Statistical Approach to the Bekenstein-Hawking Entropy

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STATISTICAL APPROACH TO THE BEKENSTEIN-HAWKING ENTROPY

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ABSTRACT

We consider a Schwarzschild black hole type in this work whose particles, only those that lies on its surface, the event horizon (r_+), contributes to the entropy and we found it by using the canonical ensemble. We don't consider any interaction between this particles, but the inner energy.

Keywords Bekenstein-Hawking entropy · Canonical ensemble · Black hole · Partition function

1 Introduction

A black hole of mass M and radius r_+ , with not charge and rotation is considered in this work. The surface of the black hole is composed by a number N fixes of indistinguishable particles of the same mass m_0 which are restricted to move on the black hole surface and does not interacts one with another.

In [4] was calculated the entropy via thermodynamic consideration, taking in account the work made by the net force on the event horizon at constant temperature. Others consideration are made to find the entropy in [3, 6, 2]. In [1] is considered that the main contribution to the entropy is given by thermally excited 'invisible' modes propagating in the close vicinity of the horizon, which is true. In [1] is considered a dynamical origin of the entropy, and in [5] is considered that black hole entropy counts only those states of a black hole that can influence the outside, which is true too. Here our goal is to find this magnitude from the statistical point of view, considering the canonical ensemble. In the first part we found the entropy without consider the internal energy, after this we consider the internal energy of the superficial mass of the black hole, which is considered to be constant.

2 Partition function

2.1 With low interacting energy

The canonical partition function for indistinguishable particles that describe this system

$$Z = \frac{Z}{N!}, \quad (1)$$

where

$$Z = \frac{1}{h^f N} \int_{-\infty}^{\infty} e^{-\beta \mathcal{H}} d^f \vec{p}_1 \cdots d^f \vec{p}_N d^f \vec{r}_1 \cdots d^f \vec{r}_N, \quad (2)$$

is the canonical partition function for distinguishable particles, and

$$\mathcal{H} = \sum_{i=1}^N \frac{p_i^2}{2m_0} + U(r), \quad (3)$$

the Hamiltonian.

$$Z = \frac{1}{h^f N} \int_{-\infty}^{\infty} e^{-\beta \frac{\vec{p}_1^2}{2m_0}} d^f \vec{p}_1 \cdots e^{-\beta \frac{\vec{p}_N^2}{2m_0}} d^f \vec{p}_N \int_{-\infty}^{\infty} e^{-\beta U(r)} d^f \vec{r}_1 \cdots d^f \vec{r}_N. \quad (4)$$

We consider that the potential energy $U(r)$ is negligible and only the particles on the event horizon contributes to the entropy.

Using spherical coordinates and considering that the particles are restricted to move on the black hole surface ($f = 2$), only the components p_θ and p_ϕ contribute to the total momentum $\vec{p}$ of each particle, that is,

$$\vec{p} = \vec{p}_\theta + \vec{p}_\phi. \quad (5)$$

Then, the Eq. 4 can be write as,

$$\begin{aligned} Z &= \frac{1}{h^{2N}} \int_{-\infty}^{\infty} e^{-\beta \frac{p_{\theta_1}^2}{2m_0}} dp_{\theta_1} \int_{-\infty}^{\infty} e^{-\beta \frac{p_{\phi_1}^2}{2m_0}} dp_{\phi_1} \cdots \int_{-\infty}^{\infty} e^{-\beta \frac{p_{\theta_N}^2}{2m_0}} dp_{\theta_N} \int_{-\infty}^{\infty} e^{-\beta \frac{p_{\phi_N}^2}{2m_0}} dp_{\phi_N} \times \\ &\times \int_0^\pi r_+^2 \sin \theta_1 d\theta_1 \int_0^{2\pi} d\phi_1 \cdots \int_0^\pi r_+^2 \sin \theta_N d\theta_N \int_0^{2\pi} d\phi_N. \end{aligned} \quad (6)$$

Doing the calculations we get,

$$\begin{aligned} Z &= \frac{1}{h^{2N}} \left[\sqrt{2\pi k_B m_0 T} \right]^{2N} [4\pi r_+^2]^N \\ &= \left[\frac{k_B m_0 T A}{2\pi \hbar^2} \right]^N, \end{aligned} \quad (7)$$

where A is the area of the event horizon and T the temperature. The Eq. 7 is the partition function of N particles.

Then, the partition function, Eq. 1, can be write as

$$\mathcal{Z} = \frac{1}{N!} \left[\frac{k_B m_0 T A}{2\pi \hbar^2} \right]^N. \quad (8)$$

The free energy, F , is equal to

$$\begin{aligned} F &= -k_B T \ln \mathcal{Z} \\ &= -k_B T \ln \left(\frac{1}{N!} \left[\frac{k_B m_0 T A}{2\pi \hbar^2} \right]^N \right). \end{aligned} \quad (9)$$

Now we can determine the entropy of this system,

$$\begin{aligned} S &= -\frac{\partial F}{\partial T} \\ &= k_B N \ln \frac{k_B m_0 T A}{2\pi \hbar^2} + c_0, \end{aligned} \quad (10)$$

where c_0 is a constant igul to $2Nk_B$.

Entropy reach its maximum value when the temperature does. This temperature corresponds to the Hawking temperature, then we get

$$S_H = Nk_B \ln \frac{Ac^3}{4\hbar GN} + Nk_B \ln \frac{m_0}{4\pi^2 M} \quad (11)$$

The terms $Ac^3/4\hbar GN$ and $m_0/4\pi^2 M$ can be approximate to $e^{Ac^3/4\hbar GN}$ and $e^{m_0/4\pi^2 M}$ respectively, then the entropy give

$$S_H = k_B \frac{Ac^3}{4\hbar G} + k_B \frac{Nm_0}{4\pi^2 M}. \quad (12)$$

The term $m = Nm_0$ corresponds to the superficial mass of the black hole which is much less than that the corresponding black hole mass M , then we can neglected this term, and we get the Bekenstein-Hawking entropy,

$$S_H = \frac{k_B c^3}{4\hbar G} A. \quad (13)$$

2.2 Considering the internal energy

Considering the Eq. 4, and that the internal energy, U , of the superficial mass of the black hole does not depend on $|\vec{r}_j - \vec{r}_i|$, where r_i and r_j are the position vectors of each particles ($U = Nm_0 c^2$), the partition function Z is,

$$Z = \frac{1}{h^{2N}} [2\pi k_B T m_0]^N e^{-\beta U} A^N, \quad (14)$$

and then Eq. 1 can be write as,

$$\mathcal{Z} = \frac{1}{N! h^{2N}} [2\pi k_B T m_0]^N e^{-\beta U} A^N. \quad (15)$$

Now we can calculate the free energy and get

$$F = -Nk_B T \left[1 + \ln \frac{k_B T m_0 A}{\hbar^2 N} - \frac{m_0 c^2}{k_B T} \right], \quad (16)$$

and the entropy

$$S = k_B N \ln \frac{k_B m_0 A T}{2\pi \hbar^2 N} + c_0, \quad (17)$$

which correspond to the same results as Eq. 10 . This means that the constant internal energy in the potential energy could not be considered in the calculus of the black hole entropy. The Eq. 17 show the dependence of the entropy from temperature T , area A and number of particles N and we see that entropy increase as temperatura does, which we hope.

3 Conclusions

The Bekenstein-Hawking entropy come from the particles that lies on the surface of the black hole, on the event horizon. This results is due to the hight density of the inner surfaces of this type of object. Particles in the inner of the black hole has low kinetic energy, that reduces the temperature. On the other hand it seems that both, internal and interaction energy does not affect the Bekenstein-Hawking entropy.

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