Sheffer Stroke Nelson Algebras

Tahsin Oner (✉ tahsin.oner@ege.edu.tr)

Ege Universitesi <https://orcid.org/0000-0002-6514-4027>

Tugce Katican

Ege University: Ege Universitesi

Arsham Borumand Saeid

Shahid Bahonar University of Kerman

Research Article

Keywords: Sheffer stroke, Nelson algebra, s-Nelson algebra, (ultra) ideal, quasi-subalgebra

Posted Date: August 31st, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-712955/v1>

License: This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Sheffer Stroke Nelson Algebras

Tahsin Oner · Tugce Katican · Arsham
Borumand Saeid ·

Received: date / Accepted: date

Abstract In this study, Sheffer stroke Nelson algebras (briefly, s-Nelson algebras), (ultra) ideals, quasi-subalgebras and quotient sets on these algebraic structures are introduced. The relationships between s-Nelson and Nelson algebras are analyzed. Also, it is shown that a s-Nelson algebra is a bounded distributive modular lattice, and the family of all ideals forms a complete distributive modular lattice. A congruence relation on s-Nelson algebra is determined by its ideal and quotient s-Nelson algebras are constructed by this congruence relation. Finally, it is indicated that a quotient s-Nelson algebra defined by the ultra ideal is totally ordered and that the cardinality of the quotient is less than or equals to 2.

Keywords Sheffer stroke · Nelson algebra · s-Nelson algebra · (ultra) ideal · quasi-subalgebra

Mathematics Subject Classification (2000) 06F05 · 03G25 · 03G10

T. Oner
Department of Mathematics,
Faculty of Science,
Ege University,
İzmir, Turkey
E-mail: tahsin.oner@ege.edu.tr

T. Katican
Department of Mathematics,
Faculty of Science,
Ege University,
İzmir, Turkey
E-mail: tugcekten@gmail.com

A. Borumand Saeid
Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shadid Bahonar University of Kerman,
Kerman,
Iran
E-mail: arsham@uk.ac.ir

1 Introduction

H. M. Sheffer introduced Sheffer stroke (or Sheffer operation) [25] which can states all mathematically definable connectives in Boolean logic [10]. In other words, This operation can be considered by itself to construct a logical formal system without using any other logical operations, but other unary or binary connective cannot satisfy this property. Therefore, there exist its some applications in computer science, technology and industry. Besides, this binary operation has been used in algebraic structures such as orthoimplication algebras [1], ortholattices [2], strong Sheffer stroke non-associative MV-algebras [3], filters [12] and neutrosophic N -structures [17], Sheffer stroke Hilbert algebras [13], fuzzy filters [14] and neutrosophic N -structures [18], (fuzzy) filters and neutrosophic N -structures of Sheffer stroke BL-algebras ([15], [8]), Sheffer stroke UP-algebras [16], Sheffer stroke BE-algebras [7], Sheffer stroke BG-algebras [24] and their fuzzy implivative ideals [20] and Sheffer stroke BCK-algebras [21].

Ewald introduced intuitionistic tense logic IKt as a generalization of intuitionistic propositional logic with the unary operators P , F , H and G [4]. Also, Figallo and Pelaitay introduced the variety IKt of IKt-algebras and showed that the IKt system has IKt-algebras as and algebraic counterpart [5]. Then H. Rasiowa introduced the concept of Nelson algebra or N -lattices [22] and called quasi-pseudo boolean algebras [23] which is the algebraic counterpart of the constructive logic with strong negation introduced by Nelson [11] and Markov [9]. Figallo et al. introduced and investigated the variety of tense Nelson algebras which are Nelson algebras with two tense operators, and analyzed the relationships between (centered tense) Nelson algebras and IKt-algebras [6].

We present basic definitions of Sheffer stroke and Nelson algebras. Then Sheffer stroke Nelson algebras (in short, s-Nelson algebras) and a partial order on this algebra are defined, and some of properties are investigated. It is proved that a s-Nelson algebra is a bounded distributive modular lattice. In addition, we indicate that every s-Nelson algebra is a Nelson algebra but the special conditions are necessary to prove the inverse. Also, an (ultra) ideal of a s-Nelson algebra is introduced and it is presented the statements which are equivalent to the definition of an (ultra) ideal of this algebraic structure. Indeed, it is demontrated that every ultra ideal of a s-Nelson algebra is the ideal but the inverse is not true in general. Besides, a congruence relation on a s-Nelson algebra is defined by its ideal, and so, it is illustrated that a quotient set of a s-Nelson algebra defined by this congruence relation is a s-Nelson algebra. Consequently, it is showbn that a partial order on the quotient s-Nelson algebra defined by an ultra ideal is a totally order on this algebraic structure.

2 Preliminaries

In this section, we present basic definitions and notions about Sheffer operation and Nelson algebras.

Definition 1 [2] Let $\mathcal{N} = \langle N, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer operation (or Sheffer stroke) if it satisfies the following conditions:

- (S1) $n_1 | n_2 = n_2 | n_1$,
 - (S2) $(n_1 | n_1) | (n_1 | n_2) = n_1$,
 - (S3) $n_1 | ((n_2 | n_3) | (n_2 | n_3)) = ((n_1 | n_2) | (n_1 | n_2)) | n_3$,
 - (S4) $(n_1 | ((n_1 | n_1) | (n_2 | n_2))) | (n_1 | ((n_1 | n_1) | (n_2 | n_2))) = n_1$,
- for all $n_1, n_2, n_3 \in N$.

Definition 2 [6] A Nelson algebra is an algebra $\mathcal{N} = \langle N, \wedge, \vee, \rightarrow, \sim, 1 \rangle$ of type $(2, 2, 2, 1, 0)$ that satisfy the conditions:

- (N1) $n \vee 1 = 1$,
 - (N2) $n_1 \wedge (n_1 \vee n_2) = n_1$,
 - (N3) $n_1 \wedge (n_2 \vee n_3) = (n_3 \wedge n_1) \vee (n_2 \wedge n_1)$,
 - (N4) $\sim \sim n = n$,
 - (N5) $\sim (n_1 \wedge n_2) = \sim n_1 \vee \sim n_2$,
 - (N6) $n_1 \wedge \sim n_1 = (n_1 \wedge \sim n_1) \wedge (n_2 \vee \sim n_2)$, (N7) $n \rightarrow n = 1$,
 - (N8) $n_1 \wedge (n_1 \rightarrow n_2) = n_1 \wedge (\sim n_1 \vee n_2)$,
 - (N9) $(n_1 \wedge n_2) \rightarrow n_3 = n_1 \rightarrow (n_2 \rightarrow n_3)$,
 - (N10) $(n_1 \rightarrow n_2) \wedge (\sim n_1 \vee n_2) = \sim n_1 \vee n_2$,
 - (N11) $n_1 \rightarrow (n_2 \wedge n_3) = (n_1 \rightarrow n_2) \wedge (n_1 \rightarrow n_3)$,
- for all $n, n_1, n_2, n_3 \in N$.

Moreover, it follows from axioms (N2) and (N3) that $\langle N, \wedge, \vee, \sim, 1 \rangle$ is a Kleene algebra.

Definition 3 [24] A modular lattice is any lattice which satisfies $x \leq y \rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)$.

Theorem 1 [24] *Every distributive lattice is a modular lattice.*

3 Sheffer stroke Nelson algebras

In this section, we introduce Sheffer stroke Nelson algebras and give some properties.

Definition 4 A Sheffer stroke Nelson algebra (shortly, s-Nelson algebra) is an algebra $\mathcal{N} = \langle N, |, 1 \rangle$ of type $(2, 0)$ satisfying the following conditions:

- (sn1) $n | (n | n) = 1$,
 - (sn2) $n_1 | (n_2 | n_3) = ((n_1 | (n_2 | n_2)) | (n_1 | (n_3 | n_3))) | ((n_1 | (n_2 | n_2)) | (n_1 | (n_3 | n_3)))$,
- for all $n, n_1, n_2, n_3 \in N$. Define $0 = 1 | 1$.

Remark 1 The axioms (sn1) and (sn2) are independent.

1. Independence of (sn1):

Given a set $N = \{0, \alpha, 1\}$ with the following Cayley table:

Table 1

	0	α	1
0	1	1	1
α	1	α	α
1	1	α	0

Then (sn2) holds while (sn1) does not since $1 \neq \alpha = \alpha|(\alpha|\alpha)$.

2. Independence of (sn2):

Consider a set $N = \{0, \alpha, 1\}$ with Cayley table as below:

Table 2

	0	α	1
0	1	1	1
α	1	0	α
1	1	α	0

Then (sn1) is satisfied (sn2) is not since $\alpha = \alpha|(0|1) \neq 0 = \alpha|\alpha = (\alpha|1)|(\alpha|1) = ((\alpha|(0|0))|(\alpha|(1|1)))|((\alpha|(0|0))|(\alpha|(1|1)))$.

Example 1 Consider a structure $(N, |, 1)$ where $N = \{0, w_1, w_2, 1\}$ and a binary operation $|$ on N with Cayley table as follows:

Table 3

	0	w_1	w_2	1
0	1	1	1	1
w_1	1	w_2	1	w_2
w_2	1	1	w_1	w_1
1	1	w_2	w_1	0

Then it is a s-Nelson algebra.

Lemma 1 Let $(N, |, 1)$ be a s-Nelson algebra. Then

1. $n|(1|1) = 1$,
2. $1|(n|n) = n$,
3. $(n|1)|(n|1) = n$,
4. $n|n = n|1$,
5. $(n|1)|1 = n$,
6. $n|(n|1) = 1$,
7. $((n_1|n_2)|(n_1|n_3))|((n_1|n_2)|(n_1|n_3)) = n_1|((n_2|n_2)|(n_3|n_3))$,

8. $((n_1|n_2)|n_2)|n_2 = n_1|n_2$,
9. $n_1|n_1 = n_1|n_2 \Leftrightarrow n_1|(n_2|n_2) = 1$,
10. $n_1|(n_1|n_2) = n_1|(n_2|n_2)$,
11. $n|0 = 1$,

for all $n, n_1, n_2, n_3 \in N$.

Proof Let $(N, |, 0, 1)$ be a s-Nelson algebra.

1. $n|(1|1) = n|((n|(n|n))|(n|(n|n))) = ((n|n)|(n|n))|(n|n) = n|(n|n) = 1$ from (sn1), (S2) and (S3).
2. $1|(n|n) = (n|(n|n))|(n|n) = n$ from (sn1), (S1) and (S2), respectively.
3. $(n|1)|(n|1) = (n|(n|(n|n))|(n|(n|(n|n)))) = (((n|n)|(n|n))|(n|n))|(((n|n)|(n|n))|(n|n)) = n$ from (sn1), (S1) and (S2).
4. $n|n = ((n|1)|(n|1))|((n|1)|(n|1)) = n|1$ from (3) and (S2), respectively.
5. $(n|1)|1 = 1|(((n|1)|(n|1))|((n|1)|(n|1))) = 1|(n|n) = n$ from (S1), (S2), (3) and (2), respectively.
6. $n|(n|1) = (n|1)|((n|1)|(n|1)) = 1$ from (S1), (3) and (sn1), respectively.
- 7.

$$\begin{aligned}
 & ((n_1|n_2)|(n_1|n_3))|((n_1|n_2)|(n_1|n_3)) \\
 &= ((n_1|((n_2|n_2)|(n_2|n_2)))|(n_1|((n_3|n_3)|(n_3|n_3)))) \\
 & \quad |((n_1|((n_2|n_2)|(n_2|n_2)))|(n_1|((n_3|n_3)|(n_3|n_3)))) \\
 &= n_1|((n_2|n_2)|(n_3|n_3))
 \end{aligned}$$

from (S2) and (sn2).

8.

$$\begin{aligned}
 & ((n_1|n_2)|n_2)|n_2 = n_2|(n_2|(n_2|n_1)) \\
 &= ((n_2|(n_2|n_2))|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
 & \quad |((n_2|(n_2|n_2))|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
 &= (1|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
 & \quad | (1|(n_2|((n_2|n_1)|(n_2|n_1)))) \\
 &= n_2|((n_2|n_1)|(n_2|n_1)) \\
 &= ((n_2|n_2)|(n_2|n_2))|n_1 \\
 &= n_1|n_2
 \end{aligned}$$

from (S1)-(S3) and (sn1)-(sn2).

9. Let $n_1|n_1 = n_1|n_2$. Then it follows from (S2), (S3), (sn1) and (1) that

$$\begin{aligned}
 n_1|(n_2|n_2) &= ((n_1|n_1)|(n_1|n_1))|(n_2|n_2) \\
 &= ((n_1|n_2)|(n_1|n_2))|(n_2|n_2) \\
 &= n_1|((n_2|(n_2|n_2))|(n_2|(n_2|n_2))) \\
 &= n_1|(1|1) \\
 &= 1,
 \end{aligned}$$

for all $n_1, n_2 \in N$.

Conversely, let $n_1|(n_2|n_2) = 1$. Thus, it is obtained from (2)-(3), (sn1)-(sn2) and (S1)-(S2) that

$$\begin{aligned}
 n_1|n_1 &= 1|((n_1|n_1)|(n_1|n_1)) \\
 &= 1|n_1 \\
 &= n_1|(n_1|(n_2|n_2)) \\
 &= ((n_1|(n_1|n_1))|(n_1|((n_2|n_2)|(n_2|n_2)))) \\
 &\quad |((n_1|(n_1|n_1))|(n_1|((n_2|n_2)|(n_2|n_2)))) \\
 &= (1|(n_1|n_2))|(1|(n_1|n_2)) \\
 &= n_1|n_2,
 \end{aligned}$$

for all $n_1, n_2 \in N$.

10. We have from (3), (sn1)-(sn2) and (S1) that

$$\begin{aligned}
 n_1|(n_1|n_2) &= ((n_1|(n_1|n_1))|(n_1|(n_2|n_2))) \\
 &\quad |((n_1|(n_1|n_1))|(n_1|(n_2|n_2))) \\
 &= (1|(n_1|(n_2|n_2))|(1|(n_1|(n_2|n_2)))) \\
 &= n_1|(n_2|n_2)
 \end{aligned}$$

for all $n_1, n_2 \in N$.

11. It follows from (1) that $n|0 = n|(1|1) = 1$, for all $n \in N$.

Lemma 2 Let $(N, |, 1)$ be a s -Nelson algebra. Then a binary operation $\leq$ on N defined by $n_1 \leq n_2 \Leftrightarrow n_1|(n_2|n_2) = 1$ is a partial order on N .

Proof Let $(N, |, 0, 1)$ be a s -Nelson algebra and a binary operation $\leq$ on N be defined by $n_1 \leq n_2 \Leftrightarrow n_1|(n_2|n_2) = 1$, for any $n_1, n_2 \in N$.

- Reflexive: since $n|(n|n) = 1$ from (sn1), we have $n \leq n$, for all $n \in N$.
- Antisymmetric: let $n_1 \leq n_2$ and $n_2 \leq n_1$. Then $n_1|(n_2|n_2) = n_2|(n_1|n_1) = 1$. Since

$$\begin{aligned}
 n_1 &= 1|(n_1|n_1) \\
 &= (n_2|(n_2|n_2))|(n_1|n_1) \\
 &= (n_1|n_1)|(n_2|(n_2|n_2)) \\
 &= (((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|((n_2|n_2)|(n_2|n_2)))) \\
 &\quad |(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|((n_2|n_2)|(n_2|n_2)))) \\
 &= (((n_1|n_1)|(n_2|n_2))|(n_2|(n_1|n_1))) \\
 &\quad |(((n_1|n_1)|(n_2|n_2))|(n_2|(n_1|n_1))) \\
 &= (n_1|n_1)|(n_2|n_2)
 \end{aligned}$$

and similarly, $n_2 = (n_1|n_1)|(n_2|n_2)$ from (S1)-(S2), (sn1)-(sn2) and (2)-(3), it follows that $n_1 = n_2$.

– Transitive: let $n_1 \leq n_2$ and $n_2 \leq n_3$. Then $n_1|(n_2|n_2) = n_2|(n_3|n_3) = 1$.
Since

$$\begin{aligned}
 1 &= n_1|(n_2|n_2) \\
 &= n_1|(n_2|1) \\
 &= n_1|(n_2|(n_2|(n_3|n_3))) \\
 &= n_1|(((n_2|(n_2|n_2))|(n_2|(n_3|n_3)|(n_3|n_3)))) \\
 &\quad |(((n_2|(n_2|n_2))|(n_2|(n_3|n_3)|(n_3|n_3)))) \\
 &= n_1|((1|(n_2|n_3))|(1|(n_2|n_3))) \\
 &= n_1|(n_2|n_3) \\
 &= ((n_1|(n_2|n_2))|(n_1|(n_3|n_3))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))) \\
 &= (1|(n_1|(n_3|n_3))|(1|(n_1|(n_3|n_3)))) \\
 &= n_1|(n_3|n_3)
 \end{aligned}$$

from (S1)-(S2), (sn1)-(sn2) and (3), it is obtained that $n_1 \leq n_3$.

It is clear that 0 is the least element and 1 is the greatest element of N .

Lemma 3 Let $(N, |, 1)$ be a s-Nelson algebra and $\leq$ be a partial order on N . Then $(N, \leq)$ (or $(N, \vee, \wedge, 0, 1)$) is a bounded distributive modular lattice where $n_1 \vee n_2 = (n_1|n_1)|(n_2|n_2)$ and $n_1 \wedge n_2 = (n_1|n_2)|(n_1|n_2)$, for all $n_1, n_2 \in N$.

Proof Let $(N, |, 0, 1)$ be a s-Nelson algebra and $\leq$ be a partial order on N . Since $((n_1|n_2)|(n_1|n_2))|(n_2|n_2) = n_1|((n_2|(n_2|n_2))|(n_2|(n_2|n_2))) = n_1|(1|1) = 1$ and $((n_1|n_2)|(n_1|n_2))|(n_1|n_1) = n_2|((n_1|(n_1|n_1))|(n_1|(n_1|n_1))) = n_2|(1|1) = 1$ from (S1), (S3) and Lemma 1 (1), we get $(n_1|n_2)|(n_1|n_2) \leq n_1$ and $(n_1|n_2)|(n_1|n_2) \leq n_2$. So, $(n_1|n_2)|(n_1|n_2)$ is a lower bound of n_1 and n_2 . Let $n_3 \leq n_1, n_2$. Then $n_3|(n_1|n_1) = n_3|(n_2|n_2) = 1$. Since

$$\begin{aligned}
 n_3|(((n_1|n_2)|(n_1|n_2))|((n_1|n_2)|(n_1|n_2))) &= n_3|(n_1|n_2) \\
 &= ((n_3|(n_1|n_1))|(n_3|(n_2|n_2)))| \\
 &\quad ((n_3|(n_1|n_1))|(n_3|(n_2|n_2))) \\
 &= (1|1)|(1|1) \\
 &= 1
 \end{aligned}$$

from (S2) and (sn2), it follows that $n_3 \leq (n_1|n_2)|(n_1|n_2) \leq n_1, n_2$. Thus, $n_1 \wedge n_2 = (n_1|n_2)|(n_1|n_2)$ is the greatest lower bound (infimum) of n_1 and n_2 . Similarly, $n_1 \vee n_2 = (n_1|n_1)|(n_2|n_2)$ is the least upper bound (supremum) of n_1 and n_2 . Hence, $(N, \leq)$ is a lattice.

Since

$$\begin{aligned}
 n_1 \vee (n_2 \wedge n_3) &= (n_1|n_1)|(n_2|n_3) \\
 &= (((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_3|n_3))) \\
 &\quad |(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_3|n_3)))) \\
 &= (n_1 \vee n_2) \wedge (n_1 \vee n_3)
 \end{aligned}$$

and

$$\begin{aligned}
 n_1 \wedge (n_2 \vee n_3) &= (n_1|((n_2|n_2)|(n_3|n_3))|(n_1|((n_2|n_2)|(n_3|n_3)))) \\
 &= (n_1|n_2)|(n_1|n_3) \\
 &= (((n_1|n_2)|(n_1|n_2))|((n_1|n_2)|(n_1|n_2))) \\
 &\quad |(((n_1|n_3)|(n_1|n_3))|((n_1|n_3)|(n_1|n_3)))) \\
 &= (n_1 \wedge n_2) \vee (n_1 \wedge n_3)
 \end{aligned}$$

from (sn2), Lemma 1 (6) and (S2), $(N, \leq)$ is distributive. By Theorem 1, it is modular lattice.

Moreover, since $n \vee 1 = (n|n)|(1|1) = 1$, $n \vee 0 = (n|n)|(0|0) = 1|(n|n) = n$ and $n \wedge 0 = (n|0)|(n|0) = (n|(1|1)|(n|(1|1))) = 1|1 = 0$, $n \wedge 1 = (n|1)|(n|1) = n$ from Lemma 1 (1)-(3) and (S1), $(N, \leq)$ is bounded.

Therefore, $(N, \leq)$ is a bounded distributive modular lattice.

Lemma 4 *Let $(N, |, 1)$ be a s-Nelson algebra. Then*

1. $n_1 \leq n_2 \Leftrightarrow n_2|n_2 \leq n_1|n_1$,
2. $n_1 \leq n_2$ implies $n_2|n_3 \leq n_1|n_3$,
3. $n_1 \leq n_2|(n_1|n_1)$,
4. $n_2|n_2 \leq n_2|(n_1|n_1)$,
5. $n_1 \leq (n_1|n_2)|n_2$,
6. $n_1 \leq n_2|(n_3|n_3) \Leftrightarrow (n_1|n_2)|(n_1|n_2) \leq n_3$,
7. $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \leq (n_1|(n_2|n_2))|(n_2|(n_3|n_3))$,
8. $n_1|(n_2|n_2) \leq (n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))$,
9. $n_1|(n_2|n_2) \leq (n_3|(n_1|n_1))|((n_3|(n_2|n_2))|(n_3|(n_2|n_2)))$,

for all $n_1, n_2, n_3 \in N$.

Proof Let $(N, |, 1)$ be a s-Nelson algebra.

1. $n_1 \leq n_2 \Leftrightarrow n_1|(n_2|n_2) = 1 \Leftrightarrow (n_2|n_2)|((n_1|n_1)|(n_1|n_1)) = n_1|(n_2|n_2) = 1 \Leftrightarrow n_2|n_2 \leq n_1|n_1$ from Lemma 2 and (S1)-(S2).
2. Let $n_1 \leq n_2$. Then $n_1|(n_2|n_2) = 1$ from Lemma 2. Since

$$\begin{aligned}
 (n_2|n_3)|((n_1|n_3)|(n_1|n_3)) &= ((n_1|n_3)|(n_1|n_3))|(n_2|n_3) \\
 &= (((n_1|n_3)|(n_1|n_3))|(n_2|n_2))| \\
 &\quad (((n_1|n_3)|(n_1|n_3))|(n_3|n_3))| \\
 &\quad (((n_1|n_3)|(n_1|n_3))|(n_2|n_2))| \\
 &\quad (((n_1|n_3)|(n_1|n_3))|(n_3|n_3)) \\
 &= ((n_3|((n_1|(n_2|n_2))|(n_1|(n_2|n_2))))| \\
 &\quad (n_1|((n_3|(n_3|n_3))|(n_3|(n_3|n_3))))| \\
 &\quad ((n_3|((n_1|(n_2|n_2))|(n_1|(n_2|n_2))))| \\
 &\quad (n_1|((n_3|(n_3|n_3))|(n_3|(n_3|n_3)))) \\
 &= ((n_3|(1|1))|(n_1|(1|1)))|((n_3|(1|1))|(n_1|(1|1))) \\
 &= 1
 \end{aligned}$$

from (S1)-(S3), (sn1)-(sn2) and Lemma 1 (1), it follows that $n_2|n_3 \leq n_1|n_3$.

3. Since

$$\begin{aligned}
 n_1|((n_2|(n_1|n_1))|(n_2|(n_1|n_1))) &= ((n_1|(n_1|n_1))|(n_1|(n_1|n_1)))|n_2 \\
 &= n_2|(1|1) \\
 &= 1
 \end{aligned}$$

from (S1), (S3) and Lemma 1 (1), it is obtained from Lemma 2 that $n_1 \leq n_2|(n_1|n_1)$.

4. $n_2|n_2 \leq (n_1|n_1)|((n_2|n_2)|(n_2|n_2)) = n_2|(n_1|n_1)$ from (3), (S1) and (S2), respectively.

5. Since $n_1|(((n_1|n_2)|n_2)|((n_1|n_2)|n_2)) = (n_1|n_2)|((n_1|n_2)|(n_1|n_2)) = 1$ from (S1), (S3) and (sn1), it follows from Lemma 2 that $n_1 \leq (n_1|n_2)|n_2$.
6. $n_1 \leq n_2|(n_3|n_3) \Leftrightarrow (n_1|n_2)|(n_1|n_2) \leq n_3$ is proved from Lemma 2 and (S3).
7. Since

$$\begin{aligned}
& ((n_1|(n_3|n_3))|(n_1|(n_3|n_3))|(((n_1|(n_2|n_2))| \\
& (n_2|(n_3|n_3))|((n_1|(n_2|n_2))|(n_2|(n_3|n_3)))) \\
& = (n_1|(n_2|n_2))|(((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3| \\
& n_3))))|((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))) \\
& = (n_1|(n_2|n_2))|((n_1|(((n_2|(n_3|n_3))|(n_3|n_3))|((n_2|(n_3|n_3))|(n_3| \\
& n_3))))|(n_1|(((n_2|(n_3|n_3))|(n_3|n_3))|((n_2|(n_3|n_3))|(n_3|n_3)))) \\
& = ((n_1|(n_1|(n_2|n_2))|(n_1|(n_1|(n_2|n_2))))|(((n_2| \\
& |(n_3|n_3))|(n_3|n_3))|((n_2|(n_3|n_3))|(n_3|n_3)))) \\
& = ((n_1|n_2)|(n_1|n_2))|(((n_2|(n_3|n_3))| \\
& (n_3|n_3))|((n_2|(n_3|n_3))|(n_3|n_3))) \\
& = (((n_1|n_2)|(n_1|n_2))|(n_3|n_3))|(((n_1| \\
& n_2)|(n_1|n_2))|(n_3|n_3))|(n_2|(n_3|n_3)) \\
& = ((n_1|((n_2|(n_3|n_3))|(n_2|(n_3|n_3))))|(n_1|((n_2| \\
& |(n_3|n_3))|(n_2|(n_3|n_3))))|(n_2|(n_3|n_3)) \\
& = n_1|(((n_2|(n_3|n_3))|((n_2|(n_3|n_3))|(n_2|(n_3|n_3)))) \\
& |((n_2|(n_3|n_3))|((n_2|(n_3|n_3))|(n_2|(n_3|n_3)))) \\
& = n_1|(1|1) \\
& = 1
\end{aligned}$$

from (S1)-(S3), (sn1), Lemma 1 (1) and (10), it follows from Lemma 2 that $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \leq (n_1|(n_2|n_2))|(n_2|(n_3|n_3))$.

8. Since

$$\begin{aligned}
& (n_1|(n_2|n_2))|(((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3| \\
& n_3))))|((n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3)))) \\
& = ((n_1|(n_3|n_3))|(n_1|(n_3|n_3))|(((n_1|(n_2|n_2)) \\
& |(n_2|(n_3|n_3))|((n_1|(n_2|n_2))|(n_2|(n_3|n_3)))) \\
& = 1
\end{aligned}$$

from (S1), (S3), (7) and Lemma 2, we have from Lemma 2 that

$$n_1|(n_2|n_2) \leq (n_2|(n_3|n_3))|((n_1|(n_3|n_3))|(n_1|(n_3|n_3))).$$

9. By substituting $[n_1 := n_2|n_2]$, $[n_2 := n_1|n_1]$ and $[n_3 := n_3|n_3]$ in (8), simultaneously, it follows from (S1) and (S2) that

$$n_1|(n_2|n_2) \leq (n_3|(n_1|n_1))|((n_3|(n_2|n_2))|(n_3|(n_2|n_2))).$$

Theorem 2 Let $(N, |, 1)$ be a s -Nelson algebra. If $n_1 \rightarrow n_2 := n_1|(n_2|n_2)$, $n_1 \vee n_2 := (n_1|n_1)|(n_2|n_2)$, $n_1 \wedge n_2 := (n_1|n_2)|(n_1|n_2)$ and $\sim n := n|n$, for all $n, n_1, n_2 \in N$, then a structure $\mathcal{N} = \langle N, \wedge, \vee, \rightarrow, \sim, 1 \rangle$ states a Nelson algebra.

Proof Let $(N, |, 0, 1)$ be a s -Nelson algebra, and $n_1 \rightarrow n_2 := n_1|(n_2|n_2)$, $n_1 \vee n_2 := (n_1|n_1)|(n_2|n_2)$, $n_1 \wedge n_2 := (n_1|n_2)|(n_1|n_2)$ and $\sim n := n|n$, for all $n, n_1, n_2 \in N$. Then

- (N1) $n \vee 1 = (n|n)|(1|1) = 1$ from Lemma 1 (1),
 (N2) $n_1 \wedge (n_1 \vee n_2) = (n_1|((n_1|n_1)|(n_2|n_2)))(n_1|((n_1|n_1)|(n_2|n_2))) = n_1$ from (S2),
 (N3) $n_1 \wedge (n_2 \vee n_3) = (n_3 \wedge n_1) \vee (n_2 \wedge n_1)$ from Lemma 3,
 (N4) $\sim \sim n = (n|n)|(n|n)$ from (S2),
 (N5) $\sim (n_1 \wedge n_2) = n_1|n_2 = ((n_1|n_1)|(n_1|n_1))((n_2|n_2)|(n_2|n_2)) = \sim n_1 \vee \sim n_2$ from (S2),
 (N6) Since $n_1 \wedge \sim n_1 = (n_1|(n_1|n_1))|(n_1|(n_1|n_1)) = 1|1 = 0$ and $n_2 \vee \sim n_2 = n_2|(n_2|n_2) = 1$ from (sn1), (S1) and (S2), it follows from Lemma 1 (3) that $n_1 \wedge \sim n_1 = 0 = (0|1)|(0|1) = 0 \wedge 1 = (n_1 \wedge \sim n_1) \wedge (n_2 \vee \sim n_2)$,
 (N7) $n \rightarrow n = n|(n|n) = 1$ from (sn1),
 (N8)

$$\begin{aligned} n_1 \wedge (n_1 \rightarrow n_2) &= (n_1|(n_1|(n_2|n_2)))(n_1|(n_1|(n_2|n_2))) \\ &= (n_1|(((n_1|n_1)|(n_1|n_1))|(n_2|n_2))) \\ &\quad |(n_1|(((n_1|n_1)|(n_1|n_1))|(n_2|n_2))) \\ &= n_1 \wedge (\sim n_1 \vee n_2) \end{aligned}$$

from (S2),

- (N9) $(n_1 \wedge n_2) \rightarrow n_3 = ((n_1|n_2)|(n_1|n_2))n_3 = n_1|((n_2|n_3)|(n_2|n_3)) = n_1 \rightarrow (n_2 \rightarrow n_3)$ from (S3),
 (N10) $(n_1 \rightarrow n_2) \wedge (\sim n_1 \vee n_2) = n_1|(n_2|n_2) = ((n_1|n_1)|(n_1|n_1))|(n_2|n_2) = \sim n_1 \vee n_2$ from (S2), and
 (N11) $n_1 \rightarrow (n_2 \wedge n_3) = n_1|(n_2|n_3) = ((n_1|(n_2|n_2))|(n_1|(n_3|n_3)))(n_1|(n_2|n_2))|(n_1|(n_3|n_3)) = (n_1 \rightarrow n_2) \wedge (n_1 \rightarrow n_3)$ from (sn2).

Example 2 Consider the s-Nelson algebra $(N, |, 1)$ in Example 1. Then $\langle N, \wedge, \vee, \rightarrow, \sim, 1 \rangle$ defined by the s-Nelson algebra is a Nelson algebra with the following Cayley tables:

Table 4

$\rightarrow$	0	w_1	w_2	1	$\vee$	0	w_1	w_2	1
0	1	1	1	1	0	0	w_1	w_2	1
w_1	w_2	1	w_2	1	w_1	w_1	w_1	1	1
w_2	w_1	w_1	1	1	w_2	w_2	1	w_2	1
1	0	w_1	w_2	1	1	1	1	1	1

p	$\sim p$	$\wedge$	0	w_1	w_2	1
0	1	0	0	0	0	0
w_1	w_2	w_1	0	w_1	0	w_1
w_2	w_1	w_2	0	0	w_2	w_2
1	0	1	0	w_1	w_2	1

Theorem 3 Let $\mathcal{N} = \langle N, \wedge, \vee, \rightarrow, \sim, 1 \rangle$ be a Nelson algebra with 0. If $n_1|n_2 := \sim (n_1 \wedge n_2) = \sim n_1 \vee \sim n_2 = n_1 \rightarrow \sim n_2$, for all $n_1, n_2 \in N$, then a structure $(N, |, 1)$ is a s-Nelson algebra

Proof Let $\mathcal{N} = \langle N, \wedge, \vee, \rightarrow, \sim, 1 \rangle$ be a Nelson algebra with 0 and $n_1|n_2 := \sim (n_1 \wedge n_2) = \sim n_1 \vee \sim n_2 = n_1 \rightarrow \sim n_2$, for all $n_1, n_2 \in N$. Then

- (S1) $n_1|n_2 = \sim n_1 \vee \sim n_2 = \sim n_2 \vee \sim n_1 = n_2|n_1$,
 (S2) $(n_1|n_1)|(n_1|n_2) = \sim (\sim n_1 \wedge (\sim n_1 \vee \sim n_2)) = \sim \sim n_1 = n_1$ from (N2) and (N4),
 (S3) $n_1|((n_2|n_3)|(n_2|n_3)) = \sim n_1 \vee (\sim n_2 \vee \sim n_3) = (\sim n_1 \vee \sim n_2) \vee \sim n_3 = ((n_1|n_2)|(n_1|n_2))|n_3$ from (N4),
 (S4) $(n_1|((n_1|n_1)|(n_2|n_2))|(n_1|((n_1|n_1)|(n_2|n_2)))) = n_1 \wedge (n_1 \vee n_2) = n_1$, from (N2) and (N4),
 (sn1) $n|(n|n) = n \rightarrow n = 1$ from (N4) and (N7), and
 (sn2)

$$\begin{aligned}
 n_1|(n_2|n_3) &= n_1 \rightarrow (n_2 \wedge n_3) \\
 &= (n_1 \rightarrow n_2) \wedge (n_1 \rightarrow n_3) \\
 &= (n_1 \rightarrow \sim \sim n_2) \wedge (n_1 \rightarrow \sim \sim n_3) \\
 &= ((n_1|(n_2|n_2))|(n_1|(n_3|n_3))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3))))
 \end{aligned}$$

from (N4) and (N11).

Example 3 Consider a Nelson algebra $\langle N, \wedge, \vee, \rightarrow, \sim, 1 \rangle$ in which $N = \{0, a, b, c, d, e, f, 1\}$, binary operations $\wedge, \vee, \rightarrow$ and an unary operation $\sim$ on N with the following Cayley tables:

Table 5

$\rightarrow$	0	a	b	c	d	e	f	1	p	$\sim p$
0	1	1	1	1	1	1	1	1	0	1
a	f	1	f	f	1	1	f	1	a	f
b	e	e	1	e	1	e	1	1	b	e
c	d	d	d	1	d	1	1	1	c	d
d	c	e	f	c	1	e	f	1	d	c
e	b	d	b	f	d	1	f	1	e	b
f	a	a	d	e	d	e	1	1	f	a
1	0	a	b	c	d	e	f	1	1	0

$\vee$	0	a	b	c	d	e	f	1	$\wedge$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1	0	0	0	0	0	0	0	0	0
a	a	a	d	e	d	e	1	1	a	0	a	0	0	a	a	0	a
b	b	d	b	f	d	1	f	1	b	0	0	b	0	b	0	b	b
c	c	e	f	c	1	e	f	1	c	0	0	0	c	0	c	c	c
d	d	d	d	1	d	1	1	1	d	0	a	b	0	d	a	b	d
e	e	e	1	e	1	e	1	1	e	0	a	0	c	a	e	c	e
f	f	1	f	f	1	1	f	1	f	0	0	b	c	b	c	f	f
1	1	1	1	1	1	1	1	1	1	0	a	b	c	d	e	f	1

Then a structure $(N, |, 1)$ defined by this Nelson algebra is a s-Nelson algebra with Cayley table as follow:

Table 6

	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

4 Ideals of s-Nelson algebras

In this section, we define a quasi-subalgebra and an ideal of a s-Nelson algebra, and examine some features. When not otherwise specified, N is a s-Nelson algebra.

Definition 5 Let N be a s-Nelson algebra. Then a nonempty subset M of N is called a quasi-subalgebra of N if $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in M$, for all $n_1, n_2 \in M$.

Example 4 Consider the s-Nelson algebra N in Example 3. Then a subset $M = \{0, b, e, 1\}$ of N is a quasi-subalgebra of N .

Definition 6 Let N be a s-Nelson algebra. Then a nonempty subset I of N is called an ideal of N if

- (NI1) $0 \in I$
- (NI2) $n_2 \in I$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ imply $n_1 \in I$, for all $n_1, n_2 \in N$.

Example 5 Consider the s-Nelson algebra N in Example 3. Then N itself, $\{0\}$, $\{0, a, d, b\}$ and $\{0, c\}$ are ideals of N .

Lemma 5 Let I be a nonempty subset of a s-Nelson algebra N . Then I is an ideal of N if and only if

- (NI3) $n_1 \leq n_2$ and $n_2 \in I$ imply $n_1 \in I$,
- (NI4) $n_1, n_2 \in I$ implies $n_1 \vee n_2 \in I$, for all $n_1, n_2 \in N$.

Proof Assume that I is an ideal of N . Let $n_1 \leq n_2$ and $n_2 \in I$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) = 1|1 = 0 \in I$ from Lemma 2 and (NI1). Thus, $n_1 \in I$ by (NI2). Let $n_1, n_2 \in I$. Since $((((n_1 \vee n_2)|(n_2|n_2))|(n_1 \vee n_2)|(n_2|n_2)))|(n_1|n_1)|(((n_1 \vee n_2)|(n_2|n_2))|((n_1 \vee n_2)|(n_2|n_2)))|(n_1 n_1)) = ((n_1 \vee n_2)|(n_1 \vee n_2))|((n_1 \vee n_2)|((n_1 \vee n_2)|(n_1 \vee n_2))) = 1|1 = 0 \in I$ from (S1), (S3), Lemma 3, (sn1) and (NI1), we have from (NI2) that $n_1 \vee n_2 \in I$.

Conversely, let I be a nonempty subset of N satisfying (NI3) and (NI4). Since $0 \leq n$ and $n \in I$, we get from (NI3) that $0 \in I$. Let $n_2 \in I$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$. Since

$$\begin{aligned} n_1 \vee n_2 &= (n_1|n_1)|(n_2|n_2) \\ &= (((n_1|n_1)|(n_2|n_2))|1)|(((n_1|n_1)|(n_2|n_2))|1) \\ &= (((n_2|n_2)|(n_1|n_1))|(n_2|n_2)|((n_2|n_2)|(n_2|n_2))) \\ &\quad |(((n_2|n_2)|(n_1|n_1))|(n_2|n_2)|((n_2|n_2)|(n_2|n_2))) \\ &= (n_1|(n_2|n_2))|(n_2|n_2) \\ &= ((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \vee n_2 \in I \end{aligned}$$

from Lemma 3, Lemma 1 (1), (S1)-(S2), (sn2) and (NI4), and $n_1 \leq n_1 \vee n_2$, it follows from (NI3) that $n_1 \in I$.

Lemma 6 *Every ideal of a s-Nelson algebra N is a quasi-subalgebra of N .*

Proof Let I be an ideal of N and $n_1, n_2 \in I$. Since

$$\begin{aligned} &((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_2|n_2))) \\ &= (n_2|n_2)|((n_1|(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_2|n_2)))) \\ &\quad n_2)))|(n_1|(((n_1|n_1)|(n_2|n_2))|((n_1|n_1)|(n_2|n_2)))) \\ &= (n_2|n_2)|(((n_1|(n_1|n_1))|(n_1|(n_1|n_1)))|(n_2|n_2| \\ &\quad n_2))|(((n_1|(n_1|n_1))|(n_1|(n_1|n_1)))|(n_2|n_2))) \\ &= (n_2|n_2)|(((n_2|n_2)|(1|1))|((n_2|n_2)|(1|1))) \\ &= 1 \end{aligned}$$

from (S1), (S3), (sn1) and Lemma 1 (1), it follows from Lemma 2 and Lemma 3 that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \leq (n_1|n_1)|(n_2|n_2) = n_1 \vee n_2$. Thus, we have from Lemma 5 that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$, and so, I is a quasi-subalgebra of N .

However, the inverse of Lemma 6 is not true in general.

Example 6 Consider the s-Nelson algebra N in Example 3. Then $I = \{0, c, d, 1\}$ is a quasi-subalgebra of N but it is not an ideal of N since $a \notin I$ when $(a|(d|d))|(a|(d|d)) = 0 \in I$ and $d \in I$.

Theorem 4 *The family $\mathcal{I}_N$ of all ideals of a s-Nelson algebra N forms a complete distributive modular lattice.*

Proof Let $\{I_i : i \in \mathbb{N}\}$ be a family of ideals of N . Since $0 \in I_i$, for all $i \in \mathbb{N}$, it follows that $0 \in \bigcup_{i \in \mathbb{N}} I_i$ and $0 \in \bigcap_{i \in \mathbb{N}} I_i$. Assume that $n_2 \in \bigcap_{i \in \mathbb{N}} I_i$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in \bigcap_{i \in \mathbb{N}} I_i$. Since $n_2 \in I_i$ and $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I_i$, for all $i \in \mathbb{N}$, it is obtained that $n_1 \in I_i$, for all $i \in \mathbb{N}$, which implies that $n_1 \in \bigcap_{i \in \mathbb{N}} I_i$. Let X be the family of all ideals of N containing $\bigcup_{i \in \mathbb{N}} I_i$. Thus, $\bigcap X$ is also an ideal of N .

If $\bigwedge_{i \in \mathbb{N}} I_i = \bigcap_{i \in \mathbb{N}} I_i$ and $\bigvee_{i \in \mathbb{N}} I_i = \bigcap X$, then $(\mathcal{I}_N, \bigvee, \bigwedge)$ is a complete lattice. Moreover, $(\mathcal{I}_N, \bigvee, \bigwedge)$ is distributive from the definitions of $\bigvee$ and $\bigwedge$, and so, it is modular by the Theorem 1.

Corollary 1 *Let V be a subset of a s-Nelson algebra N . Then there exists the minimal ideal $\langle V \rangle$ of N containing the subset V .*

Proof Let $V = \{I : I \text{ is an ideal of } N \text{ containing } M \subseteq N\}$. Then $\bigcap V = \{n \in N : n \in \bigcap_{I \in V} I\}$ is the minimal ideal of N containing $M \subseteq N$.

Definition 7 Let I be an ideal of a s-Nelson algebra N . Then I is called an ultra ideal of N if $n_1 \wedge n_2 \in I$ implies $n_1 \in I$ or $n_2 \in I$, for all $n_1, n_2 \in N$.

Example 7 Consider the s-Nelson algebra N in Example 3. Then $\{0, a, b, d\}$ is an ultra ideal of N .

Remark 2 Every ultra ideal of N is an ideal of N by Definition 7, but the inverse does not usually hold.

Example 8 Consider the s-Nelson algebra N in Example 3. Then $I = \{0, b\}$ is an ideal of N but it is not ultra since $d, f \notin I$ when $d \wedge f = b \in I$.

Proposition 1 *Let I be an ideal of a s-Nelson algebra N . Then I is an ultra ideal of N if and only if $n \in I$ or $n|n \in I$, for all $n \in N$.*

Proof ($\Rightarrow$) Let I be an ultra ideal of N . Since $n \wedge (n|n) = (n|(n|n))|(n|(n|n)) = 1|1 = 0 \in I$ from Lemma 3 and (sn1), it is obtained that $n \in I$ or $n|n \in I$, for all $n \in N$.

($\Leftarrow$) Let I be an ideal of N such that $n \in I$ or $n|n \in I$, for all $n \in N$, and $n_1 \wedge n_2 \in I$. Assume that $n_1 \notin I$, for some $n_1 \in N$. Then $n_1|n_1 \in I$. Since $(n_2|((n_1|n_1)|(n_1|n_1)))|(n_2|((n_1|n_1)|(n_1|n_1))) = (n_1|n_2)|(n_1|n_2) = n_1 \wedge n_2 \in I$ from (S1), (S2) and Lemma 3, it follows from (NI2) that $n_2 \in I$. Suppose that $n_2 \notin I$, for some $n_2 \in N$. Then $n_2|n_2 \in I$. Similarly, it is obtained from (NI2) that $n_1 \in I$ by substituting $[n_1 := n_2]$ and $[n_2 := n_1]$ in above statement, simultaneously. Thus, I is an ultra ideal of N .

Proposition 2 *Let I be an ideal of a s-Nelson algebra N . Then I is an ultra ideal of N if and only if $n_1 \notin I$ and $n_2 \notin I$ imply $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

Proof Let I be an ultra ideal of N , and $n_1, n_2 \notin I$. By Proposition 1, $n_1|n_1 \in I$ and $n_2|n_2 \in I$. Since $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \leq n_2|n_2$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \leq n_1|n_1$ from Lemma 4 (1) and (3), it follows from (NI3) that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.

Conversely, let I be an ideal of N satisfying that $n_1 \notin I$ and $n_2 \notin I$ imply $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for any $n_1, n_2 \in N$. Suppose that $n \notin I$ and $n|n \notin I$, for some $n \in N$. Then we have from (S2) that $(n|((n|n)|(n|n)))|(n|((n|n)|(n|n))) = (n|n)|(n|n) = n \in I$ and $((n|n)|(n|n))|((n|n)|(n|n)) = n|n \in I$, which are contradictions. Thus, $n \in I$ or $n|n \in I$. Thereby, I is an ultra ideal of N from Proposition 1.

Proposition 3 *Let I be an ideal of a s -Nelson algebra N . Then I is an ultra ideal of N if and only if $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

Proof Let I be an ultra ideal of N . Since

$$\begin{aligned}
 & ((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \wedge ((n_2|(n_1|n_1))|(n_2|(n_1|n_1))) \\
 &= (((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|((n_2|(n_1|n_1))|(n_2|(n_1|n_1)))) \\
 & \quad |(((n_1|(n_2|n_2))|(n_1|(n_2|n_2)))|((n_2|(n_1|n_1))|(n_2|(n_1|n_1)))) \\
 &= ((((((n_1|(n_2|n_2))|(n_1|(n_2|n_2))|(n_1|n_1))|((n_1|(n_2|n_2))| \\
 & \quad (n_1|(n_2|n_2))|(n_1|n_1)))|n_2)|(((n_1|(n_2|n_2))|(n_1|(n_2|n_2))| \\
 & \quad n_2))|(n_1|n_1))|(((n_1|(n_2|n_2))|(n_1|(n_2|n_2))|(n_1|n_1))|n_2) \\
 &= (((n_2|n_2)|((n_1|(n_1|n_1))|(n_1|(n_1|n_1))))|((n_2|n_2)|((n_1| \\
 & \quad (n_1|n_1))|(n_1|(n_1|n_1))))|n_2)|(((n_2|n_2)|((n_1|(n_1|n_1))|(n_1| \\
 & \quad (n_1|n_1))))|((n_2|n_2)|((n_1|(n_1|n_1))|(n_1|(n_1|n_1))))|n_2) \\
 &= (((n_2|n_2)|(1|1))|((n_2|n_2)|(1|1))|n_2)|(((n_2|n_2)|(1|1))|((n_2|n_2)|(1|1))|n_2) \\
 &= 1|1 \\
 &= 0 \in I
 \end{aligned}$$

from Lemma 3, (S1), (S3), (sn1) and Lemma 1 (1), it follows from Definition 7 that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.

Conversely, let I be an ideal of N such that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for any $n_1, n_2 \in N$. Assume that $n_1 \wedge n_2 \in I$. Since

$$\begin{aligned}
 & (n_1|((n_1 \wedge n_2)|(n_1 \wedge n_2))|(n_1|((n_1 \wedge n_2)|(n_1 \wedge n_2)))) \\
 &= (n_1|(((n_1|n_2)|(n_1|n_2))|(n_1|n_2))|(n_1|n_2))) \\
 & \quad |(n_1|(((n_1|n_2)|(n_1|n_2))|(n_1|n_2))|(n_1|n_2))) \\
 &= (n_1|(n_1|n_2))|(n_1|(n_1|n_2)) \\
 &= (n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I
 \end{aligned}$$

or

$$\begin{aligned}
 & (n_2|((n_1 \wedge n_2)|(n_1 \wedge n_2))|(n_2|((n_1 \wedge n_2)|(n_1 \wedge n_2)))) \\
 &= (n_2|(((n_2 \wedge n_1)|(n_2 \wedge n_1))|(n_2|(n_2 \wedge n_1))|(n_2 \wedge n_1))) \\
 &= (n_2|(((n_2|n_1)|(n_2|n_1))|(n_2|n_1))|(n_2|n_1))) \\
 & \quad |(n_2|(((n_2|n_1)|(n_2|n_1))|(n_2|n_1))|(n_2|n_1))) \\
 &= (n_2|(n_2|n_1))|(n_2|(n_2|n_1)) \\
 &= (n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I
 \end{aligned}$$

from Lemma 3, (S2) and Lemma 1 (10), it follows from (NI2) that $n_1 \in I$ or $n_2 \in I$, for all $n_1, n_2 \in N$. Therefore, I is an ultra ideal of N .

5 Quotient s -Nelson algebra by ideals

In this section, a quotient s -Nelson algebra defined by ideals and some of properties are given.

Definition 8 Let R be an equivalence relation on N and $(n_1, n_2) \in R$ imply $(n_1|n_3, n_2|n_3) \in R$, for all $n_1, n_2, n_3 \in N$. Then R is called a congruence relation on N .

Example 9 Consider the s-Nelson algebra N in Example 1. Then $R = \{(0, 0), (w_1, w_1), (w_2, w_2), (1, 1), (0, w_1), (w_1, 0), (w_2, 1), (1, w_2)\}$ is a congruence relation on N .

Lemma 7 An equivalence relation R is a congruence relation on N if and only if $(n_1, m_1) \in R$ and $(n_2, m_2) \in R$ imply $(n_1|n_2, m_1|m_2) \in R$, for all $n_1, n_2, m_1, m_2 \in N$.

Proof Let R be a congruence relation on N and $(n_1, m_1) \in R$ and $(n_2, m_2) \in R$. Since $n_1|n_2\beta m_1|n_2$ and $m_1|n_2\beta m_1|m_2$ from (S1), it is obtained from the transitivity of R that $(n_1|n_2, m_1|m_2) \in R$, for all $n_1, n_2, m_1, m_2 \in N$.

Conversely, let R be an equivalence relation on N such that $(n_1, m_1) \in R$ and $(n_2, m_2) \in R$ imply $(n_1|n_2, m_1|m_2) \in R$, for any $n_1, n_2, m_1, m_2 \in N$. Suppose that n_1, n_2 and n_3 be arbitrary elements of N such that $(n_1, n_2) \in R$. Since $(n_3, n_3) \in R$, it follows that $(n_1|n_3, n_2|n_3) \in R$, for all $n_1, n_2, n_3 \in N$. Hence, R is a congruence relation on N .

Lemma 8 Let I be an ideal of a s-Nelson algebra N and a binary relation λ_I on N be defined by

$$(n_1, n_2) \in \lambda_I \Leftrightarrow (n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I \\ \text{and } (n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I,$$

for all $n_1, n_2 \in N$. Then λ_I is a congruence relation on N .

Proof Let I be an ideal of N .

- The reflexivity property is obtained from (sn1).
- The symmetry property is obvious from the definition of λ_I .
- Transitivity: let $(n_1, n_2) \in \lambda_I$ and $(n_2, n_3) \in \lambda_I$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)), (n_2|(n_1|n_1))|(n_2|(n_1|n_1)), (n_2|(n_3|n_3))|(n_2|(n_3|n_3)), (n_3|(n_2|n_2))|(n_3|(n_2|n_2)) \in I$. Since $(n_1|(n_2|n_2))|(n_2|(n_3|n_3)) = ((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \vee ((n_2|(n_3|n_3))|(n_2|(n_3|n_3))) \in I$ from Lemma 3, (S2) and (NI4), it follows from Lemma 4 (7) and (NI3) that $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \in I$. Similarly, it is obtained from Lemma 4 (7) and (NI3) that $(n_3|(n_1|n_1))|(n_3|(n_1|n_1)) \in I$ since $(n_3|(n_2|n_2))|(n_2|(n_1|n_1)) = ((n_3|(n_2|n_2))|(n_3|(n_2|n_2))) \vee ((n_2|(n_1|n_1))|(n_2|(n_1|n_1))) \in I$ from Lemma 3, (S2) and (NI4). So, $(n_1, n_3) \in \lambda_I$.

Hence, λ_I is an equivalence relation on N .

Let $(n_1, n_2) \in \lambda_I$ and $(m_1, m_2) \in \lambda_I$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)), (n_2|(n_1|n_1))|(n_2|(n_1|n_1)), (m_1|(m_2|m_2))|(m_1|(m_2|m_2)), (m_2|(m_1|m_1))|(m_2|(m_1|m_1)) \in I$.

1. Since

$$\begin{aligned} n_1|(n_2|n_2) &= (n_2|n_2)|((n_1|n_1)|(n_1|n_1)) \\ &\leq (m_1|((n_2|n_2)|(n_2|n_2))|((m_1|((n_1|n_1)| \\ &\quad |(n_1|n_1)))|(m_1|((n_1|n_1)|(n_1|n_1)))) \\ &= (m_1|n_2)|((m_1|n_1)|(m_1|n_1)) \end{aligned}$$

and

$$\begin{aligned} n_2|(n_1|n_1) &= (n_1|n_1)|((n_2|n_2)|(n_2|n_2)) \\ &\leq (m_1|((n_1|n_1)|(n_1|n_1))|((m_1|((n_2|n_2)| \\ &\quad |(n_2|n_2))|(m_1|((n_2|n_2)|(n_2|n_2)))) \\ &= (m_1|n_1)|((m_1|n_2)|(m_1|n_2)) \end{aligned}$$

from (S1), (S2) and Lemma 4 (9), it is obtained Lemma 4 (1) and (S1) that $((n_2|m_1)|((n_1|m_1)|(n_1|m_1))|((n_2|m_1)|((n_1|m_1)|(n_1|m_1))) \leq (n_1|(n_2|n_2))|(n_1|(n_2|n_2))$ and $((n_1|m_1)|((n_2|m_1)|(n_2|m_1))|((n_1|m_1)|((n_2|m_1)|(n_2|m_1))) \leq (n_2|(n_1|n_1))|(n_2|(n_1|n_1))$. Thus, $((n_2|m_1)|((n_1|m_1)|(n_1|m_1))|((n_2|m_1)|((n_1|m_1)|(n_1|m_1))) \in I$ and $((n_1|m_1)|((n_2|m_1)|(n_2|m_1))|((n_1|m_1)|((n_2|m_1)|(n_2|m_1))) \in I$ from (NI3), and so, $(n_1|m_1, n_2|m_1) \in \lambda_I$.

2. By substituting $[n_1 := m_1]$, $[n_2 := m_2]$ and $[m_1 := n_2]$ in (1), it follows from (S1) that $(n_2|m_1, n_2|m_2) \in \lambda_I$.

Hence, $(n_1|m_1, n_2|m_2) \in \lambda_I$ from the transitivity of λ_I .

Therefore, λ_I is a congruence relation on N .

Theorem 5 *Let I be an ideal of a s-Nelson algebra N and λ_I be a congruence relation on N defined by I . Then $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is a s-Nelson algebra where the quotient set $N/\lambda_I = \{[n]_{\lambda_I} : n \in N\}$, the binary operation $|_{\lambda_I}$ is defined by $[n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I} = [n_1|n_2]_{\lambda_I}$ and $[1]_{\lambda_I}|_{\lambda_I}[1]_{\lambda_I} = [0]_{\lambda_I} = I$.*

Proof Let I be an ideal of a s-Nelson algebra N and λ_I be a congruence relation on N defined by I .

$$(S1): [n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I} = [n_1|n_2]_{\lambda_I} = [n_2|n_1]_{\lambda_I} = [n_2]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I}$$

$$(S2): ([n_1]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I})|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}) = [(n_1|n_1)|(n_1|n_2)]_{\lambda_I} = [n_1]_{\lambda_I},$$

(S3):

$$\begin{aligned} &[n_1]_{\lambda_I}|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I})|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I}) \\ &= [n_1|((n_2|n_3)|(n_2|n_3))]_{\lambda_I} \\ &= [((n_1|n_2)|(n_1|n_2))|n_3]_{\lambda_I} \\ &= (([n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I})|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}))|_{\lambda_I}[n_3]_{\lambda_I}, \end{aligned}$$

$$\begin{aligned} (S4): &([n_1]_{\lambda_I}|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I})|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}))|_{\lambda_I} \\ &([n_1]_{\lambda_I}|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}[n_1]_{\lambda_I})|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I})) \\ &= [(n_1|((n_1|n_1)|(n_2|n_2))|(n_1|((n_1|n_1)|(n_2|n_2))))]_{\lambda_I} \\ &= [n_1]_{\lambda_I}, \end{aligned}$$

$$(sn1): [n]_{\lambda_I}|_{\lambda_I}([n]_{\lambda_I}|_{\lambda_I}[n]_{\lambda_I}) = [n|(n|n)]_{\lambda_I} = [1]_{\lambda_I},$$

$$\begin{aligned} (sn2): &[n_1]_{\lambda_I}|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I}) \\ &= [n_1|(n_2|n_3)]_{\lambda_I} \\ &= [((n_1|(n_2|n_2))|(n_1|(n_3|n_3))|((n_1|(n_2|n_2))|(n_1|(n_3|n_3))))]_{\lambda_I} \\ &= (([n_1]_{\lambda_I}|_{\lambda_I}([n_2]_{\lambda_I}|_{\lambda_I}[n_2]_{\lambda_I}))|_{\lambda_I}([n_1]_{\lambda_I}|_{\lambda_I}([n_3]_{\lambda_I}|_{\lambda_I}[n_3]_{\lambda_I})))|_{\lambda_I} \end{aligned}$$

for all $n_1, n_2, n_3 \in N$.

Thereby, $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is a s-Nelson algebra.

Moreover, $[1]_{\lambda_I} |_{\lambda_I} [1]_{\lambda_I} = [1]_{\lambda_I} = [0]_{\lambda_I}$. Assume that $n \in [0]_{\lambda_I}$. Since $(n, 0) \in \lambda_I$, we have from Lemma 1 (1), (3) and (S1) that $n = (n|1)|(n|1) = (n|(0|0))|(n|(0|0)) \in I$ and $0 = 1|1 = ((n|n)|(1|1))|((n|n)|(1|1)) = (0|(n|n))|(0|(n|n)) \in I$. So, $[0]_{\lambda_I} \subseteq I$. Suppose that $n \in I$. Since $(n|(0|0))|(n|(0|0)) = (n|1)|(n|1) = n \in I$ and $(0|(n|n))|(0|(n|n)) = ((n|n)|(1|1))|((n|n)|(1|1)) = 1|1 = 0 \in I$ from Lemma 1 (1), (3) and (S1), it follows that $(n, 0) \in \lambda_I$, and so, $I \subseteq [0]_{\lambda_I}$. Thus, $[0]_{\lambda_I} = I$.

Example 10 Consider the s-Nelson algebra N in Example 3. For an ideal $I = \{0, c\}$ of N , a relation $\lambda_I = \{(0, 0), (a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (1, 1), (0, c), (c, 0), (d, 1), (1, d), (a, e), (e, a), (b, f), (f, b)\}$ is a congruence relation on N . Then $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is a s-Nelson algebra where $N/\lambda_I = \{[0]_{\lambda_I}, [b]_{\lambda_I}, [e]_{\lambda_I}, [1]_{\lambda_I}\}$ and the Sheffer operation $|_{\lambda_I}$ on N/λ_I have Cayley table as follow:

Table 7

$ _{\lambda_I}$	$[0]_{\lambda_I}$	$[b]_{\lambda_I}$	$[e]_{\lambda_I}$	$[1]_{\lambda_I}$
$[0]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$
$[b]_{\lambda_I}$	$[1]_{\lambda_I}$	$[e]_{\lambda_I}$	$[1]_{\lambda_I}$	$[e]_{\lambda_I}$
$[e]_{\lambda_I}$	$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[b]_{\lambda_I}$	$[b]_{\lambda_I}$
$[1]_{\lambda_I}$	$[1]_{\lambda_I}$	$[e]_{\lambda_I}$	$[b]_{\lambda_I}$	$[0]_{\lambda_I}$

Lemma 9 Let I be an ideal of a s-Nelson algebra N and λ_I be a congruence relation on N defined by I . Then a relation $\leq$ on N/λ_I defined by

$$[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I} \Leftrightarrow (n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$$

is a partial order on N/λ_I .

Proof Let I be an ideal of a s-Nelson algebra N and λ_I be a congruence relation on N defined by I . By Theorem 5, $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is a s-Nelson algebra.

• Since $(n|(n|n))|(n|(n|n)) = 1|1 = 0$ from (sn1), it follows that $[n]_{\lambda_I} \leq [n]_{\lambda_I}$.

• Let $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ and $[n_2]_{\lambda_I} \leq [n_1]_{\lambda_I}$. Since $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, it is obtained that $(n_1, n_2) \in \lambda_I$, which means that $[n_1]_{\lambda_I} = [n_2]_{\lambda_I}$.

• Let $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ and $[n_2]_{\lambda_I} \leq [n_3]_{\lambda_I}$. Then $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_3|n_3))|(n_2|(n_3|n_3)) \in I$. Since $(n_1|(n_2|n_2))|(n_2|(n_3|n_3)) = ((n_1|(n_2|n_2))|(n_1|(n_2|n_2))) \vee ((n_2|(n_3|n_3))|(n_2|(n_3|n_3))) \in I$ and $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \leq (n_1|(n_2|n_2))|(n_2|(n_3|n_3))$ from (S2), Lemma 3 and Lemma 4 (7), it is obtained from (NI3) that $(n_1|(n_3|n_3))|(n_1|(n_3|n_3)) \in I$. Thus, $[n_1]_{\lambda_I} \leq [n_3]_{\lambda_I}$.

Hence, a relation $\leq$ on N/λ_I is a partial order on N/λ_I .

Theorem 6 Let I be an ideal of a s-Nelson algebra N . Then I is an ultra ideal of N if and only if N/λ_I is a totally ordered s-Nelson algebra and $|N/\lambda_I| \leq 2$.

Proof ($\Rightarrow$) Let I be an ultra ideal of a s-Nelson algebra N . By Theorem 5, $(N/\lambda_I, |_{\lambda_I}, [1]_{\lambda_I})$ is a s-Nelson algebra. Since $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$ from Proposition 3, it follows that $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ and $[n_2]_{\lambda_I} \leq [n_1]_{\lambda_I}$, for all $n_1, n_2 \in N$. So, N/λ_I is a totally ordered s-Nelson algebra. Suppose that $|N/\lambda_I| > 2$. Let $[n]_{\lambda_I} \in N/\lambda_I$ such that $[0]_{\lambda_I} < [n]_{\lambda_I} < [1]_{\lambda_I}$. Since I is an ultra ideal of N , we get from Proposition 1 that $n \in I$ or $n|n \in I$. Assume that $n|n \in I$. Since $(1|(n|n))|(1|(n|n)) = n|n \in I$ and $(n|(1|1))|(n|(1|1)) = 1|1 = 0 \in I$ from Lemma 1 (1)-(2) and (NI1), it is obtained that $(n, 1) \in \lambda_I$. Thus, $[n]_{\lambda_I} = [1]_{\lambda_I}$ which is a contradiction. Hence, $|N/\lambda_I| \leq 2$.

($\Leftarrow$) Let I be an ideal of N and N/λ_I be a totally ordered s-Nelson algebra. Since $[n_1]_{\lambda_I} \leq [n_2]_{\lambda_I}$ or $[n_2]_{\lambda_I} \leq [n_1]_{\lambda_I}$, we get that $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$. Therefore, I is an ultra ideal of N by Proposition 3.

Corollary 2 *Let I be an ideal of a s-Nelson algebra N . Then N/λ_I is a totally ordered s-Nelson algebra if and only if $n \in I$ or $n|n \in I$, for all $n \in N$.*

Corollary 3 *Let I be an ideal of a s-Nelson algebra N . Then N/λ_I is a totally ordered s-Nelson algebra if and only if $n_1 \notin I$ and $n_2 \notin I$ imply $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ and $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

Corollary 4 *Let I be an ideal of a s-Nelson algebra N . Then N/λ_I is a totally ordered s-Nelson algebra if and only if $(n_1|(n_2|n_2))|(n_1|(n_2|n_2)) \in I$ or $(n_2|(n_1|n_1))|(n_2|(n_1|n_1)) \in I$, for all $n_1, n_2 \in N$.*

6 Conclusion

In the study, we have given a Sheffer stroke Nelson algebra, some types of ideals, a quasi-subalgebra and a quotient set of this algebraic structure. We introduce a Sheffer stroke Nelson algebra (for short, s-Nelson algebra) and show that the axioms are independent. Then a partial order on a s-Nelson algebra is defined and some properties are examined. We indicate that a s-Nelson algebra states a bounded distributive modular lattice. Also, it is proved that every s-Nelson algebra is a Nelson algebra and a Nelson algebra with zero is a s-Nelson algebra under the special conditions. Indeed, we describe an (ultra) ideal of a s-Nelson algebra and present some properties. It is demonstrated that every ideal of s-Nelson algebra is its quasi-subalgebra but the inverse is generally not true. In fact, it is indicated that a family of all ideals of a s-Nelson algebra forms a complete distributive modular lattice, and so, a minimal ideal of a s-Nelson algebra containing its subset is determined. Besides, it is illustrated that every ultra ideal of a s-Nelson algebra is the ideal but the inverse does not mostly hold. It is given the statemants which are equivalent to the definition of an (ultra) ideal of a s-Nelson algebra. Moreover, a congruence relation on a s-Nelson algebra is defined by means of its ideal. Thus, a quotient

s-Nelson algebra is described via this congruence relation and a partial order on the quotient structure is determined. Finally, it is proved that a quotient s-Nelson algebra is totally ordered if and only if an ideal of a s-Nelson algebra is ultra, and some results are presented.

In future works, we wish to investigate fuzzy and neutrosophic structures on s-Nelson algebras.

Acknowledgements The authors are thankful to the referees for a careful reading of the paper and for valuable comments and suggestions. The authors have made equally contributions to the study.

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Ethical approval

This article does not contain any studies with animals performed by any of the authors.

Informed consent

Informed consent was obtained from all individual participants included in the study.

Funding

This study is partially funded by Ege University Scientific Research Projects Directorate with the project number 22747.

References

1. J. C. Abbott, Implicational algebras, *Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie*, **11**(1), 3-23 (1967).
2. I. Chajda, Sheffer operation in ortholattices, *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, **44**(1), 19-23 (2005).
3. I. Chajda, R. Halaš and H. Länger, Operations and structures derived from non-associative MV-algebras, *Soft Computing*, **23**(12), 3935-3944 (2019).
4. W. B. Ewald, Intuitionistic tense and modal logic, *J. Symbolic Logic*, 51(1), 166-179 (1986).
5. A. V. Figallo and G. Pelaitay, An algebraic axiomatization of the Ewald's intuitionistic tense logic, *Soft Computing*, **18**(10), 1873-1883 (2014).
6. A. V. Figallo, G. Pelaitay and J. Sarmiento, An algebraic study of tense operators on Nelson algebras, *Stud Logica*, (2020). <https://doi.org/10.1007/s11225-020-09907-0>

7. T. Katican, T. Oner and A. Borumand Saeid, On Sheffer stroke BE-algebras, *Discussione Mathematicae General Algebra and Applications*, (in press).
8. T. Katican, T. Oner, A. Rezaei and F. Smarandache, Neutrosophic N-structures applied to Sheffer stroke BL-algebras, *Computer Modeling in Engineering and Sciences*, (in press).
9. A. A. Markov, A constructive logic, *Uspehi Matematicheskikh Nauk (N.S.)*, **5**, 187-188 (1950).
10. W. McCune, R. Veroff, B. Fitelson, K. Harris, A. Feist and L. Vos, Short single axioms for Boolean algebra, *Journal of Automated Reasoning*, **29**(1), 1-16 (2002).
11. D. Nelson, Constructible falsity, *J. Symbolic Logic*, **14**, 16-26 (1949).
12. T. Oner, T. Katican, A. Borumand Saeid and M. Terziler, Filters of strong Sheffer stroke non-associative MV-algebras, *Analele Stiintifice ale Universitatii Ovidius Constanta*, **29**(1), 143-164 (2021).
13. T. Oner, T. Katican and A. Borumand Saeid, Relation between Sheffer Stroke and Hilbert Algebras, *Categories and General Algebraic Structures with Applications*, **14**(1), 245-268 (2021).
14. T. Oner, T. Katican and A. Borumand Saeid, Fuzzy Filters of Sheffer stroke Hilbert algebras, *Journal of Intelligent and Fuzzy Systems*, **40**(1), 759-772 (2021).
15. T. Oner, T. Katican and A. Borumand Saeid, (Fuzzy) filters of Sheffer stroke BL-algebras, *Kragujevac Journal of Mathematics*, **47**(1), 39-55 (2023).
16. T. Oner, T. Katican and A. Borumand Saeid, On Sheffer stroke UP-algebras, *Discussiones Mathematicae General Algebra and Applications*, (in press).
17. T. Oner, T. Katican and A. Rezaei, Neutrosophic N-structures on strong Sheffer stroke non-associative MV-algebras, *Neutrosophic Sets and Systems*, **40**, 235-252 (2021).
18. T. Oner, T. Katican and A. Borumand Saeid, Neutrosophic N-structures on Sheffer stroke Hilbert algebras, *Neutrosophic Sets and Systems*, **42**, 221-238 (2021).
19. T. Oner, T. Kalkan and N. Kircali Gursoy, Sheffer stroke BG-algebras, *International Journal of Maps in Mathematics*, **4**(1), 27-30 (2021).
20. T. Oner, T. Kalkan, T. Katican and A. Rezaei, Fuzzy implicative ideals of Sheffer stroke BG-algebras, *Facta Universitatis Series Mathematics and Informatics*, (in press).
21. T. Oner, T. Kalkan and A. Borumand Saeid, Class of Sheffer stroke BCK-algebras, *Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica*, (in press).
22. H. Rasiowa, N-lattices and constructive logic with strong negation, *Fund. Math.*, **46**, 61-80 (1958).
23. H. Rasiowa, *An algebraic aproach to non-classic logic*, North-Holland, Amsterdam (1974).
24. H. P. Sankappanavar and S. Burris, *A course in universal algebra*, Graduate Texts Math, Vol. 78, (1981).
25. H. M. Sheffer, A set of five independent postulates for Boolean algebras, with application to logical constants, *Transactions of the American Mathematical Society*, **14**(4), 481-488 (1913).