quantum error correction codes that include the Bacon-Shor codes. We show that our scheme can lead to asymptotically good quantum codes while Bacon-Shor codes cannot. Further, the concatenation scheme allows us to derive quantum LDPC codes of distance $\Omega(N^{2/3}/\log\log N)$ which can improve Hastings's recent result [arXiv:2102.10030] by a polylogarithmic factor. Moreover, assisted by the Evra-Kaufman-Zémor distance balancing construction, our concatenation scheme can yield quantum LDPC codes with non-vanishing code rates and better minimum distance upper bound than the hypergraph product quantum LDPC codes. Finally, we derive a family of fast encodable and decodable quantum concatenated codes with parameters $Q = [[N, \Omega(\sqrt{N}), \Omega(\sqrt{N})]]$ and they also belong to the Bacon-Shor codes. We show that Q can be encoded very efficiently by circuits of size $O(N)$ and depth $O(\sqrt{N})$, and can correct any adversarial error of weight up to half the minimum distance bound in $O(\sqrt{N})$ time. To the best of our knowledge, they are the most powerful quantum codes for correcting so many adversarial errors in sublinear time by far.

INTRODUCTION

Quantum systems are vulnerable to the environment noise induced by decoherence. It is one of the biggest obstacles in quantum information processing. Similar to classical systems, one feasible solution is to exploit quantum error correction codes (QECC) to encode the primitive quantum information into a larger quantum state. It is widely believed that QECCs are necessary in realizing long-term quantum communications and in building fault-tolerant (FT) quantum computers [1, 2]. The construction and design of QECCs with excellent performance is thus one significant task. Particularly, how to construct asymptotically good QECCs with positive rates and linear distance is one of the central questions in quantum coding theory [3, 4].

It is known that QECCs can be constructed from classical linear codes, e.g., by using the stabilizer formalism [5] or the Calderbank-Shor-Steane (CSS) construction [3, 6]. But it is not an easy task to do that straightforwardly since an additional dual-containing constraint is needed. Nevertheless, there exist two different categories of quantum codes that can be constructed from arbitrary classical linear codes without the dual-containing constraint; namely, the entanglement-assisted codes [7–9] and the quantum concatenation codes [10, 11]. The former goes beyond standard QECC regimes because it requires presharing entangled states between the sender and the receiver. The latter contains important families of quantum codes, such as the Bacon-Shor (BS) codes [6, 10] and the hypergraph product (HP) quantum codes [11]. The BS code provides a very simple way of constructing quantum codes, yet it can lead to fault-tolerant subsystem codes with high threshold [12]. The HP quantum code can be seen as a variant of the BS codes because their stabilizer structure is similar. However, the HP construction can lead to quantum low-density parity-check (qLDPC) codes with non-constant minimum distance, while the BS code cannot. Further, the HP qLDPC codes of length N can maintain a non-vanishing rate and $\Omega(\sqrt{N})$ distance. It should be noted that both BS codes and the HP quantum codes have a bad asymptotic behavior because their minimum distance scales as the square root of the block length.

It is well-known that asymptotically good classical codes with parameters $[N, \Omega(N), \Omega(N)]$ exist [13]; however, construction of QECCs with the same asymptotic scaling remains one of the biggest open questions in quantum coding theory. Prior to 2020, the best QECCs with non-vanishing code rates only have the minimum distance in the order of $\Omega(\sqrt{N})$.

Recently, several studies are shown to break the $\Omega(\sqrt{N})$ minimum distance barrier, see e.g., [14–17]. The main technical tools of these achievements could be broken into two types: one combines abstract topology objects with probabilistic arguments [16, 17], while the other provides explicit construction based on high dimensional expanders [14, 15]. Explicitly, Ref. [14] applies the Evra-Kaufman-Zémor (EKZ) distance balancing method to construct qLDPC codes from unbalanced ones with asymmetric distance. Then qLDPC codes of distance $\Omega(\sqrt{N} \log N)$ and dimension $\Omega(\sqrt{N}/\log N)$ were constructed by using the high dimensional expanders. In [16], a family of qLDPC codes of distance $\Omega(N^{3/5}/\text{polylog } N)$ and dimension $N^{3/5}$ was constructed based on fiber bundles. In [17], qLDPC codes with almost linear distance $\Theta(N/\log N)$ and dimension $\Theta(\log N)$ were obtained by using a lift product construction.

In addition to the construction of QECCs with good parameters, practical QECCs need to equip with both efficient encoders and decoders. If the decoding is slower than the error accumulations, additional noise will be introduced during the syndrome decoding [18]. Moreover, a unique phenomenon in quantum coding theory, called *error degeneracy*, exists, which distinguishes quantum codes from classical ones both in theory and in physics. However, error degeneracy does not simplify the decoding of QECCs [19]. In contrast, it makes most decoding methods that are efficient for classical codes fail to fully decode QECCs, e.g., the belief-propagation (BP) algorithm [20]. How to utilize error degeneracy to correct more errors is an attractive and crucial issue, and its importance, in our opinion, is no less than the construction of good QECCs. Surface codes [21] can be decoded efficiently in polynomial time to correct degenerate errors by using the classical perfect matching algorithm [22]. Recently, the HP qLDPC codes can be decoded efficiently in linear time by using the syndrome reducing based bit-flip algorithm [23, 24]. In [25], deep neural network based BP decoders are trained for qLDPC codes by considering error degeneracy in the loss function. The EKZ distance balancing construction in [14] is also useful to the decoding issues and qLDPC codes of distance $\Omega(\sqrt{N \log N})$ can be decoded in polynomial time.

On the other hand, the encoding of QECCs is also significant but it has received much less attention compared to the decoding. Noise not only occurs in the transmission and decoding, but also in the encoding circuit [4]. Large-scale encoding circuit and deep encoding depth make the risk of qubits suffering from faulty gates increase rapidly. Consequently, the qubits might be polluted even before the transmission and the computation. The encoding circuit

of stabilizer codes of length N with g stabilizer generators generally has $O(gN)$ gates and $O(N)$ depth [26]. But in many practical applications, the encoding circuits need to be with linear size and sublinear depth [27, 28].

In this work, we generalize the concatenation scheme of Bacon-Shor codes and propose quantum serially concatenated parity-check (qSCPC) codes. We show that qSCPC codes can also be derived from any two classical linear codes without the dual-containing constraint. The BS codes can be seen as a special family of our qSCPC codes. Moreover, our concatenation scheme can lead to asymptotically good QECCs while BS codes cannot. If we use classical low-density parity-check (LDPC) codes as one of the component codes in qSCPC codes, we show that error degeneracy can greatly improve the minimum distance of qSCPC codes. In such case, the pure minimum distance of qSCPC codes without considering error degeneracy is constant to the block length. Yet, degenerate qSCPC codes can meet the quantum GV bound whose minimum distance is linear to the block length. We also show that qSCPC codes can approach the Hashing bound for asymmetric Pauli channels with a large asymmetry.

In addition to the asymptotic behavior, the X -stabilizer generators are strongly LDPC [29], while the Z -stabilizer generators have $O(N)$ weight, where N is the block length. Then by using the weight reduction procedure in [29] and the EKZ distance balancing construction, we can derive qLDPC codes with distance $\Omega(N^{2/3}/\log \log N)$ and dimension $\Omega(N^{2/3}/\ell)$, $\ell = (\log N)^{2+\epsilon}$, $\epsilon > 0$. Thus we improve Hastings's result [29] by a polylogarithmic scaling. Moreover, our codes are determinate to exist while the codes used in [29] are based on some probability arguments which only guarantee the existence with high probability.

We can also use the qSCPC concatenation scheme to derive qLDPC codes with positive rates and minimum distance which is the square root of the block length. Compared with the HP qLDPC codes in [11, 30], the minimum distance upper bound of our codes can beat that of the HP qLDPC codes. Further, we show that qSCPC codes can be decoded in linear time and correct a much larger fraction of adversarial errors than quantum expander codes in [23, 24]. Moreover, we can use arbitrary expander codes in the concatenation scheme and then our codes need a much smaller expansion than [23, 24]. As a special case, we construct a family of asymmetric qSCPC codes with parameters

$$\mathcal{Q} = [[N, \Omega(N/n_0), d_Z \geq \Omega(N/n_0)/d_X \geq n_0]] \quad (1)$$

by using expander codes as one of the component codes, where $1 \leq n_0 \leq N$. We show that $\mathcal{Q}$ also belongs to the BS code family.

Let $n_0 = \sqrt{N}$, we can derive a family of quantum codes with parameters

$$\mathcal{Q}_1 = [[N, \Omega(\sqrt{N}), \Omega(\sqrt{N})]], \quad (2)$$

which can correct all errors of weight smaller than half the minimum distance bound in $O(\sqrt{N})$ time. We show that $\mathcal{Q}_1$ can be encoded fastly by a circuit of size $O(N)$ and depth $O(\sqrt{N})$. Moreover, $\mathcal{Q}_1$ can correct a random X -error with very high probability provided $p_X < 25\%$.

If let $n_0 = \log N$, then we can derive a family of quantum codes with parameters

$$\mathcal{Q}_2 = [[N, \Omega(N/\log N), \Omega(N/\log N)/\Omega(\log N)]], \quad (3)$$

which can correct any random X -error with high probability and correct an almost linear number of Z -errors in $O(N/\log N)$ time. While fault-tolerant quantum computation [31] and communication [4, 32] prefer fast encodable and decodable codings, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are thus practical and beneficial for the future use.

RESULTS

Stabilizer codes – Let $GF(q)$ be the Galois field of size q and denote by $GF(q^m)$ a field extension of $GF(q)$, where q is a prime power and $m \geq 1$ is an integer. Denote the unitary operators $X(u)$ and $Z(v)$ on $\mathbb{C}^q$ by $X(u)|a\rangle = |a+u\rangle$ and $Z(v)|a\rangle = \xi^{\text{Tr}(va)}|a\rangle$, where “Tr” is the trace operation, $\xi = \exp(2\pi i/p)$ is a primitive p th root of unity, and $u, v \in GF(q)$. Denote by the error group $G_n = \{\xi^c X(u)Z(v) | u, v \in GF(q)^n, c \in GF(p)\}$. A stabilizer code Q is a q^k -dimensional subspace of the Hilbert space $\mathcal{H}_n = \mathbb{C}^{q^n}$, i.e.,

$$Q = \{|\varphi\rangle \in \mathcal{H}_n \mid E|\varphi\rangle = |\varphi\rangle, \forall E \in \mathcal{S}\}, \quad (4)$$

where $\mathcal{S} = \langle S_1, \dots, S_{n-k} \rangle$ with stabilizer generators $S_i (1 \leq i \leq n-k)$ is an Abelian subgroup of G_n called the stabilizer group. Then by measuring the eigenvalues of the stabilizer generators, the syndrome of errors can be revealed. An error $E \in G_n$ is detectable if it anticommutes with some stabilizer generator $S_i (1 \leq i \leq n-k)$ which results in a nonzero syndrome, otherwise it is undetectable and the syndrome is zero. The minimum distance

d of code Q is the minimum weight of an undetectable error $E \in G_n$, which does not belong to the stabilizer group. We denote by $Q = [[n, k, d]]_q$. Q is called nondegenerate if the stabilizer group $\mathcal{S}$ does not contain a nontrivial element whose weight is less than d , otherwise it is degenerate. CSS codes present a direct way to construct stabilizer codes from two classical linear codes C_X and C_Z that satisfy the orthogonal relationship so that the stabilizer generators commute with each other, i.e., $C_X^\perp \subseteq C_Z$.

qSCPC codes – Denote by $C_1 = [n_1, k_1, d_1]_q$ and $C_2 = [k_1, k_2, d_2]_q$ two classical linear codes over $GF(q)$. We use them to construct the serially concatenated parity-check (SCPC) code $\mathcal{C}_X$, where C_1 and C_2 are used as the inner and outer codes, respectively. Let the parity check matrix and generator matrix of C_i be H_i and G_i ($i = 1, 2$), respectively. We serially concatenate the parity check matrix of C_2 with the generator matrix of C_1 to derive

$$\mathcal{H}_X = H_2 G_1, \quad (5)$$

which is used as the parity check matrix of the SCPC code $\mathcal{C}_X = [n_1, k_X]_q$, where $k_X = n_1 - k_1 + k_2$. Denote by

$$\mathcal{H}_Z = H_1 \text{ and } \mathcal{C}_Z = C_1. \quad (6)$$

It is easy to see that $\mathcal{H}_X \mathcal{H}_Z^T = 0$ and $\mathcal{C}_X^\perp \subseteq \mathcal{C}_Z$. Thus we can derive qSCPC codes with parameters $\mathcal{Q} = [[n_1, k_2]]_q$ by using the CSS construction.

We show that qSCPC codes include the family of BS codes. Take $H_1 = I_{\tilde{n}_2} \otimes \tilde{H}_1$, $G_1 = I_{\tilde{n}_2} \otimes \tilde{G}_1$ and $H_2 = \tilde{H}_2 \otimes I_{\tilde{k}_1}$, where $\tilde{H}_1$ and $\tilde{G}_1$ are the parity check and generator matrix of $\tilde{C}_1 = [\tilde{n}_1, \tilde{k}_1, \tilde{d}_1]$, respectively, and $\tilde{H}_2$ is the parity check matrix of $\tilde{C}_2 = [\tilde{n}_2, \tilde{k}_2, \tilde{d}_2]$. We have

$$\mathcal{H}_X = H_2 G_1 = \tilde{H}_2 \otimes \tilde{G}_1, \text{ and } \mathcal{H}_Z = I_{\tilde{n}_2} \otimes \tilde{H}_1. \quad (7)$$

Then $\mathcal{H}_X$ and $\mathcal{H}_Z$ can lead to the BS code construction [10]. Therefore the BS code can be seen as a specific case of qSCPC codes. It is known that BS codes are asymptotically bad and have a minimum distance scaling at most as the square root of the block length. We show that qSCPC codes are asymptotically good and can meet the quantum GV bound.

Theorem 1 *There exist qSCPC codes with parameters $\mathcal{Q} = [[n_{\mathcal{Q}}, k_{\mathcal{Q}}, d_{\mathcal{Q}}]]_q$ meeting the quantum GV bound asymptotically, i.e., $\mathcal{Q}$ can achieve a quantum rate*

$$\frac{k_{\mathcal{Q}}}{n_{\mathcal{Q}}} \geq 1 - 2H_q(\delta_{\mathcal{Q}}) - o(1) \quad (8)$$

for any block length $n_{\mathcal{Q}} \rightarrow \infty$, where $\delta_{\mathcal{Q}} = d_{\mathcal{Q}}/n_{\mathcal{Q}}$ is the relative minimum distance, and $H_q(x) = x \log_q(q-1) - x \log_q x - (1-x) \log_q(1-x)$ is the q -ary entropy function. There also exist asymptotically good asymmetric qSCPC codes $\mathcal{A} = [[n_{\mathcal{A}}, k_{\mathcal{A}}, d_Z/d_X]]_q$ such that

$$\frac{k_{\mathcal{A}}}{n_{\mathcal{A}}} \geq 1 - H_q(\delta_X) - H_q(\delta_Z) - o(1) \quad (9)$$

for any block length $n_{\mathcal{A}} \rightarrow \infty$, where $\delta_X = d_X/n_{\mathcal{A}}$ and $\delta_Z = d_Z/n_{\mathcal{A}}$ are the relative distances, and $0 \leq \delta_X \leq \delta_Z \leq 1 - 1/q$.

Before the proof, we need to review some known results about classical alternant codes which are used as the outer codes in the concatenation construction. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n distinct elements of $GF(q^m)$ and let $v_1, v_2, \dots, v_n$ be n nonzero elements of $GF(q^m)$, where $1 \leq n \leq q^m$. Denote by $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$. Then for any $1 \leq k \leq n-1$, the generalized Reed-Solomon (GRS) code $\text{GRS}_k(\mathbf{a}, \mathbf{v})$ is defined by

$$\begin{aligned} \text{GRS}_k(\mathbf{a}, \mathbf{v}) \equiv \{ & (v_1 F(\alpha_1), v_2 F(\alpha_2), \dots, v_n F(\alpha_n)) \mid \\ & F(x) \in GF(q^m)[x], \deg F(x) < k \}. \end{aligned} \quad (10)$$

GRS codes are a famous family of maximum-distance-separable (MDS) codes with parameters $\text{GRS}_k(\mathbf{a}, \mathbf{v}) = [n, k, n-k+1]_{q^m}$ whose dual codes are still GRS codes $\text{GRS}_k(\mathbf{a}, \mathbf{v})^\perp = \text{GRS}_r(\mathbf{a}, \mathbf{y})$. Let u be any vector of length n with elements over $GF(q)$. According to [33, Chap. 12], we know that the number of GRS codes $\text{GRS}_k(\mathbf{a}, \mathbf{v})$ that contain u is at most $(q^m - 1)^k$, where k and $\mathbf{a}$ are fixed while $\mathbf{v}$ is varying.

Alternant codes are obtained as subfield subcodes of GRS codes, i.e., $\mathcal{A}_r(\mathbf{a}, \mathbf{y}) \equiv \text{GRS}_k(\mathbf{a}, \mathbf{v})|_{GF(q)}$, where $r = n - k$ is the degree of alternant codes. It is known that the dimension and the distance of alternant codes $\mathcal{A}_r(\mathbf{a}, \mathbf{y})$ satisfy $k_A \geq n - mr$ and $d_A \geq r + 1$, respectively, see [33, Chap. 12]. For fixed k and $\mathbf{a}$, but varying $\mathbf{y}$, it is easy to see that the number of alternant codes $\mathcal{A}_r(\mathbf{a}, \mathbf{y}) \equiv \text{GRS}_k(\mathbf{a}, \mathbf{v})|_{GF(q)}$ that contain $u \in GF(q)^n$ is also at most $(q^m - 1)^k$. Alternant codes form a large family of linear codes which include many important subclasses, e.g., Bose-Chaudhuri-Hocquenghem (BCH) codes and Goppa codes. They play a significant role in the McEliece cryptosystem, and furthermore alternant codes can meet the GV bound asymptotically [33]. We use alternant codes as the outer codes in qSCPC codes.

Proof: Let $C_1 = [n_1, k_1, d_1]_q$ be an arbitrary linear code. Let $C_2 = [k_1, k_2, d_2]_q$ be an alternant code. We construct a pair of dual-containing codes $\mathcal{C}_X^\perp \subseteq \mathcal{C}_Z$ according to (5) and (6). By

using the CSS construction, we can derive the qSCPC codes $\mathcal{Q} = [[n_1, k_2]]_q$ whose minimum distance is given by

$$\begin{aligned} d_{\mathcal{Q}} &= \min\{\text{wt}(c) | c \in (\mathcal{C}_Z \setminus \mathcal{C}_X^\perp) \cup (\mathcal{C}_X \setminus \mathcal{C}_Z^\perp)\} \\ &= \min\{\omega_1, \omega_2\}, \end{aligned} \quad (11)$$

where $\omega_1 = \min\{\text{wt}(c) | c \in \mathcal{C}_Z \setminus \mathcal{C}_X^\perp\} \geq d_1$, and $\omega_2 = \min\{\text{wt}(c) | c \in \mathcal{C}_X \setminus \mathcal{C}_Z^\perp\}$. Since $\mathcal{C}_Z = C_1$ can be chosen arbitrarily, we let C_1 be an asymptotically good linear code which can meet the classical GV bound, i.e.,

$$\frac{k_1}{n_1} \geq 1 - H_q(\delta_1) - o(1), \quad (12)$$

where $\delta_1 = d_1/n_1$ is the relative minimum distance. Next we need to determine ω_2 . Since we know nothing about the distance of the dual code $C_1^\perp$, there maybe some (very) low weight vectors coming from $C_1^\perp$. If we do not consider error degeneracy, the minimum distance of $\mathcal{C}_X$ is unknown and maybe very small. Fortunately, vectors come from $C_1^\perp$ do not affect the computation of ω_2 because they are degenerate. For any nonzero vector ν with Hamming weight less than λ_X , we denote the syndrome from the inner code by

$$S_\nu \equiv G_1 \nu^T. \quad (13)$$

If $S_\nu = 0$, then ν must belong to $C_1^\perp$ and is degenerate when we compute ω_2 . Therefore we only need to consider the case which makes $S_\nu \neq 0$. Let $1 \leq f_X \leq k_1$ and denote by $k_{f_X} = k_1 - (k_1 - f_X)/m$, where $m|(n_1 - f_X)$. For some given $\mathbf{a} \in GF(q^m)^{k_1}$ and varying $\mathbf{v} \in GF(q^m)^{k_1}$, let

$$C_2 \equiv \mathcal{A}_{k_1 - k_{f_X}}(\mathbf{a}, \mathbf{y}) = \text{GRS}_{k_{f_X}}(\mathbf{a}, \mathbf{v})|GF(q) \quad (14)$$

be an alternant code. Then the dimension of C_2 satisfies

$$k_2 \geq k_1 - m(k_1 - k_{f_X}) = f_X. \quad (15)$$

We know that the number of alternant codes C_2 that contain S_ν does not relate to the Hamming weight of S_ν and is at most

$$(q^m - 1)^{k_1 - (k_1 - f_X)/m}. \quad (16)$$

On the other hand, the total number of alternant codes C_2 is equal to the number of the choice of $\mathbf{v}$, and it is equal to $(q^m - 1)^{k_1}$. Therefore if

$$\sum_{j=1}^{\lambda_X-1} (q-1)^j \binom{n_1}{j} < (q^m - 1)^{(k_1 - f_X)/m}, \quad (17)$$

then there exists an alternant code such that there do not exist vectors $c \in \mathcal{C}_X \setminus \mathcal{C}_Z^\perp$ with weight less than λ_X . Taking the limit as $n_1 \rightarrow \infty$ in (17), we can derive

$$\frac{k_2}{n_1} \geq \frac{f_X}{n_1} = \frac{k_1}{n_1} - H_q\left(\frac{\lambda_X}{n_1}\right) - o(1). \quad (18)$$

Combining (11), (12), and (18) above together, we have

$$\begin{cases} \frac{k_1}{n_1} \geq 1 - H_q\left(\frac{d_1}{n_1}\right) - o(1), \\ \frac{k_2}{n_1} \geq \frac{k_1}{n_1} - H_q\left(\frac{\lambda_X}{n_1}\right) - o(1), \\ d_{\mathcal{Q}} \geq \min\{d_1, \lambda_X\}. \end{cases} \quad (19)$$

If we let $\lambda_X = d_1 + c$, where c is any constant, then we have

$$\frac{k_{\mathcal{Q}}}{n_1} = \frac{k_2}{n_1} \geq 1 - 2H_q\left(\frac{d_{\mathcal{Q}}}{n_1}\right) - o(1). \quad (20)$$

If we admit the asymmetry between the minimum distance ω_1 and ω_2 , and use the CSS construction for asymmetric quantum codes, then we have

$$\frac{k_{\mathcal{Q}}}{n_1} \geq 1 - H_q\left(\frac{\omega_1}{n_1}\right) - H_q\left(\frac{\omega_2}{n_1}\right) - o(1). \quad (21)$$

□

It is known that there exist families of asymptotically good LDPC codes which can achieve the classical GV bound. But the dual of LDPC codes has a low-density generator matrix (LDGM) and thus the minimum distance is constant. If we use LDPC codes as the inner code C_1 in SCPC codes, then the pure minimum distance of qSCPC codes without considering error degeneracy is constant to the block length. After introducing error degeneracy, the low weight codewords in the dual of LDPC codes do not affect the counting arguments in the proof of Theorem 1. Therefore, qSCPC codes are highly degenerate and the minimum distance is greatly improved from a constant to be linear with the block length.

Construction of qLDPC codes from qSCPC codes by using the EKZ distance balancing— In [16, 29], Hastings *et al.* randomly choose the X - and Z -stabilizer generators with average weight $\beta \log N$ and $\Theta(N)$, respectively. Then for sufficiently large β , they show that the resultant quantum codes of length N have $d_X, d_Z = \Theta(N)$ with high probability. By using the weight reduction procedure and the distance balancing construction, Hastings shows that there exists a family of qLDPC codes of distance $\Omega(N^{2/3}/\text{polylog}(N))$ and dimension $\Omega(N^{2/3}/\text{polylog}(N))$.

In our construction, we can use any asymptotically good LDPC codes as the inner codes to derive qSCPC codes with linear distance and dimension. Further, the Z -stabilizer generators satisfy the LDPC constraint. Then we can derive families of qLDPC codes by using the weight reduction procedure and the EKZ distance balancing construction. We use some notations and definitions of quantum codes in [29, 34]. For any CSS code of length N , denote w_X and w_Z by the maximum weight of X - and Z -stabilizer generators, respectively. For $1 \leq i \leq N$, denote the number of X - and Z -stabilizer generators acting on the i -th qubit by q_{X_i} and q_{Z_i} , respectively. Let $q_X = \max\{q_{X_i} | 1 \leq i \leq N\}$ and $q_Z = \max\{q_{Z_i} | 1 \leq i \leq N\}$. First we have the following result:

Lemma 1 *Let N be a positive integer and $\ell = (\log N)^{2+\epsilon}$, $\epsilon > 0$. There exists a family of weight reduced qSCPC codes $Q = [[N^2\ell, K, d_Z/d_X]]$ with $w_X, q_X = O(1)$, $w_Z, q_Z = O(\log \log N)$, $K = \Omega(N)$, and $d_X = \Omega(N^2)$, $d_Z = \Omega(N\ell)$.*

Proof: From Theorem 1, we know that there exist qSCPC codes $Q' = [[N, K, d'_Z/d'_X]]$ with $K = \Theta(N)$, and $d'_X, d'_Z = \Theta(N)$. Further, the X -stabilizer generators of Q' are LDPC, then we have $w'_X, q'_X = O(1)$. But $w'_Z, q'_Z = \Theta(N)$. We use the thickening in [29] to reduce the q'_Z of Q' and we take $\ell' = \Theta(N)$, $w = \Theta(\log \log N)$. According to [29, Lemma 3], we can reduce q'_Z to $O(\log \log N)$ while the length and X -distance of Q' are increased to $\Omega(N^2)$. By using the coning in [29, Lemma 8], w'_Z is reduced to $O(\log \log N)$, the length is increased to $N^2\ell$, and the Z -distance of Q' is increased to $\Omega(N\ell)$. Therefore, we can derive a weight reduced qSCPC code $Q = [[N^2\ell, K, d_Z/d_X]]$, where $K = \Omega(N)$, and $d_X = \Omega(N^2)$, $d_Z = \Omega(N\ell)$. $\square$

Proposition 1 *There exists a family of qLDPC codes of length N , distance $\Omega(N^{2/3}/\log \log N)$, and dimension $\Omega(N^{2/3}/\ell)$, where $\ell = (\log N)^{2+\epsilon}$, $\epsilon > 0$.*

Proof: By using the soundness parameters of quantum codes in [29, 34], we can reduce the qSCPC codes into qLDPC codes. Notice that Ref. [29] used quantum codes of $w_X, q_X, w_Z, q_Z = O(\log N)$ to derive qLDPC codes of distance $\Omega(N^{2/3}/\text{polylog} N)$. Here, our weight reduced qSCPC codes in Lemma 1 have $w_X, q_X, w_Z, q_Z = O(\log \log N)$. By using the soundness properties in [29, 34] and the EKZ distance balancing construction, we can derive qLDPC codes of distance $\Omega(N^{2/3}/\log \log N)$ and dimension $\Omega(N^{2/3}/\ell)$. $\square$

It is shown that Proposition 1 improves the result of [29, Theorem 2] by a polylogarithmic scaling. Also, the distance of Proposition 1 is slightly better than that of qLDPC codes in [17] of distance $\Omega(N^{2/3}/\log N)$.

On the other hand, if we use LDGM codes as the inner codes and asymptotically good LDPC codes as the outer codes to construct qSCPC codes, we can obtain asymmetric qLDPC codes with $d_X = \Theta(N)$ and $d_Z = O(1)$. By using the EKZ distance balancing construction, we can obtain qLDPC codes with positive rates and minimum distance proportional to the square root of the block length. They have the same minimum distance scaling with the HP qLDPC codes while maintaining a positive rate. However we show that the exact minimum distance upper bound of our construction is better than that of the HP qLDPC codes.

Suppose that we have an LDPC code $C_0 = [n_0, k_0]$ whose parity check matrix P_0 has row weight w_r and column weight w_g , where w_r and w_g are both constant and $0 < w_g < w_r$. Denote by $\alpha_0 = w_g/w_r = r_0/n_0$. Then we can derive an LDGM code $C_1 = [n_1, k_1, d_1]$ whose generator matrix and parity check matrix are given by

$$G_1 = [I, P_0], \text{ and } H_1 = [P_0^T, I],$$

respectively. It is easy to see that $n_1 = n_0 + r_0, k_1 = r_0$, and $d_1 = w_r + 1$. Next, we let $C_2 = [k_1, k_2, d_2]$ be an arbitrary asymptotically good LDPC code, i.e., $k_2 = \kappa_2 k_1, d_2 = \delta_2 k_1$, where $0 < \kappa_2, \delta_2 < 1$.

Denote the SCPC code of C_1 and C_2 by $\mathcal{C}_Z$ with a parity check matrix $\mathcal{H}_Z = H_2 G_1$. We use $\mathcal{C}_Z$ to correct Z -errors and use C_1 to correct X -errors. Then we can derive an asymmetric qLDPC code with parameters

$$\mathcal{A} = [[n_1, k_2, d_Z/d_X]], \quad (22)$$

where $d_Z \geq d_2/w_g$ and $d_X > w_r$. Therefore, we derive a family of qLDPC codes with a large asymmetry, where the Z distance $d_Z = \Omega(n_1)$ and the X distance d_X is a constant. Next, we use the distance balancing method in [14, 34] to improve the X distance. Let $C_A = [n_A, k_A, d_A]$ be an asymptotically good LDPC code with $k_A = \kappa_A n_A, d_A = \delta_A n_A$, where $0 < \kappa_A, \delta_A < 1$, and let $d_A = d_2/(w_r w_g)$ (we assume that $d_2/(w_r w_g)$ is an integer). After the distance balancing, we can get a family of qLDPC codes with parameters

$$\begin{aligned} \mathcal{Q}_B &= [[n_1 n_A + r_2 r_A, k_2 k_A, \mathcal{D} \geq d_2/w_g]] \\ &= [[N, \kappa_2 \kappa_A \alpha_0 \hat{n}_0^2, \mathcal{D} \geq \sqrt{\delta_2 \delta_A} \hat{n}_0]] \end{aligned} \quad (23)$$

where $N = (1 + \alpha_0 + \rho_2 \rho_A \alpha_0) \hat{n}_0^2$, $\rho_2 = 1 - \kappa_2, \rho_A = 1 - \kappa_A$, and $\hat{n}_0 = \sqrt{\delta_2/(\delta_A w_r^2)} n_0$. Therefore the balanced qSCPC codes have a minimum distance scaling as $\Omega(\sqrt{n_{\mathcal{Q}_B}})$ while

maintaining a positive code rate. We compare (23) with the parameters of the HP qLDPC codes in [11] with parameters

$$\mathcal{Q}_P = [(1 + \rho_P^2)n_P^2, \kappa_P^2 n_P^2, \geq \delta_P n_P], \quad (24)$$

where $0 < \kappa_P, \delta_P < 1$, n_P is the length of the corresponding classical component code. To facilitate comparisons, we take $\delta_2 = \delta_A = \delta_P$, and for asymptotically good LDPC codes, e.g., meeting the classical GV bound [13], we can also assume that $\kappa_2 = \kappa_A = \kappa_P$ and $\rho_2 = \rho_A = \rho_P$. We let $(1 + \alpha_0 + \rho_2 \rho_A \alpha_0) \hat{n}_0^2 = (1 + \rho_P^2) n_P^2$ so that the balanced qSCPC codes have the same code length with HP qLDPC codes, then

$$\sqrt{\delta_2 \delta_A} \hat{n}_0 = \sqrt{\frac{1 + \rho_2^2}{1 + \alpha_0 + \alpha_0 \rho_2^2}} \delta_P n_P. \quad (25)$$

If $\alpha_0 < \rho_2^2/(1 + \rho_2^2)$, then the minimum distance of the balanced qSCPC codes is larger than that of the HP qLDPC codes. One disadvantage is that the dimension of the balanced qSCPC codes maybe smaller than that of the HP qLDPC codes. However, the minimum distance upper bound of qLDPC codes with positive rate is improved by the balanced qSCPC codes.

We take an example to illustrate the improvement. According to the classical GV bound, the relative minimum distance δ_P is less than $1/2$ if maintaining a positive code rate. Then the minimum distance of any HP qLDPC code of length N with a positive code rate is less than $0.36\sqrt{N}$. We take $\delta_2 = \delta_A = 0.49$ and then $\rho_2 = \rho_A \approx 0.997$. We can take an arbitrary $\alpha_0 < \rho_2^2/(1 + \rho_2^2) \approx 0.498$, e.g., $\alpha_0 = 0.1$. Then the minimum distance of the balanced qSCPC codes of length N is $\approx 0.447\sqrt{N}$ which is larger than $0.36\sqrt{N}$. In more general, we have the following result.

Proposition 2 *There exists a family of qLDPC codes of length N and minimum distance $0 < d \leq (0.5 - \epsilon)\sqrt{N}$, where $\epsilon > 0$ is constant, while maintaining a positive code rate.*

Proof: By the GV bound, we take $\alpha_0 = \epsilon_0$, $\delta_2 = \delta_A = 0.5 - \epsilon_1$ and then $\rho_2 = \rho_A = 1 - \epsilon_2$, where $\epsilon_0, \epsilon_1, \epsilon_2 > 0$. According to (23), we have

$$\frac{d_{\mathcal{Q}_B}}{\sqrt{N_{\mathcal{Q}_B}}} \geq \frac{0.5 - \epsilon_1}{\sqrt{1 + \epsilon_0 + (1 - \epsilon_2)\epsilon_0}} = 0.5 - \epsilon. \quad (26)$$

□

Next, we discuss the decoding complexity of our codes. For correcting Z -errors, we let the component code C_2 be a classical expander code in [35]. Let $G = (\mathbf{L} \cup \mathbf{R}, E)$ be a (c, d) -regular bipartite graph with each variable node in $\mathbf{L}$ has degree c , each constraint node in $\mathbf{R}$ has degree d , and $d > c$. G is called a (δ, ϑ) -left expander graph if for every subset $S \subseteq \mathbf{L}$ with $|S| \leq \delta|\mathbf{L}|$, there is $|\mathcal{N}(S)| \geq \vartheta c|S|$, where $\mathcal{N}(S) \subseteq \mathbf{R}$ is the set of neighbors of S . Similarly, if for every subset $T \subseteq \mathbf{R}$ with $|T| \leq \tilde{\delta}|\mathbf{R}|$, there is $|\mathcal{N}(T)| \geq \tilde{\vartheta}d|T|$, then G is called a $(\tilde{\delta}, \tilde{\vartheta})$ -right expander graph. Let C_d be a classical linear code of length d . The classical expander code $C(G, C_d)$ is the linear code of length $n = |\mathbf{L}|$ such that for every constraint node j in $\mathbf{R}$, the neighbors of j consist a codeword of C_d . Expander codes are asymptotically good codes and can be decoded very efficiently [35].

Lemma 2 ([35]) *Let $G = (\mathbf{L} \cup \mathbf{R}, E)$ be a (c, d) -regular $(\delta, 3/4 + \epsilon)$ -left expander graph for any $\epsilon > 0$. Let $C_d = [d, d - 1, 2]$ be the code consisting of codewords of even weight. Then the expander code $C(G, C_d)$ of block length $n = |\mathbf{L}|$ has a rate $1 - c/d$ and can decode any $\delta_0 < \delta(1 + 4\epsilon)/2$ fraction of adversarial errors in $O(n)$ time. Further, $C(G, C_d)$ can also be decoded in $O(\log(n))$ time by using a parallel decoder that uses a circuit of size $O(n \log(n))$ and depth $O(\log(n))$.*

We let the component code C_2 in the SCPC code be an expander code and we have the following result.

Lemma 3 *Let $G = (\mathbf{L} \cup \mathbf{R}, E)$ be a (c, d) -regular $(\delta, 3/4 + \epsilon)$ -left expander graph for any $\epsilon > 0$. There exist a family of asymmetric qLDPC codes with parameters $\mathcal{Q} = [[(1 + \alpha_0)n_0, (1 - c/d)\alpha_0 n_0]]$, where $0 < \alpha_0 < 1$ and $n_0 > 0$ is an integer. $\mathcal{Q}$ can correct any pattern of Z -errors of weight less than $\delta(1 + 4\epsilon)n_0\alpha_0/2$ and simultaneously correct any pattern of X -errors of some constant weight in linear time. Further, we can decode both Z -errors and X -errors in logarithmic time with a parallel decoder that uses an almost linear number of classical processors.*

Proof: Let C_1 be an LDGM code and let C_2 be an expander code of length k_1 with a (c_2, d_2) -regular (δ_2, ϑ_2) -left expander graph. The expansion is given by $\vartheta_2 = 3/4 + \epsilon_2$ for any $\epsilon_2 > 0$. Then C_2 is asymptotically good and can be decoded efficiently according to Lemma 2. We can obtain an asymmetric qLDPC code $\mathcal{A}$ with parameters as in (22). Suppose that there is an adversarial Z -error e_Z of weight less than $\delta_2(1 + 4\epsilon_2)k_1/(2w_g)$ happening in the received

codeword. By measuring the ancilla, we can obtain the syndrome information as follows

$$S_Z = H_2 G_1 e_Z^T. \quad (27)$$

If $G_1 e_Z^T = 0$, which results in a zero syndrome, then e_Z is degenerate and we do not need to correct it. Thus we only need to consider $G_1 e_Z^T \neq 0$. Here, we regard $\tilde{S}_Z \equiv G_1 e_Z^T$ as a logical error sequence happening to the outer code C_2 and we do the decoding by using the expander code C_2 . Since the weight of $\tilde{S}_Z$ must be less than $\delta_2(1 + 4\epsilon_2)k_1/2$, we can always correct it efficiently in linear time according to Lemma 2. After the decoding, the decoded $\tilde{S}_Z$ can be seen as the syndrome of $G_1 e_Z^T$, i.e., $G_1 e_Z^T = \tilde{S}_Z$. We just let $\tilde{e}_Z = (\tilde{S}_Z^T, 0, \dots, 0)$ and notice that $G_1 \tilde{e}_Z \stackrel{also}{=} \tilde{S}_Z$. Thus $\tilde{e}_Z$ and e_Z are degenerate and we utilize the error degeneracy phenomenon to facilitate the decoding. By Lemma 2, we can also decode Z -errors in logarithmic time with a parallel decoder that uses an almost linear number of classical processors.

Next, we decode X -errors by using the LDGM code C_1 . If we use the one-step majority-logic (OSMLG) decoding algorithm to decode the LDGM code, then we can correct any adversarial X -error of weight less than $\lfloor (d_1 - 1)/2 \rfloor$ in $O(n_0)$ time. Further, the OSMLG algorithm can be easily carried out in parallel, and then we can decode X -errors in constant time and require a classical hardware overhead linear to n_0 . $\square$

The qLDPC codes in Lemma 3 have a large asymmetry for correcting Z -errors and X -errors, respectively. We apply the distance balancing methods in [14, 34] to improve the X -error correction ability of $\mathcal{A}$ by using an additional classical LDPC code.

Theorem 2 *Let $G_2 = (\mathbf{L}_2 \cup \mathbf{R}_2, E_2)$ be a (c_2, d_2) -regular $(\delta_2, 3/4 + \epsilon_2)$ -left expander graph, and let $G_A = (\mathbf{L}_A \cup \mathbf{R}_A, E_A)$ be a (c_A, d_A) -regular $(\delta_A, 3/4 + \epsilon_A)$ -left expander graph for any $\epsilon_2, \epsilon_A > 0$. Let $\rho_2 = c_2/d_2$, $\rho_A = c_A/d_A$ and $0 < \alpha_0 < 1$ be any constant. Then there exist a family of qLDPC codes of length $(1 + \alpha_0 + \rho_2 \rho_A \alpha_0) \hat{n}_0^2$ which can correct any quantum error pattern of weight less than*

$$\sqrt{\delta_2 \delta_A} (1 + 4\epsilon_2) \hat{n}_0 / 2$$

in linear time, where $\hat{n}_0 = \sqrt{\delta_2 / (\delta_A w_r^2)} n_0$, $n_0 > 0$ is an integer.

Proof: We can derive the conclusion by combining Lemma 3 and the distance balancing construction in [14]. $\square$

We compare Theorem 2 with the quantum expander codes in [23]. Let $G_P = (\mathbf{L}_P \cup \mathbf{R}_P, E)$ be a (c_P, d_P) -regular $(\delta_{\mathbf{L}_P}, 5/6 + \epsilon_{\mathbf{L}_P})$ -left and $(\delta_{\mathbf{R}_P}, 5/6 + \epsilon_{\mathbf{R}_P})$ -right expander graph for any $\epsilon_{\mathbf{L}_P}, \epsilon_{\mathbf{R}_P} > 0$. Denote by $\rho_P = c_P/d_P$. Let H be the adjacency matrix of graph G_P of size $r_P \times n_P$ with $r_P = \rho_P n_P$. Denote by

$$\begin{aligned} H_X &= (I_{n_P} \otimes H, H^T \otimes I_{r_P}), \\ H_Z &= (H \otimes I_{n_P}, I_{r_P} \otimes H^T). \end{aligned}$$

the parity check matrices of two classical linear codes C_X and C_Z , respectively.

Proposition 3 [23, Theorem 2] *There exists a decoding algorithm for the associated quantum code $\mathcal{Q}_P = (C_X, C_Z)$ that runs in time linear in the code length of $(1 + \rho_P^2)n_P^2$, and that decodes any quantum error pattern of weight less than*

$$\frac{1}{3(1 + d_P)} \min\{\delta_{\mathbf{L}_P} n_P, \delta_{\mathbf{R}_P} \rho_P n_P\}. \quad (28)$$

Let $(1 + \alpha_0 + \rho_2 \rho_A \alpha_0) \hat{n}_0^2 = (1 + \rho_P^2) n_P^2$ and let $(c_2, d_2) = (c_A, d_A)$, $\epsilon_2 = \epsilon_A$, and $\delta_2 = \delta_A$. Then the balanced qSCPC codes in Theorem 2 can correct much more adversarial errors than the quantum expander codes in Proposition 3 of the same length. Further, the quantum expander codes in Proposition 3 need the corresponding expander graph to be left and right expanding at the same time, which is quite strict for expander graphs. The balanced qSCPC codes in Theorem 2 only need left expander graphs. More importantly, the required expansion in Theorem 2 is smaller than that in Proposition 3. Therefore the restriction to the expander graph of our codes is much more relaxed than that of the quantum expander codes.

A family of fast encodable and decodable qSCPC codes— In practice, we need quantum codes to be efficiently encoded and decoded. In the following, we consider a special case of qSCPC codes which can be fast encoded and decoded. First, we have the following result about asymmetric qSCPC codes.

Theorem 3 *There exists a family of asymmetric qSCPC codes $\mathcal{A} = [[N, \Omega(N/n_0), d_Z \geq \Omega(N/n_0)/d_X \geq n_0]]$ which can be decoded in $O(N/n_0 + n_0)$ time. Moreover, $\mathcal{A}$ can be encoded by a circuit of size $O((N/n_0)^2 + N)$ and depth $O(N/n_0 + n_0)$.*

Proof: Let $C_0 = [n_0, 1, n_0]$ be a binary repetition code whose parity check matrix and generator matrix are given by:

$$H_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix}, \text{ and } G_0 = (1 \ 1 \ \cdots \ 1), \quad (29)$$

respectively. Let $C_1 = [N, N/n_0, n_0]$ be a linear code with the parity check matrix $H_1 = \mathbf{I} \otimes H_0$ and the generator matrix $G_1 = \mathbf{I} \otimes G_0$, where $\mathbf{I}$ is an identity matrix of size N/n_0 . Let $C_2 = [N/n_0, k_2, d_2]$ be an asymptotically good expander code which can be decoded in linear time [35], i.e., $k_2 = \gamma_2 N/n_0$, and $d_2 = \delta_2 N/n_0$, where $0 < \gamma_2, \delta_2 < 1$. Let the parity check and generator matrices of C_2 be H_2 and G_2 , respectively. Then we can construct an asymmetric qSCPC code $\mathcal{A} = [[N, k_2, d_Z/d_X]]$ with

$$\mathcal{H}_Z = H_2 G_1, \text{ and } \mathcal{H}_X = H_1. \quad (30)$$

It is not difficult to verify that $d_Z \geq d_2$ and $d_X \geq n_0$.

Next, we consider the decoding of $\mathcal{A}$. For the Z -error e_Z , the syndrome is given by

$$S_Z \equiv \mathcal{H}_Z e_Z^T = H_2 S_Z^{(1)}, \quad (31)$$

where $S_Z^{(1)} = G_1 e_Z^T$. Suppose that $\text{wt}_H(e_Z) \leq \lfloor (d_2 - 1)/2 \rfloor$, then we must have $\text{wt}_H(S_Z^{(1)}) \leq \lfloor (d_2 - 1)/2 \rfloor$. Thus we can decode $S_Z^{(1)} = (S_{Z_1}^{(1)}, \dots, S_{Z_n}^{(1)})$ by using the expander code C_2 in $O(N/n_0)$ time. Let $\tilde{e}_Z = (S_{Z_1}^{(1)}, 0, \dots, S_{Z_n}^{(1)}, \dots, 0)$ be the decoded Z -error, where the 0s correspond to the redundant bit positions in G_1 . Then we have $G_1(e_Z + \tilde{e}_Z)^T = 0$ and $\mathcal{Q}$ is degenerate with respect to $e_Z + \tilde{e}_Z$.

Then we use C_1 to correct the X -error e_X and suppose that $\text{wt}_H(e_X) \leq \lfloor (n_0 - 1)/2 \rfloor$. We divide the parity check matrix H_1 into n_0 sub-blocks by columns, and each sub-block corresponds to a diagonal block H_0 in H_1 . We know that there are at most $\lfloor (n_0 - 1)/2 \rfloor$ erroneous sub-blocks in e_X and each erroneous sub-block has at most $\lfloor (n_0 - 1)/2 \rfloor$ errors. For each erroneous sub-block, we use the OSMLG decoding principle to check the error bit that corresponds to the last column in H_0 . The running time is $O(1)$. Thus the running time for checking all the erroneous sub-blocks is $O(n_0)$. We know that H_0 is in a systematic form and every error bit that corresponds to the last column of each sub-block is already

known. Then we can get other error bits directly without any additional decoding. Thus the total decoding of X -errors of weight up to $\lfloor (n_0 - 1)/2 \rfloor$ can be done in $O(n_0)$ time. Overall, $\mathcal{A}$ can be decoded in $O(N/n_0 + n_0)$ time.

Next, we consider the encoding of $\mathcal{A}$ and we use the standard encoding method of CSS codes in [26] to do that. We can encode the primitive pure state as follows

$$|u + \mathcal{C}_X^\perp\rangle \equiv \frac{1}{\sqrt{|\mathcal{C}_X^\perp|}} \sum_{v \in \mathcal{C}_X^\perp} |u + v\rangle, \quad (32)$$

where $u \in \mathcal{C}_Z \setminus \mathcal{C}_X^\perp$. It is not difficult to verify that the generator matrix of $\mathcal{C}_Z$ is given by

$$\mathcal{G}_Z = \begin{pmatrix} H_1 \\ \mathbf{0} & G_2 \end{pmatrix}, \quad (33)$$

where $\mathbf{0}$ is a zero matrix of size $k_2 \times (N - N/n_0)$, and $G_2 = [I \ P_2]$ is in the standard form. Denote by $P_2 = [\mathbf{P}_1, \dots, \mathbf{P}_{r_2}]$, where each $\mathbf{P}_i (1 \leq i \leq r_2)$ is a column of P_2 and $r_2 = N/n_0 - k_2$. Then the encoding circuit of the qSCPC code $\mathcal{A}$ is given in Fig. 1. In Stage I of the encoding circuit, the number of gates is determined by the density of P_2 and it is at most $O((N/n_0)^2)$. It is easy to see that the depth of Stage I is k_2 . While in Stage II, the number of C-NOT gates is exactly $N - N/n_0$ and the depth is n_0 . Therefore the size of the encoding circuit is $O((N/n_0)^2 + N)$ and the depth is $k_2 + n_0 = O(N/n_0 + n_0)$. $\square$

Notice that the last column in H_0 is quite dense and then H_1 is also dense, but we can transform H_0 into a sparse matrix by multiplying a series of elementary matrices in the left. Therefore the Z -stabilizer generators of $\mathcal{A}$ satisfy the LDPC constraint. In the fault tolerant settings, we require quantum codes to correct a linear number of random errors. Then we consider the correction of random errors by using qSCPC codes. We assume that the noise model used is the independent error model in [24, 31, 36]. Denote by p_x the probability of each X -error. It is shown that $\mathcal{A}$ can correct any adversarial X -error e_X of weight less than half the X distance bound in Theorem 3. We count all the uncorrectable errors of weight larger than half the minimum distance bound. It is easy to see that if there is at least one sub-block that has X -errors of weight larger than $d_0 = \lfloor (n_0 - 1)/2 \rfloor$, then e_X is undetectable. All the uncorrectable X -errors are counted as follows:

$$\begin{aligned}\mathcal{P}_X &\leq C_{\tilde{N}}^1 C_{n_0}^{d_0+1} (p_x^{d_0+1} (1-p_x)^{N-d_0-1} + C_{N-d_0-1}^1 p_x^{d_0+2} (1-p_x)^{N-d_0-2} + \dots + C_{N-d_0-1}^{N-d_0-1} p_x^N) \\ &= C_{\tilde{N}}^1 C_{n_0}^{d_0+1} p_x^{d_0+1} \sum_{i=0}^{N-d_0-1} C_{N-d_0-1}^i p_x^i (1-p_x)^{N-d_0-1-i}\end{aligned}\quad (34)$$

$$\leq \tilde{N} 2^{n_0-1} p_x^{d_0+1}, \quad (35)$$

where $\mathcal{P}_X$ is probability of uncorrectable X -errors and $\tilde{N} = N/n_0$. Denote $c > 0$ by any constant. If we let $n_0 = \Omega(\log N)$ and $p_x < 1/4^{c+1}$, then $N^c \mathcal{P}_X$ vanish as $N \rightarrow \infty$. If we let $n_0 = N^{c_0}$ and $p_x < 25\%$, where $0 < c_0 < 1$ is constant, then $N^c \mathcal{P}_X$ vanish as $N \rightarrow \infty$. Therefore the qSCPC code $\mathcal{A}$ in Theorem 3 can correct a linear number of random X -errors with high probability in $O(n_0)$ time as long as $n_0 = \Omega(\log N)$. However, for the correction of random Z -errors, we cannot get a non vanishing threshold.

If we let $n_0 = \sqrt{N}$ or $n_0 = \log N$, we have the following two special results:

Corollary 1 *There exists a family of quantum codes with parameters*

$$\mathcal{Q}_1 = [[N, \Omega(\sqrt{N}), \Omega(\sqrt{N})]] \quad (36)$$

which can correct all errors of weight smaller than half the minimum distance bound in $O(\sqrt{N})$ time. We show that $\mathcal{Q}_1$ can be encoded efficiently by a circuit of size $O(N)$ and depth $O(\sqrt{N})$. Moreover, $\mathcal{Q}_1$ can correct a random X -error with very high probability provided $p_x < 25\%$.

Corollary 2 *There exists a family of quantum codes with parameters*

$$\mathcal{Q}_2 = [[N, \Omega(N/\log N), \Omega(N/\log N)/\Omega(\log N)]] \quad (37)$$

which can correct an almost linear number of Z -errors and can also correct a linear number of X -errors with high probability in $O(N/\log N)$ time.

Asymmetric qSCPC codes can approach the Hashing bound—It is shown that asymmetric qSCPC codes in Theorem 1 can meet the asymmetric quantum GV bound for asymmetric quantum codes [37]. Further, if we let C_1 be LDPC codes which can approach the Shannon capacity [38], then asymmetric qSCPC codes can approach the capacity of asymmetric Pauli channels as the channel asymmetry goes to large. Suppose that we transmit quantum information over the Pauli channel

$$\varrho \mapsto p_I \varrho + p_x X \varrho X + p_y Y \varrho Y + p_z Z \varrho Z$$

FIG. 1. The encoding circuit of a family of qSCPC codes. The box indicated by P_i ($1 \leq i \leq r_2$) means that the qubits pass it are controlled by a qubit in quantum state $|\psi\rangle$ and if there is a “1” in the j th position of the i th column of the check part P_2 in G_2 , then there is a C-NOT target “ $\otimes$ ” in the j th qubit of the box P_i .

for an input state ρ , where X, Y, Z are the Pauli operators. Denote by the total error probability $\mathbf{p} = p_x + p_y + p_z$, and denote by $\zeta = (p_z + p_y)/(p_x + p_y)$ the asymmetry for the probabilities of Z -errors and X -errors. Usually, we let $p_x = p_y$, then we have $p_x = \mathbf{p}/(2\zeta + 1)$ and $p_z = \mathbf{p}(2\zeta - 1)/(2\zeta + 1)$. Since $\mathcal{C}_X$ can be decoded up to the GV bound under the bounded minimum distance decoder (BMDD) and LDPC codes can approach the Shannon capacity under the BP decoding [38, 39], then the rate of qSCPC codes $\mathcal{Q}$ can approach

$$1 - H_2(4p_x) - H_2(p_z + p_y) - o(1). \quad (38)$$

In Fig. 2, we compare the the limit of qSCPC codes in (38) with the Hashing bound given by $R_{\mathcal{Q}} = 1 - H_2(\mathbf{p})$ over Pauli channels with an asymmetry $\zeta = (p_z + p_y)/(p_x + p_y)$. If $\zeta = 1$, then the channel is symmetric with $p_x = p_y = p_z = \mathbf{p}/3$. It is shown that as the asymmetry ζ grows, the gap between qSCPC codes and the Hashing bound becomes increasingly smaller, e.g., when the asymmetry $\zeta = 10^2$ and $\zeta = 10^3$, the code rate gaps are less than 3×10^{-2} and 4×10^{-3} , respectively.

DISCUSSIONS AND CONCLUSIONS

The qSCPC codes in Theorem 1 can also be used to prove the security of the famous BB84 quantum key distribution (QKD) protocol directly [40]. By a simple calculation, we

FIG. 2. The comparison of the asymptotic bound of asymmetric qSCPC codes (solid lines) and the hashing bound (dashed lines) with different channel asymmetries $\zeta = 1, 100$, and 1000 . The horizontal axis is the total error probability in the Pauli channel and the vertical axis is the quantum code rate.

can obtain a total qubit error rate (QBER) for the security of BB84 over Pauli channels is less than 7.56%. Although it is lower than the bound of 11% in [40], our codes are not randomly generated and the LDPC component codes can be efficiently decoded. Since classical LDPC codes can approach the Shannon bound by using efficient BP decoder, qSCPC codes are particularly applicable to efficient and practical QKD. Furthermore, if we consider asymmetric errors in Pauli channels, we can further improve the total QBER for the security of BB84. For example, if we admit an asymmetry $\zeta = 100$ between Z -errors and X -errors, then the total QBER is less than 35.56%. Our codes are competitive with quantum Polar codes in [41, 42] that can achieve the coherent information. But Ref. [41] needs some preshared entanglement assistance.

We show that the qSCPC structure can lead to qLDPC codes of length N with distance $\Omega(N^{2/3}/\log \log N)$ and dimension $\Omega(N^{2/3}/\ell)$, where $\ell = (\log N)^{2+\epsilon}$, $\epsilon > 0$. The qSCPC scheme can also be used to derive qLDPC codes with positive rate and minimum distance upper bound better than the HP qLDPC codes. By using expander codes as the outer component codes, we show that our codes can be decoded in linear time and are more powerful than the quantum expander codes in correcting adversarial quantum errors. One interesting future work is that whether our scheme can be applied in FT quantum computation for correcting a linear number of random quantum errors, see [31, 36, 43].

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- [1] I. H. Deutsch, PRX Quantum **1**, 020101 (2020).
 - [2] J. Preskill, Quantum **2**, 79 (2018).
 - [3] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).
 - [4] M. Christandl and A. Müller-Hermes, arXiv preprint arXiv:2009.07161 (2020).
 - [5] A. Calderbank, E. Rains, P. Shor, and N. Sloane, IEEE Trans. Inf. Theory **44**, 1369 (1998).
 - [6] P. W. Shor, Phys. Rev. A **52**, R2493 (1995).
 - [7] T. Brun, I. Devetak, and M.-H. Hsieh, Science **314**, 436 (2006).
 - [8] M.-H. Hsieh, W.-T. Yen, and L.-Y. Hsu, IEEE Trans. Inf. Theory **57**, 1761 (2011).
 - [9] M. M. Wilde, M.-H. Hsieh, and Z. Babar, IEEE Trans. Inf. Theory **60**, 1203 (2013).
 - [10] D. Bacon and A. Casaccino, arXiv preprint quant-ph/0610088 (2006).
 - [11] J.-P. Tillich and G. Zémor, IEEE Trans. Inf. Theory **60**, 1193 (2014).
 - [12] P. Aliferis and A. W. Cross, Phys. Rev. Lett. **98**, 220502 (2007).
 - [13] D. J. MacKay, IEEE Trans. Inf. Theory **45**, 399 (1999).
 - [14] S. Evra, T. Kaufman, and G. Zémor, arXiv preprint arXiv:2004.07935 (2020).
 - [15] N. P. Breuckmann and J. N. Eberhardt, arXiv preprint arXiv:2012.09271 (2020).
 - [16] M. B. Hastings, J. Haah, and R. O’Donnell, arXiv preprint arXiv:2009.03921 (2020).
 - [17] P. Panteleev and G. Kalachev, arXiv preprint arXiv:2012.04068 (2020).
 - [18] A. Holmes, M. R. Jokar, G. Pasandi, Y. Ding, M. Pedram, and F. T. Chong, in *2020 ACM/IEEE 47th Annual International Symposium on Computer Architecture (ISCA)* (IEEE, 2020) pp. 556–569.
 - [19] M.-H. Hsieh and F. Le Gall, Phys. Rev. A **83**, 052331 (2011).
 - [20] D. Poulin and Y. Chung, Quantum Info. Comput. **8**, 987 (2008).
 - [21] A. Y. Kitaev, Ann. Phys. **303**, 2 (2003).
 - [22] J. Edmonds, Canadian Journal of mathematics **17**, 449 (1965).
 - [23] A. Leverrier, J.-P. Tillich, and G. Zémor, in *2015 IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, 2015) pp. 810–824.
 - [24] O. Fawzi, A. Grospellier, and A. Leverrier, in *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing* (2018) pp. 521–534.
 - [25] Y.-H. Liu and D. Poulin, Phys. Rev. Lett. **122**, 200501 (2019).

- [26] R. Cleve and D. Gottesman, Phys. Rev. A **56**, 76 (1997).
- [27] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. **43**, 4452 (2002).
- [28] O. Higgott, M. Wilson, J. Hefford, J. Dborin, F. Hanif, S. Burton, and D. E. Browne, arXiv preprint arXiv:2002.00362 (2020).
- [29] M. Hastings, arXiv preprint arXiv:2102.10030 (2021).
- [30] W. Zeng and L. P. Pryadko, Phys. Rev. Lett. **122**, 230501 (2019).
- [31] D. Gottesman, Quantum Inf. Comput. **14**, 1338 (2014).
- [32] S. Muralidharan, J. Kim, N. Lütkenhaus, M. D. Lukin, and L. Jiang, Phys. Rev. Lett. **112**, 250501 (2014).
- [33] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes* (The Netherlands: North-Holland, Amsterdam, 1981).
- [34] M. B. Hastings, Quantum Inf. Comput. **17**, 1307 (2017).
- [35] M. Sipser and D. A. Spielman, IEEE Trans. Inf. Theory **42**, 1710 (1996).
- [36] I. Dumer, A. A. Kovalev, and L. P. Pryadko, Phys. Rev. Lett. **115**, 050502 (2015).
- [37] R. Matsumoto, Quantum Inf. Process. **16**, 285 (2017).
- [38] T. J. Richardson and R. L. Urbanke, IEEE Trans. Inf. Theory **47**, 599 (2001).
- [39] D. J. MacKay, *Information theory, inference and learning algorithms* (Cambridge university press, 2003).
- [40] P. W. Shor and J. Preskill, Phys. Rev. Lett. **85**, 441 (2000).
- [41] J. M. Renes, F. Dupuis, and R. Renner, Phys. Rev. Lett. **109**, 050504 (2012).
- [42] J. M. Renes, D. Sutter, F. Dupuis, and R. Renner, IEEE Trans. Inf. Theory **61**, 6395 (2015).
- [43] O. Fawzi, A. Grospellier, and A. Leverrier, in *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE, 2018) pp. 743–754.