Greedy Randomized and Maximal Weighted Residual Kaczmarz Methods with Oblique Projection

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Greedy Randomized and Maximal Weighted Residual Kaczmarz Methods with Oblique Projection

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Abstract For solving large-scale consistent linear system, a greedy randomized Kaczmarz method with oblique projection and a maximal weighted residual Kaczmarz method with oblique projection are proposed. By using oblique projection, these two methods greatly reduce the number of iteration steps and running time to find the minimum norm solution, especially when the rows of matrix A are close to linear correlation. Theoretical proof and numerical results show that the greedy randomized Kaczmarz method with oblique projection and the maximal weighted residual Kaczmarz method with oblique projection are more effective than the greedy randomized Kaczmarz method and the maximal weighted residual Kaczmarz method respectively.

Keywords Oblique projection · Convergence property · Kaczmarz method · Correlation · Large linear system

Mathematics Subject Classification (2020) 15A06 · 65F10 · 90C06

1 Introduction

Consider to solve a large-scale consistent linear system

$$Ax = b, \quad (1.1)$$

where the matrix $A \in R^{m \times n}$, $b \in R^m$. One of the solutions of the system (1.1) is $x^* = A^\dagger b$, which is the least Euclidean norm solution. Especially, when the coefficient matrix A is full column rank, x^* is the unique solution of the system (1.1).

There are many researches on solving the system (1.1) through iterative methods, among which the Kaczmarz method is a representative and efficient row-action method. The Kaczmarz method [19] selects the rows of the matrix A by using the cyclic rule, and in each iteration, the current iteration point is orthogonally projected onto the corresponding hyperplane. Due to its simplicity and performance, the Kaczmarz method has been applied to many fields, such as computerized tomography [15,20], image reconstruction [7,22,39,43], distributed computing [17], and signal processing [6,15,20]; and so on [26,13,18,46]. Since the Kaczmarz method cycles through the rows of A , the performance may depend heavily on the ordering of these rows. A poor ordering may result in a very slow convergence rate. McCormick [30] proposed a maximal weighted residual Kaczmarz (MWRK) method, and proved its convergence. In recent work, a new theoretical convergence estimate was proposed for the MWRK method in [11]. Strohmer and Vershynin [42] proposed a randomized Kaczmarz (RK) method which selects a given row with proportional to the Euclidean norm of the rows of the coefficient matrix A , and proved its convergence. After the above work, research on the Kaczmarz-type methods was reignited recently, see for example, the randomized block Kaczmarz-type methods [33,35,31,12], the greedy version of Kaczmarz-type methods [25,49,47,1,2], the extended version of Kaczmarz-type methods [29,4,10], and many others [24,14,11,38,34,41]. Kaczmarz's research also accelerated the development of column action iterative methods represented by the coordinate descent method [21]. See [48,44,40,36,32,28,8,3], etc.

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Fig. 1: Oblique projection in different directions d

Recently, Bai and Wu [1] proposed a new randomized row index selection strategy, which is aimed at grasping larger entries of the residual vector at each iteration, and constructed a greedy randomized Kaczmarz (GRK) method. They proved that the convergence of the GRK method is faster than that of the RK method. In [37, 38], Constantin Popa gave the definition of oblique projection, which broke the limitation of orthogonal projection. In [27], Dirk A. Lorenez and Sean Rose et al. used oblique projection to construct random Kaczmarz methods with mismatched adjoint (RKMA). Recently, Li and Wang [23] proposed a Kaczmarz method based on oblique projection, which can solve the problem that the rows of the coefficient matrix A are close to linear correlation as well as the algorithms proposed in [45, 34]. In this paper, we give a kind of oblique projection, which is determined by the hyperplane selected in the current iteration and the last iteration, and propose an algorithm framework based on this type of oblique projection. Based on the row index selection rules of two representative randomized and non-randomized Kaczmarz-type methods – the GRK method and the MWRK method, we propose two new Kaczmarz-type methods with oblique projection (KO-type) – the greedy randomized Kaczmarz method with oblique projection (GRKO) and the maximal weighted residual Kaczmarz method with oblique projection (MWRKO) respectively, and their convergence is proved theoretically and numerically. We emphasize the efficiency of our proposed methods when the rows of the matrix A are nearly linearly correlated, and find that Kaczmarz-type method based on orthogonal projection performed poorly when applied to this kind of matrices.

The organization of this paper is as follows. In Section 2, we introduce the KO-type method, and give its two lemmas. In Section 3, we propose the GRKO method and MWRKO method naturally and prove the convergence of the two methods. In Section 4, some numerical examples are provided to illustrate the efficiency of our new methods. Finally, some brief concluding remarks are described in Section 5.

In this paper, $\langle \cdot \rangle$ stands for the scalar product. $\|x\|$ is the Euclid norm of $x \in R^n$. For a given matrix $G = (g_{ij}) \in R^{m \times n}$, g_i^T , G^T , $G^\dagger$, $R(G)$, $N(G)$, $\|G\|_F$ and $\lambda_{\min}(G)$, are used to denote the i th row, the transpose, the Moore-Penrose pseudoinverse [5], the range space, the null space, the Frobenius norm, and the smallest nonzero eigenvalue of G respectively. $P_C(x)$ is the orthogonal projection of x onto C , $\tilde{x}$ is any solution of the system (1.1); $x^* = A^\dagger b$ is the least-norm solution of the system (1.1). Let E_k denote the expected value conditional on the first k iterations, that is,

$$E_k[\cdot] = E[\cdot | j_0, j_1, \dots, j_{k-1}],$$

where j_s ($s = 0, 1, \dots, k-1$) is the column chosen at the s th iteration.

2 Kaczmarz-type Method with Oblique Projection and its Lemmas

The sets $H_i = \{x \in R^n, \langle a_i, x \rangle = b_i\}$ ($i = 1, 2, \dots, m$) are the hyperplanes which associated to the i th equation of the system (1.1). To project the current iteration point $x^{(k)}$ to one of the hyperplanes, the oblique projection [37, 38] can be expressed as follows:

$$x^{(k+1)} = P_{H_i}^d(x^{(k)}) = x^{(k)} - \frac{\langle a_i, x \rangle - b_i}{\langle d, a_i \rangle} d, \quad (2.1)$$

where $d \in R^n$ is a given direction. In Figure 1, $x^{(k+1)}$ is obtained by oblique projection of the current iteration point $x^{(k)}$ to the hyperplane $H_{i_{k+1}}$ along the direction d , i.e. $x^{(k+1)} = P_{H_{i_{k+1}}}^d(x^{(k)})$. $y^{(k+1)}$ is the iteration point obtained when the direction $d = a_{i_{k+1}}$, i.e. $y^{(k+1)} = P_{H_{i_{k+1}}}^{a_{i_{k+1}}}(x^{(k)})$. When the direction $d = a_i$ ($i = \text{mod}(m, k) + 1$),

it is the classic Kaczmarz method. However, when the hyperplanes are close to linear parallel, the Kaczmarz method based on orthogonal projection has a slow iteration speed. In this paper, we propose a iteration direction $d = w^{(i_k)} = a_{i_{k+1}} - \frac{\langle a_{i_k}, a_{i_{k+1}} \rangle}{\|a_i\|^2} a_{i_k}$, to make the current iteration point approach to the intersection of two hyperplanes, i.e. $z^{(k+1)} = P_{H_{i_{k+1}}}^{w^{(i_k)}}(x^{(k)})$.

The framework of KO-type method is given as follows.

Algorithm 1 Kaczmarz-type Method with Oblique Projection

Require: $A \in R^{m \times n}$, $b \in R^m$, $x^{(0)} \in R^n$, $K, \varepsilon > 0$

- 1: For $i = 1 : m$, $M(i) = \|a_i\|^2$
 - 2: Choose i_1 based on a certain selection rule
 - 3: Compute $x^{(1)} = x^{(0)} + \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1}$
 - 4: **for** $k = 1, 2, \dots, K$ **do**
 - 5: Choose i_{k+1} based on a certain selection rule
 - 6: Compute $D_{i_k} = \langle a_{i_k}, a_{i_{k+1}} \rangle$ and $r_{i_{k+1}}^{(k)} = b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle$
 - 7: Compute $w^{(i_k)} = a_{i_{k+1}} - \frac{D_{i_k}}{M(i_k)} a_{i_k}$ and $h_{i_k} (= \|w^{(i_k)}\|^2) = M(i_{k+1}) - \frac{D_{i_k}^2}{M(i_k)}$
 - 8: $\alpha_{i_k}^{(k)} = \frac{r_{i_{k+1}}^{(k)}}{h_{i_k}}$ and $x^{(k+1)} = x^{(k)} + \alpha_{i_k}^{(k)} w^{(i_k)}$
 - 9: **end for**
 - 10: Output $x^{(K+1)}$
-

Obviously, the Algorithm 1 can be used as a special form of the RKMA method [27], but the oblique projection direction in [27] is more defined in a way of mismatched adjoint. We will give two lemmas of KO-type method. The selection rule of its row index i_{k+1} in step 5 of Algorithm 1 does not affect the lemmas.

Lemma 1 For the Kaczmarz-type method with oblique projection, the residual satisfies the following equations:

$$r_{i_k}^{(k)} = 0 \quad (\forall k > 0), \quad (2.2)$$

$$r_{i_{k-1}}^{(k)} = 0 \quad (\forall k > 1). \quad (2.3)$$

Proof From the definition of the KO-type method, for $k = 1$, we have

$$\begin{aligned} r_{i_1}^{(1)} &= b_{i_1} - \langle a_{i_1}, x^{(1)} \rangle \\ &= b_{i_1} - \langle a_{i_1}, x^{(0)} + \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1} \rangle \\ &= 0. \end{aligned}$$

For $k > 1$, we have

$$\begin{aligned} r_{i_k}^{(k)} &= b_{i_k} - \langle a_{i_k}, x^{(k)} \rangle \\ &= b_{i_k} - \langle a_{i_k}, x^{(k-1)} + \alpha_{i_{k-1}}^{(k-1)} w^{(i_{k-1})} \rangle \\ &= b_{i_k} - \langle a_{i_k}, x^{(k-1)} \rangle - \alpha_{i_{k-1}}^{(k-1)} \langle a_{i_k}, w^{(i_{k-1})} \rangle \\ &= r_{i_k}^{(k-1)} - \frac{r_{i_k}^{(k-1)}}{h_{i_{k-1}}} \langle a_{i_k}, w^{(i_{k-1})} \rangle \\ &= r_{i_k}^{(k-1)} - r_{i_k}^{(k-1)} \\ &= 0. \end{aligned}$$

The fifth equality holds due to $\langle a_{i_k}, w^{(i_{k-1})} \rangle = \|a_{i_k}\|^2 - \frac{\langle a_{i_{k-1}}, a_{i_k} \rangle^2}{\|a_{i_{k-1}}\|^2} = h_{i_{k-1}}$. Thus, the equation (2.2) holds.

Since $k > 1$,

$$x^{(k)} = x^{(k-1)} + \alpha_{i_{k-1}}^{(k)} w^{(i_{k-1})}.$$

We get

$$(b - Ax^{(k)})_{i_{k-1}} = (b - Ax^{(k-1)})_{i_{k-1}} - (A\alpha_{i_{k-1}}^{(k)} w^{(i_{k-1})})_{i_{k-1}},$$

that is,

$$\begin{aligned} r_{i_{k-1}}^{(k)} &= r_{i_{k-1}}^{(k-1)} - \alpha_{i_{k-1}}^{(k)} \langle a_{i_{k-1}}, w^{(i_{k-1})} \rangle \\ &= \alpha_{i_{k-1}}^{(k)} \langle a_{i_{k-1}}, a_{i_k} - \frac{\langle a_{i_{k-1}}, a_{i_k} \rangle}{\|a_{i_{k-1}}\|^2} a_{i_{k-1}} \rangle \\ &= 0. \end{aligned}$$

The second equality holds due to the equation (2.2). Thus, the equation (2.3) holds.

Lemma 2 *The iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by the Kaczmarz-type method with oblique projection, satisfies the following equations:*

$$\|x^{(k+1)} - \tilde{x}\|^2 = \|x^{(k)} - \tilde{x}\|^2 - \|x^{(k+1)} - x^{(k)}\|^2 \quad (\forall k \geq 0), \quad (2.4)$$

where $\tilde{x}$ is an arbitrary solution of the system (1.1). Especially, when $P_{N(A)}(x^{(0)}) = P_{N(A)}(\tilde{x})$, $x^{(k)} - \tilde{x} \in R(A^T)$.

Proof For $k = 0$, we have

$$\begin{aligned} \langle a_{i_1}, x^{(1)} - \tilde{x} \rangle &= \langle a_{i_1}, x^{(0)} - \tilde{x} + \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1} \rangle \\ &= \langle a_{i_1}, x^{(0)} \rangle - b_{i_1} + \langle a_{i_1}, \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1} \rangle \\ &= 0, \end{aligned}$$

which shows that $x^{(1)} - \tilde{x}$ is orthogonal to a_{i_1} . Therefore, we know

$$(x^{(1)} - x^{(0)})^T (x^{(1)} - \tilde{x}) = 0.$$

It follows that

$$\|x^{(1)} - \tilde{x}\|^2 = \|x^{(0)} - \tilde{x}\|^2 - \|x^{(1)} - x^{(0)}\|^2. \quad (2.5)$$

For $k > 0$, we have

$$\begin{aligned} \langle w^{(i_k)}, x^{(k+1)} - \tilde{x} \rangle &= \langle w^{(i_k)}, x^{(k)} - \tilde{x} + \alpha_{i_k}^{(k)} w^{(i_k)} \rangle \\ &= \langle a_{i_{k+1}} - \frac{D_{i_k}}{M(i_k)} a_{i_k}, x^{(k)} - \tilde{x} \rangle + \langle w^{(i_k)}, \frac{r_{i_{k+1}}^{(k)}}{h_{i_k}} w^{(i_k)} \rangle \\ &= -r_{i_{k+1}}^{(k)} + \frac{D_{i_k}}{M(i_k)} r_{i_k}^{(k)} + r_{i_{k+1}}^{(k)} \\ &= 0. \end{aligned}$$

The third and last equalities hold due to $h_{i_k} = \|w^{(i_k)}\|^2$, and the equation (2.2) respectively. Thus we get that $x^{(k+1)} - \tilde{x}$ is orthogonal to $w^{(i_k)}$. Therefore, we get that

$$(x^{(k+1)} - x^{(k)})^T (x^{(k+1)} - \tilde{x}) = 0.$$

It follows that

$$\|x^{(k+1)} - \tilde{x}\|^2 = \|x^{(k)} - \tilde{x}\|^2 - \|x^{(k+1)} - x^{(k)}\|^2 \quad (\forall k > 0). \quad (2.6)$$

Thus, from the above proof, the equation (2.4) holds.

According to the iterative formula

$$\begin{cases} x^{(1)} = x^{(0)} + \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1}, \\ x^{(k+1)} = x^{(k)} + \alpha_{i_k}^{(k)} w^{(i_k)} \quad (\forall k > 0), \end{cases}$$

we can get $P_{N(A)}(x^{(k)}) = P_{N(A)}(x^{(k-1)}) = \dots = P_{N(A)}(x^{(0)})$, and by the fact that $P_{N(A)}(x^{(0)}) = P_{N(A)}(\tilde{x})$, we can deduce that $x^{(k)} - \tilde{x} \in R(A^T)$.

3 Greedy Randomized and Maximal Weighted Residual Kaczmarz methods with Oblique Projection

In this section, we combine the oblique projection with the GRK method [1] and the MWRK method [30] to obtain the GRKO method and the MWRKO method, and prove their convergence. Theoretical results show that the KO-type method can accelerate the convergence when there are suitable row index selection strategies.

3.1 Greedy Randomized Kaczmarz Method with Oblique projection

The core of the GRK method [1] is a new probability criterion, which can grasp the large items of the residual vector in each iteration, and randomly select the item with probability in proportion to the retained residual norm. Theories and experiments prove that it can speed up convergence speed. This paper uses its the row index selection rule in combination with the Algorithm 1 to obtain the GRKO method, and the algorithm is as follows:

Algorithm 2 Greedy Randomized Kaczmarz Method with Oblique Projection

Require: $A \in R^{m \times n}$, $b \in R^m$, $x^{(0)} \in R^n$, $K, \varepsilon > 0$

- 1: For $i = 1 : m$, $M(i) = \|a_i\|^2$
- 2: Randomly select i_1 , and compute $x^{(1)} = x^{(0)} + \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1}$
- 3: **for** $k = 1, 2, \dots, K - 1$ **do**
- 4: Compute $\varepsilon_k = \frac{1}{2} \left(\frac{1}{\|b - Ax^{(k)}\|^2} \max_{1 \leq i_{k+1} \leq m} \left\{ \frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\|a_{i_{k+1}}\|^2} \right\} + \frac{1}{\|A\|_F^2} \right)$
- 5: Determine the index set of positive integers

$$\mathcal{U}_k = \{i_{k+1} | |b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2 \geq \varepsilon_k \|b - Ax^{(k)}\|^2 \|a_{i_{k+1}}\|^2\}$$

- 6: Compute the i th entry $\tilde{r}_i^{(k)}$ of the vector $\tilde{r}^{(k)}$ according to

$$\tilde{r}_i^{(k)} = \begin{cases} b_i - \langle a_i, x^{(k)} \rangle, & \text{if } i \in \mathcal{U}_k \\ 0 & \text{otherwise} \end{cases}$$

- 7: Select $i_{k+1} \in \mathcal{U}_k$ with probability $Pr(row = i_{k+1}) = \frac{|\tilde{r}_{i_{k+1}}^{(k)}|^2}{\|\tilde{r}^{(k)}\|^2}$
- 8: Compute $D_{i_k} = \langle a_{i_k}, a_{i_{k+1}} \rangle$
- 9: Compute $w^{(i_k)} = a_{i_{k+1}} - \frac{D_{i_k}}{M(i_k)} a_{i_k}$ and

$$h_{i_k} (= \|w^{(i_k)}\|^2) = M(i_{k+1}) - \frac{D_{i_k}}{M(i_k)} D_{i_k}$$

- 10: $\alpha_{i_k}^{(k)} = \frac{\tilde{r}_{i_{k+1}}^{(k)}}{h_{i_k}} \left(= \frac{r_{i_{k+1}}^{(k)}}{h_{i_k}} \right)$ and $x^{(k+1)} = x^{(k)} + \alpha_{i_k}^{(k)} w^{(i_k)}$

11: **end for**

12: Output $x^{(K)}$

The convergence of the GRKO method is provided as follows.

Theorem 1 Consider the consistent linear system (1.1), where the coefficient matrix $A \in R^{m \times n}$, $b \in R^m$. Let $x^{(0)} \in R^n$ be an arbitrary initial approximation, $\tilde{x}$ is a solution of system (1.1) such that $P_{N(A)}(\tilde{x}) = P_{N(A)}(x^{(0)})$. Then the iteration sequence $\{x^{(k)}\}_{k=1}^{\infty}$ generated by the GRKO method obeys

$$E\|x^{(k)} - \tilde{x}\|^2 \leq \prod_{s=0}^{k-1} \zeta_s \|x^{(0)} - \tilde{x}\|^2. \quad (3.1)$$

where $\zeta_0 = 1 - \frac{(\lambda_{\min}(A^T A))}{m\|A\|_F^2}$, $\zeta_1 = 1 - \frac{1}{2}(\frac{1}{\gamma_1}\|A\|_F^2 + 1)\frac{\lambda_{\min}(A^T A)}{\Delta \cdot \|A\|_F^2}$, $\zeta_k = 1 - \frac{1}{2}(\frac{1}{\gamma_2}\|A\|_F^2 + 1)\frac{\lambda_{\min}(A^T A)}{\Delta \cdot \|A\|_F^2}$ ($\forall k > 1$), which

$$\gamma_1 = \max_{1 \leq i \leq m} \sum_{\substack{s=1 \\ s \neq i}}^m \|a_s\|^2, \quad (3.2)$$

$$\gamma_2 = \max_{\substack{1 \leq i, j \leq m \\ i \neq j}} \sum_{\substack{s=1 \\ s \neq i, j}}^m \|a_s\|^2, \quad (3.3)$$

$$\Delta = \max_{j \neq k} \sin \langle a_j, a_k \rangle^2 (\in (0, 1]). \quad (3.4)$$

In addition, if $x^{(0)} \in R(A^T)$, the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ converges to the least-norm solution of the system (1.1), i.e. $\lim_{k \rightarrow \infty} x^{(k)} = x^* = A^+ b$.

Proof When $k = 1$, we can get

$$\begin{aligned} \varepsilon_1 \|A\|_F^2 &= \frac{\max_{1 \leq i_2 \leq m} \left\{ \frac{|b_{i_2} - \langle a_{i_2}, x^{(1)} \rangle|^2}{\|a_{i_2}\|^2} \right\}}{2 \sum_{i_2=1}^m \frac{\|a_{i_2}\|^2}{\|A\|_F^2} \cdot \frac{|b_{i_2} - \langle a_{i_2}, x^{(1)} \rangle|^2}{\|a_{i_2}\|^2}} + \frac{1}{2} \\ &= \frac{\max_{1 \leq i_2 \leq m} \left\{ \frac{|b_{i_2} - \langle a_{i_2}, x^{(1)} \rangle|^2}{\|a_{i_2}\|^2} \right\}}{2 \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^m \frac{\|a_{i_2}\|^2}{\|A\|_F^2} \cdot \frac{|b_{i_2} - \langle a_{i_2}, x^{(1)} \rangle|^2}{\|a_{i_2}\|^2}} + \frac{1}{2} \\ &\geq \frac{1}{2} \left(\frac{\|A\|_F^2}{\sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^m \|a_{i_2}\|^2} + 1 \right) \\ &\geq \frac{1}{2} \left(\frac{1}{\gamma_1} \|A\|_F^2 + 1 \right). \end{aligned} \quad (3.5)$$

The second equality holds due to the equation (2.2).

When $k > 1$, we get

$$\begin{aligned} \varepsilon_k \|A\|_F^2 &= \frac{\max_{1 \leq i_{k+1} \leq m} \left(\frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\|a_{i_{k+1}}\|^2} \right)}{2 \sum_{i_{k+1}=1}^m \frac{\|a_{i_{k+1}}\|^2}{\|A\|_F^2} \cdot \frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\|a_{i_{k+1}}\|^2}} + \frac{1}{2} \\ &= \frac{\max_{1 \leq i_{k+1} \leq m} \left(\frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\|a_{i_{k+1}}\|^2} \right)}{2 \sum_{\substack{i_{k+1}=1 \\ i_{k+1} \neq i_k, i_{k-1}}}^m \frac{\|a_{i_{k+1}}\|^2}{\|A\|_F^2} \cdot \frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\|a_{i_{k+1}}\|^2}} + \frac{1}{2} \\ &\geq \frac{1}{2} \left(\frac{\|A\|_F^2}{\sum_{\substack{i_{k+1}=1 \\ i_{k+1} \neq i_k, i_{k-1}}}^m \|a_{i_{k+1}}\|^2} + 1 \right) \\ &\geq \frac{1}{2} \left(\frac{1}{\gamma_2} \|A\|_F^2 + 1 \right). \end{aligned} \quad (3.6)$$

The second equality holds due to the equation (2.2) and the equation (2.3).

Under the GRKO method, Lemma 2.2 still holds, so we can take the full expectation on both sides of the equation (2.4), and get that for $k = 0$,

$$\begin{aligned}
E\|x^{(1)} - \tilde{x}\|^2 &= \|x^{(0)} - \tilde{x}\|^2 - E\|x^{(1)} - x^{(0)}\|^2 \\
&= \|x^{(0)} - \tilde{x}\|^2 - \frac{1}{m} \sum_{i_1=1}^m \left\| \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1} \right\|^2 \\
&\leq \|x^{(0)} - \tilde{x}\|^2 - \frac{1}{m} \frac{\|b - Ax^{(0)}\|^2}{\|A\|_F^2} \\
&\leq \left(1 - \frac{\lambda_{\min}(A^T A)}{m\|A\|_F^2} \right) \|x^{(0)} - \tilde{x}\|^2 \\
&= \zeta_0 \|x^{(0)} - \tilde{x}\|^2,
\end{aligned} \tag{3.7}$$

and for $k > 0$,

$$\begin{aligned}
E_k\|x^{(k+1)} - \tilde{x}\|^2 &= \|x^{(k)} - \tilde{x}\|^2 - E_k\|x^{(k+1)} - x^{(k)}\|^2 \\
&= \|x^{(k)} - \tilde{x}\|^2 - \sum_{i_{k+1} \in \mathcal{U}_k} \frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\sum_{i_{k+1} \in \mathcal{U}_k} |b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2} \cdot \frac{|r_{i_{k+1}}^{(k)}|^2}{\|w^{(i_k)}\|^2} \\
&\leq \|x^{(k)} - \tilde{x}\|^2 - \sum_{i_{k+1} \in \mathcal{U}_k} \frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\sum_{i_{k+1} \in \mathcal{U}_k} |b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2} \cdot \frac{|r_{i_{k+1}}^{(k)}|^2}{\Delta \cdot \|a_{i_{k+1}}\|^2} \\
&\leq \|x^{(k)} - \tilde{x}\|^2 - \frac{\varepsilon_k}{\Delta} \|b - Ax^{(k)}\|^2 \\
&= \|x^{(k)} - \tilde{x}\|^2 - \frac{\varepsilon_k}{\Delta} \|A(\tilde{x} - x^{(k)})\|^2 \\
&\leq \left(1 - \frac{\varepsilon_k \lambda_{\min}(A^T A)}{\Delta} \right) \|x^{(k)} - \tilde{x}\|^2.
\end{aligned} \tag{3.8}$$

The first inequality of the equation (3.7) is achieved with the use of the fact that $\frac{|b_1|}{|a_1|} + \frac{|b_2|}{|a_2|} \geq \frac{|b_1|+|b_2|}{|a_1|+|a_2|}$ (if $|a_1| > 0, |a_2| > 0$), and the first inequality of the equation (3.8) is achieved with the use of the fact that $\|w_{i_k}\|^2 = \|a_{i_{k+1}}\|^2 - \frac{\langle a_{i_k}, a_{i_{k+1}} \rangle^2}{\|a_{i_{k+1}}\|^2} = \sin^2 \langle a_{i_k}, a_{i_{k+1}} \rangle \|a_{i_{k+1}}\|^2 \leq \Delta \cdot \|a_{i_{k+1}}\|^2$, and the second inequality of the equation (3.8) is achieved with the use of the definition of $\mathcal{U}_k$ which lead to

$$|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2 \geq \varepsilon_k \|b - Ax^{(k)}\|^2 \|a_{i_{k+1}}\|^2, \forall i_{k+1} \in \mathcal{U}_k.$$

Here in the last inequalities of the equation (3.7) and (3.8), we have used the estimate $\|Au\|_2^2 \geq \lambda_{\min}(A^T A)\|u\|^2$, which holds true for any $u \in C^n$ belonging to the column space of A^T . According to the lemma 2.2, it holds.

By making use of the equation (3.5), (3.6) and (3.8), we get

$$\begin{aligned}
E_1\|x^{(2)} - \tilde{x}\|^2 &\leq \left[1 - \frac{1}{2} \left(\frac{1}{\gamma_1} \|A\|_F^2 + 1 \right) \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \|A\|_F^2} \right] \|x^{(1)} - \tilde{x}\|^2 \\
&= \zeta_1 \|x^{(1)} - \tilde{x}\|^2, \\
E_k\|x^{(k+1)} - \tilde{x}\|^2 &\leq \left[1 - \frac{1}{2} \left(\frac{1}{\gamma_2} \|A\|_F^2 + 1 \right) \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \|A\|_F^2} \right] \|x^{(k)} - \tilde{x}\|^2 \\
&= \zeta_k \|x^{(k)} - \tilde{x}\|^2 \quad (\forall k > 1).
\end{aligned}$$

Finally, by recursion and taking the full expectation, the equation (3.1) holds.

Remark 1. In the GRKO method, h_{i_k} is not zero. Suppose $h_{i_k} = 0$, which means $\exists \lambda > 0, \lambda a_{i_k} = a_{i_{k+1}}$. Due to the system is consistent, it holds $\langle a_{i_{k+1}}, x^* \rangle = \lambda \langle a_{i_k}, x^* \rangle = \lambda b_{i_k} = b_{i_{k+1}}$. According to the equation (2.2), it holds $r_{i_{k+1}}^{(k)} = \lambda r_{i_k}^{(k)} = 0$. From step 5 of Algorithm 3.1, we can know that such index i_{k+1} will not be selected.

Remark 2. Set $\tilde{\zeta}_k = 1 - \frac{1}{2} \left(\frac{1}{\gamma_1} \|A\|_F^2 + 1 \right) \frac{\lambda_{\min}(A^T A)}{\|A\|_F^2}$ ($\forall k > 0$), and the convergence of GRK method in [1] meets:

$$E_k\|x^{(k+1)} - x^*\|^2 \leq \tilde{\zeta}_k \|x^{(k)} - x^*\|^2.$$

Obviously, $\zeta_1 \leq \tilde{\zeta}_1, \zeta_k < \tilde{\zeta}_k$ ($\forall k > 1$) is satisfied, so the convergence speed of GRKO method is faster than GRK method.

3.2 Maximal Weighted Residual Kaczmarz Method with Oblique Projection

The selection strategy for the index i_k used in the maximal weighted residual Kaczmarz (MWRK) method [30] is: Set

$$i_k = \arg \max_{i \in \{1, 2, \dots, m\}} \frac{|a_i^T x^{(k)} - b_i|}{\|a_i\|}.$$

McCormick proved the exponential convergence of the MWRK method. In [11], a new convergence conclusion of the MWRK method is given. We use its row index selection rule combined with KO-type method to obtain MWRKO method, and the algorithm is as follows:

Algorithm 3 Maximal Weighted Residual Kaczmarz Method with Oblique Projection (MWRKO)

Require: $A \in R^{m \times n}$, $b \in R^m$, $x^{(0)} \in R^n$, $K, \varepsilon > 0$

- 1: For $i = 1 : m$, $M(i) = \|a_i\|^2$
 - 2: Compute $i_1 = \arg \max_{i \in \{1, 2, \dots, m\}} \frac{|a_i^T x^{(0)} - b_i|}{\|a_i\|}$, and $x^{(1)} = x^{(0)} + \frac{b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle}{M(i_1)} a_{i_1}$
 - 3: **for** $k = 1, 2, \dots, K$ **do**
 - 4: Compute $i_{k+1} = \arg \max_{i \in \{1, 2, \dots, m\}} \frac{|a_i^T x^{(k)} - b_i|}{\|a_i\|}$
 - 5: Compute $D_{i_k} = \langle a_{i_k}, a_{i_{k+1}} \rangle$ and $r_{i_{k+1}}^{(k)} = b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle$
 - 6: Compute $w^{(i_k)} = a_{i_{k+1}} - \frac{D_{i_k}}{M(i_k)} a_{i_k}$ and $h_{i_k} (= \|w^{(i_k)}\|^2) = M(i_{k+1}) - \frac{D_{i_k}}{M(i_k)} D_{i_k}$
 - 7: $\alpha_{i_k}^{(k)} = \frac{r_{i_{k+1}}^{(k)}}{h_{i_k}}$ and $x^{(k+1)} = x^{(k)} + \alpha_{i_k}^{(k)} w^{(i_k)}$
 - 8: **end for**
 - 9: Output $x^{(K+1)}$
-

The convergence of the MWRKO method is provided as follows.

Theorem 2 Consider the consistent linear system (1.1), where the coefficient matrix $A \in R^{m \times n}$, $b \in R^m$. Let $x^{(0)} \in R^n$ be an arbitrary initial approximation, $\tilde{x}$ is a solution of system (1.1) such that $P_{N(A)}(\tilde{x}) = P_{N(A)}(x^{(0)})$. Then the iteration sequence $\{x^{(k)}\}_{k=1}^{\infty}$ generated by the MWRKO method obeys

$$\|x^{(k)} - \tilde{x}\|^2 \leq \prod_{s=0}^{k-1} \rho_s \|x^{(0)} - \tilde{x}\|^2, \quad (3.9)$$

where $\rho_0 = 1 - \frac{\lambda_{\min}(A^T A)}{\|A\|_F^2}$, $\rho_1 = 1 - \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \gamma_1}$, $\rho_k = 1 - \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \gamma_2}$ ($\forall k > 1$), which γ_1 , γ_2 and Δ are defined by equations (3.2), (3.3) and (3.4) respectively.

In addition, if $x^{(0)} \in R(A^T)$, the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ converges to the least-norm solution of the system (1.1), i.e. $\lim_{k \rightarrow \infty} x^{(k)} = x^* = A^\dagger b$.

Proof Under the MWRKO method, Lemma 2.2 still holds. For $k = 1$, we have

$$\begin{aligned} \|x^{(1)} - \tilde{x}\|^2 &= \|x^{(0)} - \tilde{x}\|^2 - \|x^{(1)} - x^{(0)}\|^2 \\ &= \|x^{(0)} - \tilde{x}\|^2 - \frac{|b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle|^2}{M(i_1)} \\ &= \|x^{(0)} - \tilde{x}\|^2 - \frac{|b_{i_1} - \langle a_{i_1}, x^{(0)} \rangle|^2}{M(i_1)} \cdot \frac{\|b - Ax^{(0)}\|^2}{\sum_{i=1}^m \frac{|b_i - \langle a_i, x^{(0)} \rangle|^2}{M(i)} \cdot M(i)} \\ &\leq \|x^{(0)} - \tilde{x}\|^2 - \frac{\|A(\tilde{x} - x^{(0)})\|^2}{\|A\|_F^2} \\ &\leq \|x^{(0)} - \tilde{x}\|^2 - \frac{\lambda_{\min}(A^T A)}{\|A\|_F^2} \|x^{(0)} - \tilde{x}\|^2 \\ &= \left(1 - \frac{\lambda_{\min}(A^T A)}{\|A\|_F^2}\right) \|x^{(0)} - x^*\|^2 \\ &= \rho_0 \|x^{(0)} - x^*\|^2. \end{aligned} \quad (3.10)$$

For $k = 1$, we have

$$\begin{aligned}
\|x^{(2)} - \tilde{x}\|^2 &= \|x^{(1)} - \tilde{x}\|^2 - \|x^{(2)} - x^{(1)}\|^2 \\
&= \|x^{(1)} - \tilde{x}\|^2 - \frac{|b_{i_2} - \langle a_{i_2}, x^{(1)} \rangle|^2}{\|w^{(i_1)}\|^2} \\
&\leq \|x^{(1)} - \tilde{x}\|^2 - \frac{|b_{i_2} - \langle a_{i_2}, x^{(1)} \rangle|^2}{\Delta \cdot M(i_2)} \cdot \frac{\|b - Ax^{(1)}\|^2}{\sum_{i=1, i \neq i_1}^m \frac{|b_i - \langle a_i, x^{(1)} \rangle|^2}{M(i)} \cdot M(i)} \\
&\leq \|x^{(1)} - \tilde{x}\|^2 - \frac{\|A(\tilde{x} - x^{(1)})\|^2}{\Delta \cdot \gamma_1} \\
&\leq \|x^{(1)} - \tilde{x}\|^2 - \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \gamma_1} \|x^{(1)} - \tilde{x}\|^2 \\
&= \left(1 - \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \gamma_1}\right) \|x^{(1)} - \tilde{x}\|^2 \\
&= \rho_1 \|x^{(1)} - \tilde{x}\|^2.
\end{aligned} \tag{3.11}$$

For $k > 1$, we have

$$\begin{aligned}
\|x^{(k+1)} - \tilde{x}\|^2 &= \|x^{(k)} - \tilde{x}\|^2 - \|x^{(k+1)} - x^{(k)}\|^2 \\
&= \|x^{(k)} - \tilde{x}\|^2 - \frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\|w^{(i_k)}\|^2} \\
&\leq \|x^{(k)} - \tilde{x}\|^2 - \frac{|b_{i_{k+1}} - \langle a_{i_{k+1}}, x^{(k)} \rangle|^2}{\Delta \cdot M(i_{k+1})} \cdot \frac{\|b - Ax^{(k)}\|^2}{\sum_{i=1, i \neq i_k, i_{k-1}}^m \frac{|b_i - \langle a_i, x^{(k)} \rangle|^2}{M(i)} \cdot M(i)} \\
&\leq \|x^{(k)} - \tilde{x}\|^2 - \frac{\|A(\tilde{x} - x^{(k)})\|^2}{\Delta \cdot \gamma_2} \\
&\leq \|x^{(k)} - \tilde{x}\|^2 - \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \gamma_2} \|x^{(k)} - \tilde{x}\|^2 \\
&= \left(1 - \frac{\lambda_{\min}(A^T A)}{\Delta \cdot \gamma_2}\right) \|x^{(k)} - \tilde{x}\|^2 \\
&= \rho_k \|x^{(k)} - \tilde{x}\|^2.
\end{aligned} \tag{3.12}$$

Here in the last inequalities of the equation (3.10), (3.11) and (3.12), we have used the estimate

$$\|Au\|_2^2 \geq \lambda_{\min}(A^T A) \|u\|^2,$$

which holds true for any $u \in C^n$ belonging to the column space of A^T . According to the lemma 2.2, it holds.

From the equation (3.10), (3.11) and (3.12), the equation (3.9) holds.

Remark 3. When multiple indicators i_{k+1} are met in Step 2 of Algorithm 3.2 in the iterative process, we randomly select any one of them.

Remark 4. In the MWRKO method, the reason of $h_{i_k} \neq 0$ is similar to Remark 1.

Remark 5. Set $\tilde{\rho}_0 = 1 - \frac{\lambda_{\min}(A^T A)}{\|A\|_F^2}$, $\tilde{\rho}_k = 1 - \frac{\lambda_{\min}(A^T A)}{\gamma_1}$ ($\forall k > 0$), and the convergence of MWRK method in [11] meets:

$$\|x^{(k)} - x^*\|^2 \leq \prod_{s=0}^{k-1} \tilde{\rho}_s \|x^{(0)} - x^*\|^2,$$

Obviously, $\rho_k < \tilde{\rho}_k$ ($\forall k > 1$), $\rho_1 \leq \tilde{\rho}_1$ and $\rho_0 = \tilde{\rho}_0$, so the convergence speed of MWRKO method is faster than MWRK method. Note that $\tilde{\rho}_k < \tilde{\zeta}_k$, $\rho_k < \zeta_k$ ($\forall k > 0$), that is $V_{MWRK} < V_{MWRKO}$, $V_{GRK} < V_{GRKO}$, $V_{GRK} < V_{MWRK}$, $V_{GRKO} < V_{MWRKO}$, where V represents the convergence speed.

4 Numerical Experiments

In this section, some numerical examples are provided to illustrate the effectiveness of the greedy randomized Kaczmarz (GRK) method, the greedy randomized Kaczmarz method with oblique projection (GRKO), the maximal weighted residual Kaczmarz method (MWRK), and the maximal weighted residual Kaczmarz method (MWRKO). All experiments are carried out using MATLAB (version R2019b) on a personal computer with 1.60 GHz central processing unit (Intel(R) Core(TM) i5-10210U CPU), 8.00 GB memory, and Windows operating system (64 bit Windows 10).

In our implementations, the right vector $b = Ax^*$ such that the exact solution $x^* \in R^n$ is a vector generated by the *rand* function. Define the relative residual error (RRE) at the k th iteration as follows:

$$\text{RRE} = \frac{\|b - Ax^{(k)}\|^2}{\|b\|^2}.$$

The initial point $x^{(0)} \in R^n$ is set to be a zero vector, and the iterations are terminated once the relative solution error satisfies $\text{RRE} < \omega$ or the number of iteration steps exceeds 100,000. If the number of iteration steps exceeds 100,000, it is denoted as "-".

We will compare the numerical performance of these methods in terms of the number of iteration steps (denoted as "IT") and the computing time in seconds (denoted as "CPU"). Here the CPU and IT mean the arithmetical averages of the elapsed running times and the required iteration steps with respect to 50 trials repeated runs of the corresponding method.

4.1 Experiments for Random Matrix Collection in $[0, 1]$

The random matrix collection in $[0, 1]$ is randomly generated by using the MATLAB function *rand*, and the numerical results are reported in Tables 1-2 and Figures 2-3. In this subsection, we let $\omega = 0.5 \times 10^{-8}$. According to the characteristics of the matrix generated by MATLAB function *rand*, Table 1 and Table 2 are the experiments for the overdetermined consistent linear systems, underdetermined consistent linear systems respectively. Under the premise of convergence, all methods can find the unique least Euclidean norm solution.

From Table 1 and Figure 2, we can see that when the linear system is overdetermined, with the increase of m , the IT of all methods decreases, but the CPU shows an increasing trend. Our new methods – the GRKO method and the MWRKO method, perform better than the GRK method and the MWRK method respectively in both iteration steps and running time. Among the four methods, the MWRKO method performs best. From Table 2 and Figure 3, we can see that in the case of underdetermined linear system, with the increase of m , the IT and CPU of all methods decrease.

In this group of experiments, whether it is an overdetermined or underdetermined linear system, whether in terms of the IT or CPU, the GRKO method and the MWRKO method perform very well compared with the GRK method and the MWRK method. These experimental phenomena are consistent with the theoretical convergence conclusions we got.

Table 1: IT and CPU of GRK, GRKO, MWRK and MWRKO for $m \times n$ matrices A with $n = 500$ and different m when the consistent linear system is overdetermined

m	IT				CPU			
	GRK	GRKO	MWRK	MWRKO	GRK	GRKO	MWRK	MWRKO
1000	12072	2105	11265	1913	1.2824	0.2099	0.7192	0.1089
2000	4726	1088	4292	898	1.4792	0.3413	1.1107	0.2157
3000	3362	897	3234	771	1.7550	0.5172	1.5711	0.3575
4000	2663	859	2517	668	1.9415	0.6396	1.6634	0.4807
5000	2398	826	2282	605	2.4134	0.8160	2.1528	0.5801
6000	2100	772	2018	586	2.6235	0.8912	2.0975	0.6486
7000	1970	752	1829	562	2.6019	1.0720	2.5441	0.7822
8000	1861	747	1703	555	3.1035	1.2421	2.4987	0.8390
9000	1750	747	1612	530	3.0223	1.3055	2.6148	0.8730

Fig. 2: Matrix $A \in \mathbb{R}^{m \times 500}$ is generated by the *rand* function in the interval $[0, 1]$. (a): IT of the GRK, GRKO, MWRK, MWRKO methods changes with m . (b): CPU of the GRK, GRKO, MWRK, MWRKO methods changes with m .

Table 2: IT and CPU of GRK, GRKO, MWRK and MWRKO for $m \times n$ matrices A with $n = 2000$ and different m when the consistent linear system is underdetermined

m	IT				CPU			
	GRK	GRKO	MWRK	MWRKO	GRK	GRKO	MWRK	MWRKO
100	802	286	848	272	0.0496	0.0223	0.0258	0.0165
200	1968	523	1948	481	0.1648	0.0496	0.0831	0.0276
300	3104	759	3148	709	0.3982	0.1090	0.2404	0.0664
400	4586	1002	4612	930	1.0539	0.2594	0.8433	0.1920
500	6233	1250	6336	1215	1.9528	0.4409	1.6836	0.3576
600	8671	1576	8882	1497	3.6363	0.7493	3.1625	0.5957
700	11895	2063	11575	1879	5.8642	1.1078	5.0029	0.9087
800	14758	2451	14888	2394	8.4280	1.5350	7.7007	1.6405
900	18223	3250	18608	2945	12.0469	2.2750	10.9511	1.8884

Fig. 3: Matrix $A \in \mathbb{R}^{m \times 2000}$ is generated by the *rand* function in the interval $[0, 1]$. (a): IT of the GRK, GRKO, MWRK, MWRKO methods changes with m . (b): CPU of the GRK, GRKO, MWRK, MWRKO methods changes with m .

4.2 Experiments for Random Matrix Collection in $[c, 1]$

In this subsection, the entries of our coefficient matrix are randomly generated in the interval $[c, 1]$. This set of experiments was also done in [34] and [45], and pointed out that when the value of c is close to 1, the rows of matrix A is closer to linear correlation. Theorem 3.1 and theorem 3.2 have shown the effectiveness of the GRKO method and the MWRKO method in this case. In order to verify this phenomenon, we construct several 1000×500 and 500×1000 matrices A , which entries is independent identically distributed uniform random variables on some interval $[c, 1]$. Note that there is nothing special about this interval, and other intervals yield the same results when the interval length remains the same. In the experiment of this subsection, we take $\omega = 0.5 \times 10^{-8}$.

From Table 3 and Figure 4, it can be seen that when the linear system is overdetermined, with c getting closer to 1, the GRK method and the MWRK method have a significant increase in the number of iterations and running time. When c increases to 0.7, the GRK method and the MWRK method exceeds the maximum number of iterations. But the IT and CPU of the GRKO method and the MWRKO method have decreasing trends. From Table 4 and Figure 5, we can get that the numerical experiment of the coefficient matrix A in the underdetermined case has similar laws to the numerical experiment in the overdetermined case.

In this group of experiments, it can be observed that when the rows of the matrix are close to linear correlation, the GRKO method and the MWRKO method can find the least Euclidean norm solution more quickly than the GRK method and the MWRK method.

Table 3: IT and CPU of GRK, GRKO, MWRK and MWRKO for matrices $A \in R^{1000 \times 500}$ generated by the *rand* function in the interval $[c, 1]$

c	IT				CPU			
	GRK	GRKO	MWRK	MWRKO	GRK	GRKO	MWRK	MWRKO
0.1	14757	2036	14594	1830	1.5811	0.2180	0.9419	0.0969
0.2	21103	1840	20717	1714	2.1684	0.2287	1.1828	0.1003
0.3	27375	1708	26986	1569	3.5926	0.1789	1.5865	0.1195
0.4	36293	1708	35595	1394	3.6751	0.1802	2.0682	0.0885
0.5	53485	1428	52853	1310	5.3642	0.1486	3.0024	0.0847
0.6	84204	1353	81647	1185	9.0879	0.1388	4.5468	0.0767
0.7	-	1227	-	1036	-	0.1298	-	0.0564
0.8	-	1080	-	926	-	0.1107	-	0.0580
0.9	-	715	-	583	-	0.0707	-	0.0324

Fig. 4: Matrix $A \in R^{1000 \times 500}$ is generated by the *rand* function in the interval $[c, 1]$. (a): IT of the GRK, GRKO, MWRK, MWRKO methods changes with c . (b): CPU of the GRK, GRKO, MWRK, MWRKO methods changes with c .

Table 4: IT and CPU of GRK, GRKO, MWRK and MWRKO for matrices $A \in R^{500 \times 1000}$ generated by the *rand* function in the interval $[c, 1]$

c	IT				CPU			
	GRK	GRKO	MWRK	MWRKO	GRK	GRKO	MWRK	MWRKO
0.1	16828	1968	16913	1795	1.7612	0.2103	0.9353	0.1083
0.2	23518	2003	23234	1857	2.3037	0.2066	1.3119	0.1230
0.3	30875	1661	31017	1688	2.9310	0.1635	1.7373	0.0997
0.4	41242	1511	40986	1515	4.3004	0.1726	2.2899	0.1025
0.5	60000	1399	59750	1349	5.4754	0.1252	2.8920	0.0727
0.6	97045	1270	95969	1264	8.5229	0.1173	4.8380	0.0688
0.7	-	1082	-	1022	-	0.1168	-	0.0646
0.8	-	858	-	863	-	0.0960	-	0.0585
0.9	-	549	-	598	-	0.0582	-	0.0353

Fig. 5: Matrix $A \in R^{500 \times 1000}$ is generated by the *rand* function in the interval $[c, 1]$. (a): IT of the GRK, GRKO, MWRK, MWRKO methods changes with c . (b): CPU of the GRK, GRKO, MWRK, MWRKO methods changes with c .

4.3 Experiments for Sparse Matrix

In this subsection, we will give three examples to illustrate the effectiveness of our new methods applied to sparse matrix. The coefficient matrices A of these three examples are the practical problems from [9] and the two test problems from [16]. We uniformly take $\omega = 0.5 \times 10^{-5}$ in these three numerical examples.

Example 1 We solve the problem (1.1) with the coefficient matrix $A \in R^{m \times n}$ chosen from the University of Florida sparse matrix collection [9]. the matrices are *divorce*, *photogrammetry*, *Ragusa18*, *Trec8*, *Stranke94*, and *well1033*. In Table 5, we list some properties of these matrices, where density is defined as follows:

$$\text{density} = \frac{\text{number of nonzeros of } m\text{-by-}n \text{ matrix}}{mn}.$$

In order to solve Example 1, we list the IT, CPU and historical convergence of the GRK, GRKO, MWRK, and MWRKO methods in Figure 6 and Table 6, respectively. It can be seen that MWRKO's IT and CPU are the least. Although the GRKO method is not faster than the MWRK method for most of the experiments in Table 6, it is always faster than the GRK method.

Table 5: The properties of different sparse matrices.

A	divorce	photogrammetry	Ragusa18	Trec8	Stranke94	well1033
$m \times n$	50×9	1388×390	23×23	23×84	10×10	1033×320
rank	9	390	15	23	10	320
cond(A)	19.3908	4.35e+8	3.48e+35	26.8949	51.7330	166.1333
density	50.00%	2.18%	12.10%	28.42%	90.00%	1.43%

Fig. 6: (a): Convergence history of methods for sparse matrices 'divorce'. (b): Convergence history of methods for sparse matrices 'photogrammetry'. (c): Convergence history of methods for sparse matrices 'Ragusa18'. (d): Convergence history of methods for sparse matrices 'Trec8'. (e): Convergence history of methods for sparse matrices 'Stranke94'. (f): Convergence history of methods for sparse matrices 'well1033'.

Table 6: IT and CPU of GRK, GRKO, MWRK and MWRKO for different sparse matrices

A	IT				CPU			
	GRK	GRKO	MWRK	MWRKO	GRK	GRKO	MWRK	MWRKO
divorce	51	28	54	22	0.0053	0.0037	0.0017	0.0013
photogrammetry	85938	48933	90480	27084	9.9917	8.0424	3.9809	2.5026
Ragusa18	744	262	727	280	0.0577	0.0270	0.0121	0.0098
Trec8	465	152	538	139	0.0382	0.0168	0.0111	0.0062
Stranke94	1513	197	1453	181	0.1291	0.0187	0.0208	0.0082
well1033	22924	9825	25250	8655	2.4278	1.5112	0.8491	0.5827

Example 2 We consider $f_{\text{ancurvedtomo}}(N, \theta, P)$ test problem from the MATLAB package AIR Tools [16], which generates sparse matrix A , an exact solution x^* and $b = Ax^*$. We set $N = 60$, $\theta = 0 : 0.5 : 179.5^\circ$,

$P = 50$, then resulting matrix is of size 32400×3600 . We test RRE every 10 iterations and run these four methods until $\text{RRE} < \omega$ is satisfied, where $\omega = 0.5 \times 10^{-5}$.

We first remove the rows of A where the entries are all 0, and perform row unitization processing on A and b . We emphasized that this will not cause a change in x^* . In Figure 7, we give 60×60 images of the exact phantom and the approximate solutions obtained by the GRK, GRKO, MWRK, MWRKO methods. In Figure 7, these four methods can basically restore the original image, but in the subgraph (f) of Figure 7, we can see that the MWRKO methods needs the least iterative steps, and the GRKO method has less iterative steps than GRK method. It can be observed from Table 7 that the MWRKO method is the best in terms of IT and CPU.

Fig. 7: Performance of GRK, GRKO, MWRK, MWRKO methods for $f_{\text{ancurvedtomo}}$ test problem. (a):Exact phantom. (b): GRK. (c):GRKO. (d): MWRK. (e):MWRKO. (f): Convergence history of GRK, GRKO, MWRK, MWRKO methods.

Table 7: IT and CPU of GRK, GRKO, MWRK and MWRKO for $f_{\text{ancurvedtomo}}$ test problem

method	IT	CPU
GRK	13550	581.17
GRKO	12750	538.82
MWRK	12050	504.83
MWRKO	10790	452.86

Example 3 We use an example from 2D seismic travel-time tomography reconstruction, implemented in the function $\text{seismictomo}(N, s, p)$ in the MATLAB package AIR Tools [16], which generates sparse matrix A , an exact solution x^* and $b = Ax^*$. We set $N = 12$, $s = 24$, $p = 35$, then resulting matrix is of size 840×144 . We run these four methods until $\text{RRE} < \omega$ is satisfied, where $\omega = 0.5 \times 10^{-5}$.

We first remove the rows of A where the entries are all 0, and perform row unitization processing on A and b . In Figure 8, we give 12×12 images of the exact phantom and the approximate solutions obtained by

the GRK, GRKO, MWRK, MWRKO methods. From the subgraph (f) of Figure 8 and Table 8, we can see that the MRKO method, the GRKO method, and the MWRK method perform similarly in the number of iteration steps, and are twice as small as the number of iteration steps of the GRK method. It can be observed from Table 8 that MWRKO method is the best in terms of IT and CPU.

Fig. 8: Performance of GRK, GRKO, MWRK, MWRKO methods for *seismictomo* test problem. (a):Exact phantom. (b): GRK. (c):GRKO. (d): MWRK. (e):MWRKO. (f): Convergence history of GRK, GRKO, MWRK, MWRKO methods.

Table 8: IT and CPU of GRK, GRKO, MWRK and MWRKO for *seismictomo* test problem

method	IT	CPU
GRK	831	0.0443
GRKO	452	0.0273
MWRK	447	0.0125
MWRKO	420	0.0108

5 Conclusion

Combined with the representative randomized and non-randomized row index selection strategies, two Kaczmarz-type methods with oblique projection for solving large-scale consistent linear systems are proposed, namely the GRKO method and the MWRKO method. The exponential convergence of the GRKO method and the MWRKO method are deduced. Theoretical and experimental results show that the convergence rates of the GRKO method and the MWRKO method are better than GRK method and the MWRK method respectively. Numerical experiments show the effectiveness of these two methods, especially when the rows of the coefficient matrix A are close to linear correlation.

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