

Trapezoidal Fully Fuzzy Sylvester Matrix Equation with Arbitrary Coefficients

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Abstract

Almost every existing method for solving trapezoidal fully fuzzy Sylvester matrix equation restricts the coefficient matrix and the solution to be positive fuzzy numbers only. In this paper, we develop new analytical methods to solve a trapezoidal fully fuzzy Sylvester matrix equation with restricted and unrestricted coefficients. The trapezoidal fully fuzzy Sylvester matrix equation is transferred to a system of crisp equations based on the sign of the coefficients by using Ahmd arithmetic multiplication operations between trapezoidal fuzzy numbers. The constructed method not only obtain a simple crisp system of linear equation that can be solved by any classical methods but also provide a widen the scope of the trapezoidal fully fuzzy Sylvester matrix equation in scientific applications. Furthermore, these methods have less steps and conceptually easy to understand when compared with existing methods. To illustrate the proposed methods numerical examples are solved.

Keywords Arithmetic fuzzy multiplication operations · Bartels Stewart · Sylvester matrix equations · Trapezoidal fuzzy numbers.

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1 Introduction

Sylvester matrix equations played a prominent role in various areas including control theory (Datta, 2004), medical imaging acquisition systems, model reduction of linear time invariant systems (Antoulas, 2005; Sorensen & Antoulas, 2002) and stochastic control, signal processing, image processing and filtering (Bouhamidi & Jbilou, 2007). When information is imprecise and only some vague knowledge about the actual values of the parameters is available, it is convenient to use fuzzy numbers (D. J. Dubois, 1980; Zadeh, 1965). Considering to any uncertainty problems that can be occurred at any time, the crisp Sylvester matrix equation has to be adapted to the fuzzy environment.

Triangular fully fuzzy Sylvester matrix equation (TFFSME) has been studied in (Dubois & Prade, 1978; Malkawi et al., 2015; Shang et al., 2015) where, the TFFSME is converted into a system of crisp linear equations by applying Dubois and Prade's arithmetic operator for multiplication. Then Kronecker product and the matrix inversion method has been applied to obtain the positive fuzzy solution. Nevertheless, these methods are restricted to non-singular positive TFFSME only.

To overcome the shortcoming in these methods, Daud, Ahmad and Malkawi (2018c) obtained positive solution for singular TFFSME where the solution is obtained by using the method of pseudoinverse. Moreover, Daud, et al., (2018b) proposed another algorithm for solving TFFSME with arbitrary coefficients which utilized the Kaufmann and Gupta's arithmetic multiplication operator (Kauffman & Gupta, 1991). However, they were able to find positive fuzzy solution only. Recently, in (Elsayed et al., 2020), the authors considered solution of trapezoidal fully fuzzy Sylvester matrix equation (TrFFSME) by transforming TrFFSME to a system of four equations where the positive and negative fuzzy solutions are obtained by applying Kronecker product and Vec-operator method.

A study was conducted by Dookhitram et al., (2015) on the TFFSME in the form $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$, which used the α -cuts expansion approach in the parameters. The method proposed has an advantage in the sense that it provides maximal and minimal symmetric solutions of the TFFSME, however, the method required long fuzzy operations process in obtaining the solution. Similarly, Daud et al., (2018a) proposed an algorithm for solving TFFSME with arbitrary coefficients. However, the method was restricted only for positive fuzzy solutions.

Most of the analytical methods proposed for solving TFFSME in the literature are based on Dubois and Prade's arithmetic operator for multiplication which is restricted only for positive fuzzy numbers with very small fuzziness (Fortin et al., 2008) and therefore, most of the analytical methods are restricted to positive coefficients and positive fuzzy solution only. In addition, many researchers have applied Kaufmann and Gupta's arithmetic multiplication operator for solving TFFSME with arbitrary coefficients however, their methods are limited to positive fuzzy solution only for systems with triangular fuzzy numbers (TFNs) only.

Sign and size of fuzzy systems are restricted in most of the existing studies due to the nature of analytical methods used and arithmetic operations applied. Where, many methods are limited only for small sized systems with positive fuzzy numbers. Therefore, in order to deal with this shortcoming, in this paper we develop new methods for solving TrFFSME with arbitrary coefficients by using Ahmd arithmetic multiplication operations between trapezoidal fuzzy numbers (TrFNs). The main intent of the proposed methods is to avoid the complexity procedure in the previous methods by introducing new systems of linear equations equivalent to the fuzzy systems which can be solved in one step by MATLAB or Mathematica. The proposed method is able to solve large size systems. In addition, it can also be applied to TFFSME and fully fuzzy matrix equation (FFME) with both (TFNs) and (TrFNs).

This paper is organized as follows, In section 2, the preliminary concepts and arithmetic operations of trapezoidal fuzzy numbers are discussed. In section 3, new methods for solving TrFFSME are developed. In section 4, numerical examples are solved. In section 5, conclusion about the proposed methods and achieved results will be drawn.

2 Preliminaries

The following are basic definitions and results related to TrFNs (Dubois & Prade, 1978; Kaufman & Gupta, 1991; Lee, 2004).

Definition 2.1 Let X be a universal set. Then, we define the fuzzy subset \tilde{A} of X by its membership function $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval

$[0, 1]$, where the function value of $\mu_{\tilde{A}}(x)$ represents the grade of membership of x in \tilde{A} . A fuzzy set \tilde{A} is written as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X, \mu_{\tilde{A}}(x) \in [0, 1]\}$.

Definition 2.2 A fuzzy set \tilde{A} , defined on the universal set of real number R ; is said to be a fuzzy number if its membership function has the following characteristics:

- (I) \tilde{A} is convex, i.e.,

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \forall x_1, x_2 \in R, \forall \lambda \in [0, 1].$$
- (II) \tilde{A} is normal, i.e.,

$$\exists x_0 \in R \text{ such that } \mu_{\tilde{A}}(x_0) = 1.$$
- (III) $\mu_{\tilde{A}}$ is piecewise continuous.

Definition 2.3 A fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$. is a TrFN if its membership function is:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < a_1 \\ \frac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \frac{a_4 - x}{a_4 - a_3} & a_3 \leq x \leq a_4 \\ 0 & x > a_4 \end{cases}$$

Figure 2.1. Representation of TrFN in the form (a_1, a_2, a_3, a_4) .

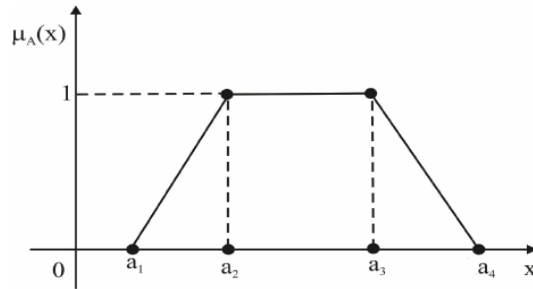


Figure 2.4. Representation of TrFN (a_1, a_2, a_3, a_4) .

Definition 2.4 The sign of the TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$. can be classified as:

- \tilde{A} is positive (negative) iff $a_1 \geq 0, (a_4 \leq 0)$.
- \tilde{A} is zero iff $(a_1, a_2, a_3 \text{ and } a_4 = 0)$.
- \tilde{A} is near zero iff $a_1 \leq 0 \leq a_4$.

Definition 2.5 Operation of TrFNs.

The arithmetic operations of TrFNs is presented as follows,

let $\tilde{A} = (a_1, a_2, a_3, a_4), \tilde{B} = (b_1, b_2, b_3, b_4)$ be two TrFNs then:

- (I) Addition

$$\tilde{A} + \tilde{B} = (a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4). \quad (2.1)$$
- (II) Subtraction

$$\tilde{A} - \tilde{B} = (a_1, a_2, a_3, a_4) - (b_1, b_2, b_3, b_4) = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1).$$
- (III) Symmetric image

$$-\tilde{A} = (-a_4, -a_3, -a_2, -a_1).$$
- (IV) Scalar multiplication: Let $\lambda \in \mathbb{R}$ then,

$$\lambda \otimes (a_1, a_2, a_3, a_4) = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4) & \lambda \geq 0 \\ (\lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1) & \lambda < 0 \end{cases}$$
- (V) Multiplication: Ahmd arithmetic multiplication operators:

Case I) If $\tilde{A} = (a_1, a_2, a_3, a_4), \tilde{B} = (b_1, b_2, b_3, b_4)$ be two arbitrary TrFNs then:

$$\tilde{A}\tilde{B} = (a, h, m, d). \quad (2.2)$$

where,

$$a = \min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4),$$

$$\begin{aligned} h &= \min(a_2b_2, a_2b_3, a_3b_2, a_3b_3), \\ m &= \max(a_2b_2, a_2b_3, a_3b_2, a_3b_3) \end{aligned}$$

and

$$d = \max(a_1b_1, a_1b_4, a_4b_1, a_4b_4).$$

Case II) If $\tilde{A} = (a_1, a_2, a_3, a_4) > 0$ and $\tilde{B} = (b_1, b_2, b_3, b_4) > 0$ then:

$$\tilde{A}\tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4). \quad (2.3)$$

Case III) If $\tilde{A} = (a_1, a_2, a_3, a_4) < 0$ and $\tilde{B} = (b_1, b_2, b_3, b_4) < 0$ then:

$$\tilde{A}\tilde{B} = (a_4b_4, a_3b_3, a_2b_2, a_1b_1). \quad (2.4)$$

Case IV) If $\tilde{A} = (a_1, a_2, a_3, a_4) > 0$ and $\tilde{B} = (b_1, b_2, b_3, b_4) < 0$ then:

$$\tilde{A}\tilde{B} = (a_4b_1, a_3b_2, a_2b_3, a_1b_4). \quad (2.5)$$

Case V) If $\tilde{A} = (a_1, a_2, a_3, a_4) < 0$ and $\tilde{B} = (b_1, b_2, b_3, b_4) > 0$ then:

$$\tilde{A}\tilde{B} = (a_1b_4, a_2b_3, a_3b_2, a_4b_1). \quad (2.6)$$

(VI) Equality: The fuzzy numbers $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are equal iff
 $a_1 = b_1, a_2 = b_2, a_3 = b_3$ and $a_4 = b_4$ (2.7)

Definition 2.6

A matrix $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ is called a trapezoidal fuzzy matrix, if each element of \tilde{A} is a TrFN.

Definition 2.7 A fuzzy matrix \tilde{A} will be:

- I) Positive (negative) and denoted by $\tilde{A} > 0$, ($\tilde{A} < 0$) if each element of \tilde{A} is positive (negative) TrFN.
- II) Non-negative (non-positive) and denoted by $\tilde{A} \geq 0$, ($\tilde{A} \leq 0$) if each element of \tilde{A} is non-negative (non-positive) TrFNs.
- III) Arbitrary, if at least one element of \tilde{A} is near zero TrFNs.

Definition 2.8 The fully fuzzy matrix equation that can be written as

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C} \quad (2.8)$$

where, $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$, $\tilde{B} = (\tilde{b}_{ij})_{m \times m}$, $\tilde{C} = (\tilde{c}_{ij})_{n \times m}$ and $\tilde{X} = (\tilde{x}_{ij})_{n \times m}$, $\forall 1 \leq i, j \leq n, m$, is called fully fuzzy Sylvester matrix equation (FFSME), and it can also be written as:

$$\sum_{\substack{i,j=1 \\ k,l=1,\dots,4}}^n a_{ij}^{(k)} x_{ij}^{(l)} + \sum_{\substack{i,j=1 \\ k,l=1,\dots,4}}^m x_{ij}^{(k)} b_{ij}^{(l)} = c_{ij}^{(k)} \quad (2.9)$$

Definition 2.9 (Paige & Van Loan, 1981). The Schur factorization of a matrix A is the factorization:

$$A = QRQ^T.$$

where,

- R is upper triangular matrix which is called a Schur form of A .
- Q is a unitary matrix ($QQ^T = I$).

Definition 2.10 The Kronecker sum of two matrices \oplus can be considered as a matrix sum defined by

$$A \oplus B = A \otimes I_b + I_a \otimes B$$

where A is a square matrix of order a and B is a square matrices of order b , and I_a, I_b are identity matrices of order a and b respectively and \otimes represents the Kronecker product.

Definition 2.11 The Vec-operator generates a column vector from a matrix A by stacking the column

$$\text{vectors of } A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \text{ as, } \text{Vec}(A) = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

3 Proposed method

In this section the TrFFSME in Eq. (2.8) is converted to an equivalent systems of crisp Sylvester matrix equations based on the sign of fuzzy numbers used by applying Ahmd arithmetic multiplication operators in Eq.(2.2) to Eq. (2.6). Where, the complexity procedure in the literature of TrFFSME can be avoided and the fuzzy solution can be obtained in one step by MATLAB R2020a or Mathematica12. The proposed methods are discussed in the following section.

3.1 TrFFSME with arbitrary coefficients.

In the following theorem we obtain a system of nonlinear equations equivalent to the TrFFSME in Eq. (2.8) where the coefficients are arbitrary, and the fuzzy solution is positive.

Theorem 3.1 If $\tilde{A} = (\tilde{a}_{ij})_{n \times n}, \tilde{B} = (\tilde{b}_{ij})_{m \times m}$ are arbitrary and $\tilde{X} = (\tilde{x}_{ij})_{n \times m} > 0, \forall 1 \leq i, j \leq n, m$. Then the TrFFSME in Eq.(2.8) can be written as follows:

$$\begin{cases} \min(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(1)} x_{ij}^{(4)}) + \min(x_{ij}^{(4)} b_{ij}^{(1)}, x_{ij}^{(1)} b_{ij}^{(1)}) = c_{ij}^{(1)} \\ \min(a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(2)} x_{ij}^{(3)}) + \min(x_{ij}^{(2)} b_{ij}^{(2)}, x_{ij}^{(3)} b_{ij}^{(2)}) = c_{ij}^{(2)} \\ \max(a_{ij}^{(3)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}) + \max(x_{ij}^{(2)} b_{ij}^{(3)}, x_{ij}^{(3)} b_{ij}^{(3)}) = c_{ij}^{(3)} \\ \max(a_{ij}^{(4)} x_{ij}^{(1)}, a_{ij}^{(4)} x_{ij}^{(4)}) + \max(x_{ij}^{(1)} b_{ij}^{(4)}, x_{ij}^{(4)} b_{ij}^{(4)}) = c_{ij}^{(4)} \end{cases} \quad (3.1)$$

Proof:

Assuming $\tilde{A} = (\tilde{a}_{ij})_{n \times n} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)})$, $\tilde{B} = (\tilde{b}_{ij})_{m \times m} = (b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}, b_{ij}^{(4)})$ are arbitrary and $\tilde{X} = (\tilde{x}_{ij})_{n \times m} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$ is positive, $\forall 1 \leq i, j \leq n, m$. Then, from

Definition 2.5 and by Eq. (2.2),

$$\tilde{A}\tilde{X} =$$

$$\left(\min(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(1)} x_{ij}^{(4)}), \min(a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(2)} x_{ij}^{(3)}), \max(a_{ij}^{(3)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}), \max(a_{ij}^{(4)} x_{ij}^{(1)}, a_{ij}^{(4)} x_{ij}^{(4)}) \right)$$

and,

$$\tilde{X}\tilde{B} =$$

$$\left(\min(x_{ij}^{(4)} b_{ij}^{(1)}, x_{ij}^{(1)} b_{ij}^{(1)}), \min(x_{ij}^{(2)} b_{ij}^{(2)}, x_{ij}^{(3)} b_{ij}^{(2)}), \max(x_{ij}^{(2)} b_{ij}^{(3)}, x_{ij}^{(3)} b_{ij}^{(3)}), \max(x_{ij}^{(1)} b_{ij}^{(4)}, x_{ij}^{(4)} b_{ij}^{(4)}) \right)$$

Adding $\tilde{A}\tilde{X}$ and $\tilde{X}\tilde{B}$ using Eq. (2.1) in Definition 2.5 we get:

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B}$$

$$\begin{aligned} &= \left(\min(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(1)} x_{ij}^{(4)}), \min(a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(2)} x_{ij}^{(3)}), \max(a_{ij}^{(3)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}), \max(a_{ij}^{(4)} x_{ij}^{(1)}, a_{ij}^{(4)} x_{ij}^{(4)}) \right) \\ &+ \left(\min(x_{ij}^{(4)} b_{ij}^{(1)}, x_{ij}^{(1)} b_{ij}^{(1)}), \min(x_{ij}^{(2)} b_{ij}^{(2)}, x_{ij}^{(3)} b_{ij}^{(2)}), \max(x_{ij}^{(2)} b_{ij}^{(3)}, x_{ij}^{(3)} b_{ij}^{(3)}), \max(x_{ij}^{(1)} b_{ij}^{(4)}, x_{ij}^{(4)} b_{ij}^{(4)}) \right) \end{aligned}$$

By Eq.(2.9) in Definition 2.8, the TrFFSME in Eq. (2.8) can be written as:

$$\begin{cases} \min(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(1)} x_{ij}^{(4)}) + \min(x_{ij}^{(4)} b_{ij}^{(1)}, x_{ij}^{(1)} b_{ij}^{(1)}) = c_{ij}^{(1)} \\ \min(a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(2)} x_{ij}^{(3)}) + \min(x_{ij}^{(2)} b_{ij}^{(2)}, x_{ij}^{(3)} b_{ij}^{(2)}) = c_{ij}^{(2)} \\ \max(a_{ij}^{(3)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}) + \max(x_{ij}^{(2)} b_{ij}^{(3)}, x_{ij}^{(3)} b_{ij}^{(3)}) = c_{ij}^{(3)} \\ \max(a_{ij}^{(4)} x_{ij}^{(1)}, a_{ij}^{(4)} x_{ij}^{(4)}) + \max(x_{ij}^{(1)} b_{ij}^{(4)}, x_{ij}^{(4)} b_{ij}^{(4)}) = c_{ij}^{(4)} \end{cases}$$

Remark 3.1

The left-hand side of the non-linear equations obtained in Eq. (3.1) can be reduced to linear system by applying the following:

$$\begin{aligned} \min(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(1)} x_{ij}^{(4)}) &= \begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} & \text{if } a_{ij}^{(1)} \geq 0, \\ a_{ij}^{(1)} x_{ij}^{(4)} & \text{if } a_{ij}^{(1)} \leq 0. \end{cases} \\ \min(x_{ij}^{(4)} b_{ij}^{(1)}, x_{ij}^{(1)} b_{ij}^{(1)}) &= \begin{cases} x_{ij}^{(1)} b_{ij}^{(1)} & \text{if } b_{ij}^{(1)} \geq 0, \\ x_{ij}^{(4)} b_{ij}^{(1)} & \text{if } b_{ij}^{(1)} \leq 0. \end{cases} \\ \min(a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(2)} x_{ij}^{(3)}) &= \begin{cases} a_{ij}^{(2)} x_{ij}^{(2)} & \text{if } a_{ij}^{(2)} \geq 0, \\ a_{ij}^{(2)} x_{ij}^{(3)} & \text{if } a_{ij}^{(2)} \leq 0. \end{cases} \\ \min(x_{ij}^{(2)} b_{ij}^{(2)}, x_{ij}^{(3)} b_{ij}^{(2)}) &= \begin{cases} x_{ij}^{(2)} b_{ij}^{(2)} & \text{if } b_{ij}^{(2)} \geq 0, \\ x_{ij}^{(3)} b_{ij}^{(2)} & \text{if } b_{ij}^{(2)} \leq 0. \end{cases} \\ \max(a_{ij}^{(3)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}) &= \begin{cases} a_{ij}^{(3)} x_{ij}^{(3)} & \text{if } a_{ij}^{(3)} \geq 0, \\ a_{ij}^{(3)} x_{ij}^{(2)} & \text{if } a_{ij}^{(3)} \leq 0. \end{cases} \\ \max(x_{ij}^{(2)} b_{ij}^{(3)}, x_{ij}^{(3)} b_{ij}^{(3)}) &= \begin{cases} x_{ij}^{(3)} b_{ij}^{(3)} & \text{if } b_{ij}^{(3)} \geq 0, \\ x_{ij}^{(2)} b_{ij}^{(3)} & \text{if } b_{ij}^{(3)} \leq 0. \end{cases} \\ \max(a_{ij}^{(4)} x_{ij}^{(1)}, a_{ij}^{(4)} x_{ij}^{(4)}) &= \begin{cases} a_{ij}^{(4)} x_{ij}^{(4)} & \text{if } a_{ij}^{(4)} \geq 0, \\ a_{ij}^{(4)} x_{ij}^{(1)} & \text{if } a_{ij}^{(4)} \leq 0. \end{cases} \\ \max(x_{ij}^{(1)} b_{ij}^{(4)}, x_{ij}^{(4)} b_{ij}^{(4)}) &= \begin{cases} x_{ij}^{(4)} b_{ij}^{(4)} & \text{if } b_{ij}^{(4)} \geq 0, \\ x_{ij}^{(1)} b_{ij}^{(4)} & \text{if } b_{ij}^{(4)} \leq 0. \end{cases} \end{aligned}$$

Remark 3.2

Any $n \times m$ TrFFSME with arbitrary coefficient can converted to a linear system of $2n \times 2m$ equations.

It is worth mention that the systems of equation in Eq. (3.1) after reducing it by Remark 3.1 to linear system, can be solved by many classical methods. However, in this paper the matrix inversion method will be used, where the solution is presented in the form:

$$\bar{X} = \begin{pmatrix} x_{11}^{(1)} \\ \vdots \\ x_{nm}^{(1)} \\ x_{11}^{(2)} \\ \vdots \\ x_{nm}^{(2)} \\ x_{11}^{(3)} \\ \vdots \\ x_{nm}^{(3)} \\ x_{11}^{(4)} \\ \vdots \\ x_{nm}^{(4)} \end{pmatrix}$$

Which can be written in matrix form as:

$$\bar{X} = \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) & \dots & (x_{1m}^{(1)}, x_{1m}^{(2)}, x_{1m}^{(3)}, x_{1m}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) & \dots & (x_{2m}^{(1)}, x_{2m}^{(2)}, x_{2m}^{(3)}, x_{2m}^{(4)}) \\ \vdots & \vdots & \ddots & \vdots \\ (x_{n1}^{(1)}, x_{n1}^{(2)}, x_{n1}^{(3)}, x_{n1}^{(4)}) & (x_{n2}^{(1)}, x_{n2}^{(2)}, x_{n2}^{(3)}, x_{n2}^{(4)}) & \dots & (x_{nm}^{(1)}, x_{nm}^{(2)}, x_{nm}^{(3)}, x_{nm}^{(4)}) \end{pmatrix}.$$

3.2 Restricted TrFFSME

In the following theorem we obtain a system of crisp Sylvester equation equivalent to the TrFFSME in Eq. (2.8) when the coefficients are positive, and the fuzzy solution is positive as well.

Theorem 3.1 *If $\tilde{A} = (\tilde{a}_{ij})_{n \times n} > 0, \tilde{B} = (\tilde{b}_{ij})_{m \times m} > 0$ and $\tilde{X} = (\tilde{x}_{ij})_{n \times m} > 0, \forall 1 \leq i, j \leq n, m$. Then the TrFFSME in Eq.(2.8) can be written as follows:*

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} + x_{ij}^{(1)} b_{ij}^{(1)} = c_{ij}^{(1)} \\ a_{ij}^{(2)} x_{ij}^{(2)} + x_{ij}^{(2)} b_{ij}^{(2)} = c_{ij}^{(2)} \\ a_{ij}^{(3)} x_{ij}^{(3)} + x_{ij}^{(3)} b_{ij}^{(3)} = c_{ij}^{(3)} \\ a_{ij}^{(4)} x_{ij}^{(4)} + x_{ij}^{(4)} b_{ij}^{(4)} = c_{ij}^{(4)} \end{cases} \quad (3.2)$$

Proof:

Assuming $\tilde{A} = (\tilde{a}_{ij})_{n \times n} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)})$, $\tilde{B} = (\tilde{b}_{ij})_{m \times m} = (b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}, b_{ij}^{(4)})$, $\tilde{X} = (\tilde{x}_{ij})_{n \times m} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$ and $\tilde{C} = (\tilde{c}_{ij})_{n \times m} = (c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)})$ positive TrFNs $\forall 1 \leq i, j \leq n, m$. Then, from Definition 2.5 and by Eq. (2.3),

$$\tilde{A}\tilde{X} = (a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}, a_{ij}^{(4)} x_{ij}^{(4)}) \text{ and } \tilde{X}\tilde{B} = (x_{ij}^{(1)} b_{ij}^{(1)}, x_{ij}^{(2)} b_{ij}^{(2)}, x_{ij}^{(3)} b_{ij}^{(3)}, x_{ij}^{(4)} b_{ij}^{(4)}).$$

Adding $\tilde{A}\tilde{X}$ and $\tilde{X}\tilde{B}$ using Eq. (2.1) in Definition 2.5 we get:

$$\begin{aligned} \tilde{A}\tilde{X} + \tilde{X}\tilde{B} &= (a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}, a_{ij}^{(4)} x_{ij}^{(4)}) + (x_{ij}^{(1)} b_{ij}^{(1)}, x_{ij}^{(2)} b_{ij}^{(2)}, x_{ij}^{(3)} b_{ij}^{(3)}, x_{ij}^{(4)} b_{ij}^{(4)}) \\ &= (a_{ij}^{(1)} x_{ij}^{(1)} + x_{ij}^{(1)} b_{ij}^{(1)}, a_{ij}^{(2)} x_{ij}^{(2)} + x_{ij}^{(2)} b_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)} + x_{ij}^{(3)} b_{ij}^{(3)}, a_{ij}^{(4)} x_{ij}^{(4)} + x_{ij}^{(4)} b_{ij}^{(4)}). \end{aligned}$$

By Eq.(2.9) in Definition 2.8, the TrFFSME in Eq. (2.8) can be written as follows:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} + x_{ij}^{(1)} b_{ij}^{(1)} = c_{ij}^{(1)} \\ a_{ij}^{(2)} x_{ij}^{(2)} + x_{ij}^{(2)} b_{ij}^{(2)} = c_{ij}^{(2)} \\ a_{ij}^{(3)} x_{ij}^{(3)} + x_{ij}^{(3)} b_{ij}^{(3)} = c_{ij}^{(3)} \\ a_{ij}^{(4)} x_{ij}^{(4)} + x_{ij}^{(4)} b_{ij}^{(4)} = c_{ij}^{(4)} \end{cases}$$

In the following theorem we obtain a system of crisp Sylvester equation equivalent to the TrFFSME in Eq. (2.8) when the coefficients are negative, and the fuzzy solution is negative as well.

Theorem 3.2 *If $\tilde{A} = (\tilde{a}_{ij})_{n \times n} < 0, \tilde{B} = (\tilde{b}_{ij})_{m \times m} < 0$ and $\tilde{X} = (\tilde{x}_{ij})_{n \times m} < 0, \forall 1 \leq i, j \leq n, m$. Then the TrFFSME in Eq.(2.8) can be written as follows:*

$$\begin{cases} a_{ij}^{(4)} x_{ij}^{(4)} + x_{ij}^{(4)} b_{ij}^{(4)} = c_{ij}^{(1)} \\ a_{ij}^{(3)} x_{ij}^{(3)} + x_{ij}^{(3)} b_{ij}^{(3)} = c_{ij}^{(2)} \\ a_{ij}^{(2)} x_{ij}^{(2)} + x_{ij}^{(2)} b_{ij}^{(2)} = c_{ij}^{(3)} \\ a_{ij}^{(1)} x_{ij}^{(1)} + x_{ij}^{(1)} b_{ij}^{(1)} = c_{ij}^{(4)} \end{cases} \quad (3.3)$$

Proof: Straightforward similar to Theorem 3.2.

In the following theorem a system of crisp Sylvester equation equivalent to the TrFFSME in Eq. (2.8) when the coefficients are positive, and the fuzzy solution is negative.

Theorem 3.3 *If $\tilde{A} = (\tilde{a}_{ij})_{n \times n} > 0, \tilde{B} = (\tilde{b}_{ij})_{m \times m} > 0$ and $\tilde{X} = (\tilde{x}_{ij})_{n \times m} < 0, \forall 1 \leq i, j \leq n, m$. Then the TrFFSME in Eq.(2.8) can be written as follows:*

$$\begin{cases} a_{ij}^{(4)} x_{ij}^{(1)} + x_{ij}^{(1)} b_{ij}^{(4)} = c_{ij}^{(1)} \\ a_{ij}^{(3)} x_{ij}^{(2)} + x_{ij}^{(2)} b_{ij}^{(3)} = c_{ij}^{(2)} \\ a_{ij}^{(2)} x_{ij}^{(3)} + x_{ij}^{(3)} b_{ij}^{(2)} = c_{ij}^{(3)} \\ a_{ij}^{(1)} x_{ij}^{(4)} + x_{ij}^{(4)} b_{ij}^{(1)} = c_{ij}^{(4)} \end{cases} \quad (3.4)$$

Proof: Straightforward similar to Theorem 3.2.

In the following theorem a system of crisp linear Sylvester matrix equation equivalent to the TrFFSME in Eq. (2.8) when the coefficients are negative, and the fuzzy solution is positive.

Theorem 3.4 If $\tilde{A} = (\tilde{a}_{ij})_{n \times n} < 0$, $\tilde{B} = (\tilde{b}_{ij})_{m \times m} < 0$ and $\tilde{X} = (\tilde{x}_{ij})_{n \times m} > 0$, $\forall 1 \leq i, j \leq n, m$.

Then the TrFFSME in Eq.(2.8) can be written as follows:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(4)} + x_{ij}^{(4)} b_{ij}^{(1)} = c_{ij}^{(1)} \\ a_{ij}^{(2)} x_{ij}^{(3)} + x_{ij}^{(3)} b_{ij}^{(2)} = c_{ij}^{(2)} \\ a_{ij}^{(3)} x_{ij}^{(2)} + x_{ij}^{(2)} b_{ij}^{(3)} = c_{ij}^{(3)} \\ a_{ij}^{(4)} x_{ij}^{(1)} + x_{ij}^{(1)} b_{ij}^{(4)} = c_{ij}^{(4)} \end{cases} \quad (3.5)$$

Proof: Straightforward similar to Theorem 3.2.

In the following subsection we apply Bartle's Stewart method (BSM) to the system of equations in Eq.(3.2) to obtain the positive fuzzy solution. It worth mentioning that, the same method can be also applied to the other systems of equations in Eq. (3.3) to Eq. (3.5).

3.2.1 Generalized BSM for TrFFSME

In this section the generalized BSM is amended to solve the system of equations in Eq. (3.2) as follows:

Step 1: Suppose $a_{ij}^{(1)}, b_{ij}^{(1)}, a_{ij}^{(2)}, b_{ij}^{(2)}, a_{ij}^{(3)}, b_{ij}^{(3)}, a_{ij}^{(4)}$ and $b_{ij}^{(4)}$ are real and have real Schur decompositions $a_{ij}^{(1)} = U_1 R_1 U_1^T$, $b_{ij}^{(1)} = V_1 S_1 V_1^T$, $a_{ij}^{(2)} = U_2 R_2 U_2^T$, $b_{ij}^{(2)} = V_2 S_2 V_2^T$, $a_{ij}^{(3)} = U_3 R_3 U_3^T$, $b_{ij}^{(3)} = V_3 S_3 V_3^T$, $a_{ij}^{(4)} = U_4 R_4 U_4^T$ and $b_{ij}^{(4)} = V_4 S_4 V_4^T$ where U and V are orthogonal and R and S are upper quasi-triangular. Then the system of equations in Eq.(3.2) can be transformed to:

$$\begin{cases} U_1^T a_{ij}^{(1)} U_1 \cdot U_1^T x_{ij}^{(1)} V_1 + U_1^T x_{ij}^{(1)} V_1 \cdot V_1^T b_{ij}^{(1)} V_1 = U_1^T c_{ij}^{(1)} V_1, \\ U_2^T a_{ij}^{(2)} U_2 \cdot U_2^T x_{ij}^{(2)} V_2 + U_2^T x_{ij}^{(2)} V_2 \cdot V_2^T b_{ij}^{(2)} V_2 = U_2^T c_{ij}^{(2)} V_2, \\ U_3^T a_{ij}^{(3)} U_3 \cdot U_3^T x_{ij}^{(3)} V_3 + U_3^T x_{ij}^{(3)} V_3 \cdot V_3^T b_{ij}^{(3)} V_3 = U_3^T c_{ij}^{(3)} V_3, \\ U_4^T a_{ij}^{(4)} U_4 \cdot U_4^T x_{ij}^{(4)} V_4 + U_4^T x_{ij}^{(4)} V_4 \cdot V_4^T b_{ij}^{(4)} V_4 = U_4^T c_{ij}^{(4)} V_4. \end{cases}$$

Consequently, they can be written as:

$$\begin{cases} R_1 W_1 + W_1 S_1 = D_1, \\ R_2 W_2 + W_2 S_2 = D_2, \\ R_3 W_3 + W_3 S_3 = D_3, \\ R_4 W_4 + W_4 S_4 = D_4. \end{cases}$$

where,

$$\begin{aligned} R_1 &= U_1^T a_{ij}^{(1)} U_1, R_2 = U_2^T a_{ij}^{(2)} U_2, R_3 = U_3^T a_{ij}^{(3)} U_3, \text{ and } R_4 = U_4^T a_{ij}^{(4)} U_4, \\ W_1 &= U_1^T x_{ij}^{(1)} V_1, W_2 = U_2^T x_{ij}^{(2)} V_2, W_3 = U_3^T x_{ij}^{(3)} V_3, \text{ and } W_4 = U_4^T x_{ij}^{(4)} V_4, \\ S_1 &= V_1^T b_{ij}^{(1)} V_1, S_2 = V_2^T b_{ij}^{(2)} V_2, S_3 = V_3^T b_{ij}^{(3)} V_3, \text{ and } S_4 = V_4^T b_{ij}^{(4)} V_4, \\ D_1 &= U_1^T c_{ij}^{(1)} V_1, D_2 = U_2^T c_{ij}^{(2)} V_2, D_3 = U_3^T c_{ij}^{(3)} V_3 \text{ and } D_4 = U_4^T c_{ij}^{(4)} V_4. \end{aligned}$$

Applying Kronecker product and Vec-operator gives:

$$\begin{cases} P_1 w_1 = d_1, \\ P_2 w_2 = d_2, \\ P_3 w_3 = d_3, \\ P_4 w_4 = d_4. \end{cases}$$

where,

$$\begin{aligned} P_1 &= I \otimes R_1 + S_1^T \otimes I, w_1 = \text{vec}(W_1) \text{ and } d_1 = \text{vec}(D_1), \\ P_2 &= I \otimes R_2 + S_2^T \otimes I, w_2 = \text{vec}(W_2) \text{ and } d_2 = \text{vec}(D_1), \\ P_3 &= I \otimes R_3 + S_3^T \otimes I, w_3 = \text{vec}(W_3) \text{ and } d_3 = \text{vec}(D_3), \\ P_4 &= I \otimes R_4 + S_4^T \otimes I, w_4 = \text{vec}(W_4) \text{ and } d_4 = \text{vec}(D_4). \end{aligned}$$

Many analytical method can be used to solve for w_1, w_2, w_3 and w_4 . However, in this paper we will apply Gaussian elimination and back substitution method.

Step 2: The values of $x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}$ and $x_{ij}^{(4)}$ can be computed as follows:

$$\begin{aligned} x_{ij}^{(1)} &= U_1 W_1 V_1^T, \\ x_{ij}^{(2)} &= U_2 W_2 V_2^T, \\ x_{ij}^{(3)} &= U_3 W_3 V_3^T, \\ x_{ij}^{(4)} &= U_4 W_4 V_4^T. \end{aligned}$$

Step 3: Combining $x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}$ and $x_{ij}^{(4)}$ which obtained in step 2. The solution of TrFFSME is represented by:

$$\tilde{X} = \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix}.$$

Remark 3.3.

The system of equations in Eq. (3.2) can also be solved using the built-in function on Mathematica, namely “LyapunovSolve[a,b,c]” or MATLAB function “Sylvester(a,b,c)”.

Definition 3.1 Trapezoidal positive fuzzy solution matrix in general form.

Let $\tilde{X} = (\tilde{x}_{ij})_{n \times m} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$ be a trapezoidal fuzzy matrix. If $\tilde{X} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$ is an exact solution of Eq. (3.1) such that $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)} > 0, \forall 1 \leq i, j \leq n, m$, then $\tilde{X} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$ is called a positive fuzzy solution of Eq. (2.8).

4. Numerical Examples

In this section, the proposed methods are illustrated by solving two examples.

Example 4.1) Solve the following TrFFSME:

$$\begin{aligned} &\begin{pmatrix} (2, 3, 4, 5) & (1, 2, 3, 6) \\ (-6, -5, -3, -1) & (-3, -2, 1, 5) \end{pmatrix} \cdot \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} \\ &+ \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} \cdot \begin{pmatrix} (-5, -4, -2, -1) & (1, 2, 4, 6) \\ (-4, -3, 2, 3) & (-2, -1, 3, 5) \end{pmatrix} \\ &= \begin{pmatrix} (-37, -7, 29, 72) & (-3, 13, 48, 122) \\ (-106, -60, 1, 48) & (-57, -22, 34, 105) \end{pmatrix} \end{aligned}$$

Solution:

Since \tilde{A}, \tilde{B} are arbitrary fuzzy matrices and \tilde{X} is assumed to be positive, we can convert the given TrFFSME to the system of equations in Eq. (3.1), then by Remark 3.1 and Remark 3.2 the following 16 equations can be obtained:

$$\left\{ \begin{array}{l} 2x_{11}^{(1)} + x_{21}^{(1)} - 5x_{11}^{(4)} - 4x_{12}^{(4)} = -37 \\ 2x_{12}^{(1)} + x_{22}^{(1)} + x_{11}^{(1)} - 2x_{12}^{(4)} = -3 \\ -6x_{11}^{(4)} - 3x_{21}^{(4)} - 5x_{21}^{(4)} - 4x_{22}^{(4)} = -106 \\ -6x_{12}^{(4)} - 3x_{22}^{(4)} + x_{21}^{(1)} - 2x_{22}^{(4)} = -57 \\ 3x_{11}^{(2)} + 2x_{21}^{(2)} - 4x_{11}^{(3)} - 3x_{12}^{(3)} = -7 \\ 3x_{12}^{(2)} + 2x_{22}^{(2)} + 2x_{11}^{(2)} - x_{12}^{(3)} = 13 \\ -5x_{11}^{(3)} - 2x_{21}^{(3)} - 4x_{21}^{(3)} - 3x_{22}^{(3)} = -60 \\ -5x_{12}^{(3)} - 2x_{22}^{(3)} + 2x_{21}^{(2)} - x_{22}^{(3)} = -22 \\ 4x_{11}^{(3)} + 3x_{21}^{(3)} - 2x_{11}^{(2)} + 2x_{12}^{(3)} = 29 \\ 4x_{12}^{(3)} + 3x_{22}^{(3)} + 4x_{11}^{(3)} + 3x_{12}^{(3)} = 48 \\ -3x_{11}^{(2)} + x_{21}^{(3)} - 2x_{21}^{(2)} + 2x_{22}^{(3)} = 1 \\ -3x_{12}^{(2)} + x_{22}^{(3)} + 4x_{21}^{(3)} + 3x_{22}^{(3)} = 34 \\ 5x_{11}^{(4)} + 6x_{21}^{(4)} - x_{11}^{(1)} + 3x_{12}^{(4)} = 72 \\ 5x_{21}^{(4)} + 6x_{22}^{(4)} + 6x_{11}^{(4)} + 5x_{12}^{(4)} = 122 \\ -x_{11}^{(1)} + 5x_{21}^{(4)} - x_{21}^{(1)} + 3x_{22}^{(4)} = 48 \\ -x_{12}^{(1)} + 5x_{22}^{(4)} + 6x_{21}^{(4)} + 5x_{22}^{(4)} = 105 \end{array} \right.$$

This system of equations can be solved by many analytical methods. However, in this paper we will consider the matrix inversion method as follows:

We first rewrite it in the form, $MX = N$, where,

$$M = \begin{pmatrix} 2 & 0 & 0 & -5 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & -5 \\ 0 & 3 & -4 & 0 & 0 & 0 & -3 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -3 \\ 0 & -2 & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \\ -1 & 0 & 0 & 5 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 5 & 0 & 0 & 6 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 5 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 10 \end{pmatrix},$$

$$X = \begin{pmatrix} x_{11}^{(1)} \\ x_{11}^{(2)} \\ x_{11}^{(3)} \\ x_{11}^{(4)} \\ x_{12}^{(1)} \\ x_{12}^{(2)} \\ x_{12}^{(3)} \\ x_{12}^{(4)} \\ x_{21}^{(1)} \\ x_{21}^{(2)} \\ x_{21}^{(3)} \\ x_{21}^{(4)} \\ x_{22}^{(1)} \\ x_{22}^{(2)} \\ x_{22}^{(3)} \\ x_{22}^{(4)} \end{pmatrix} \text{ and } N = \begin{pmatrix} -37 \\ -3 \\ -106 \\ -57 \\ -7 \\ 13 \\ -60 \\ -22 \\ 29 \\ 48 \\ 1 \\ 34 \\ 72 \\ 122 \\ 48 \\ 105 \end{pmatrix}.$$

Multiplying both sides of $MX = N$ by M^{-1} , we get:

$$\begin{pmatrix} x_{11}^{(1)} \\ x_{11}^{(2)} \\ x_{11}^{(3)} \\ x_{11}^{(4)} \\ x_{12}^{(1)} \\ x_{12}^{(2)} \\ x_{12}^{(3)} \\ x_{12}^{(4)} \\ x_{21}^{(1)} \\ x_{21}^{(2)} \\ x_{21}^{(3)} \\ x_{21}^{(4)} \\ x_{22}^{(1)} \\ x_{22}^{(2)} \\ x_{22}^{(3)} \\ x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \\ 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 4 \\ 5 \\ 6 \\ 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}.$$

Which can be written in matrix form as follows:

$$\tilde{X} = \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} = \begin{pmatrix} (1,2,3,5) & (1,2,3,4) \\ (2,4,5,6) & (2,3,5,7) \end{pmatrix}.$$

Figure 1 shows the positive fuzzy solution \tilde{X} .

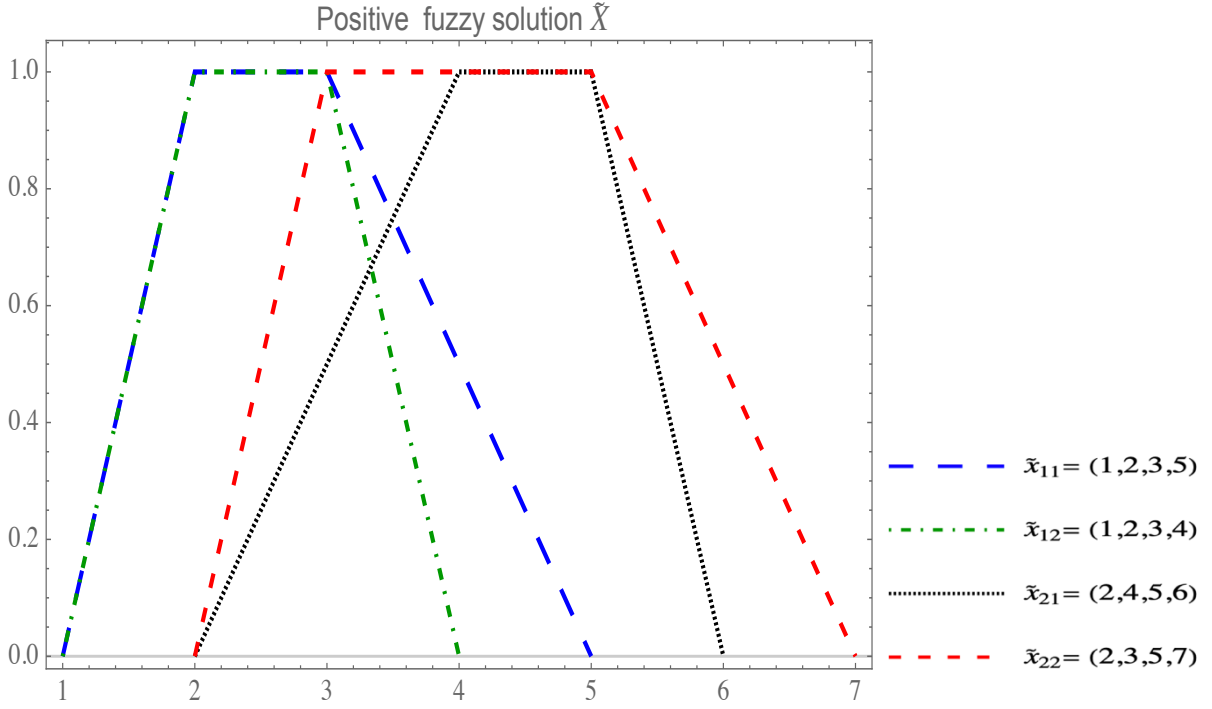


Fig. 1 The positive fuzzy solution \tilde{X} of Example 4.1

Example 4.2) Consider the following TrFFSME:

$$\begin{pmatrix} (1,2,3,4) & (2,3,5,7) \\ (2,4,5,6) & (1,3,4,5) \end{pmatrix} \cdot \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} \\ + \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} \cdot \begin{pmatrix} (1,2,4,5) & (1,3,5,6) \\ (4,5,6,8) & (1,3,5,7) \end{pmatrix} \\ = \begin{pmatrix} (14,37,89,143) & (9,31,82,140) \\ (22,57,103,162) & (12,47,95,156) \end{pmatrix}$$

Solution:

Step 1:

Since $\tilde{A} \geq 0, \tilde{B} \geq 0$ and $\tilde{X} \geq 0$. To solve the given TrFFSME we will first convert it to the linear system in Eq. (3.2) as follows:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} + x_{ij}^{(1)} b_{ij}^{(1)} = c_{ij}^{(1)} \\ a_{ij}^{(2)} x_{ij}^{(2)} + x_{ij}^{(2)} b_{ij}^{(2)} = c_{ij}^{(2)} \\ a_{ij}^{(3)} x_{ij}^{(3)} + x_{ij}^{(3)} b_{ij}^{(3)} = c_{ij}^{(3)} \\ a_{ij}^{(4)} x_{ij}^{(4)} + x_{ij}^{(4)} b_{ij}^{(4)} = c_{ij}^{(4)} \end{cases}$$

where,

$\tilde{A} = \begin{pmatrix} (1,2,3,4) & (2,3,5,7) \\ (2,4,5,6) & (1,3,4,5) \end{pmatrix}$ and the crisp matrices $a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}$ and $a_{ij}^{(4)}$ are:

$$a_{ij}^{(1)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, a_{ij}^{(2)} = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}, a_{ij}^{(3)} = \begin{pmatrix} 3 & 5 \\ 5 & 4 \end{pmatrix} \text{ and } a_{ij}^{(4)} = \begin{pmatrix} 4 & 7 \\ 6 & 5 \end{pmatrix}.$$

$\tilde{B} = \begin{pmatrix} (1,2,4,5) & (1,3,5,6) \\ (4,5,6,8) & (1,3,5,7) \end{pmatrix}$ and the crisp matrices $b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}$ and $b_{ij}^{(4)}$ are:

$$b_{ij}^{(1)} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}, b_{ij}^{(2)} = \begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix}, b_{ij}^{(3)} = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \text{ and } b_{ij}^{(4)} = \begin{pmatrix} 5 & 6 \\ 8 & 7 \end{pmatrix}.$$

$\tilde{C} = \begin{pmatrix} (14,37,89,143) & (9,31,82,140) \\ (22,57,103,162) & (12,47,95,156) \end{pmatrix}$ and the crisp matrices $c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}$ and $c_{ij}^{(4)}$ are:

$$c_{ij}^{(1)} = \begin{pmatrix} 14 & 9 \\ 22 & 12 \end{pmatrix}, c_{ij}^{(2)} = \begin{pmatrix} 37 & 31 \\ 57 & 47 \end{pmatrix}, c_{ij}^{(3)} = \begin{pmatrix} 89 & 82 \\ 103 & 95 \end{pmatrix} \text{ and } c_{ij}^{(4)} = \begin{pmatrix} 143 & 140 \\ 162 & 156 \end{pmatrix}.$$

$\tilde{X} = \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix}$ and the crisp matrices $x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}$ and $x_{ij}^{(4)}$

are:

$$x_{ij}^{(1)} = \begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \end{pmatrix}, x_{ij}^{(2)} = \begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \end{pmatrix}, x_{ij}^{(3)} = \begin{pmatrix} x_{11}^{(3)} & x_{12}^{(3)} \\ x_{21}^{(3)} & x_{22}^{(3)} \end{pmatrix} \text{ and } x_{ij}^{(4)} = \begin{pmatrix} x_{11}^{(4)} & x_{12}^{(4)} \\ x_{21}^{(4)} & x_{22}^{(4)} \end{pmatrix}.$$

We decompose the following matrices by applying Definition 2.11 as follows:

$$a_{ij}^{(1)} = U_1 R_1 U_1^T, b_{ij}^{(1)} = V_1 S_1 V_1^T, a_{ij}^{(2)} = U_2 R_2 U_2^T, b_{ij}^{(2)} = V_2 S_2 V_2^T, a_{ij}^{(3)} = U_3 R_3 U_3^T, b_{ij}^{(3)} = V_3 S_3 V_3^T, a_{ij}^{(4)} = U_4 R_4 U_4^T \text{ and } b_{ij}^{(4)} = V_4 S_4 V_4^T.$$

Where,

$$U_1 = \begin{pmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{pmatrix}, U_1^T = \begin{pmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{pmatrix} \text{ and } R_1 = \begin{pmatrix} 3 & 0 \\ 0 & -0.99 \end{pmatrix}.$$

$$U_2 = \begin{pmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{pmatrix}, U_2^T = \begin{pmatrix} -0.7071 & 0.7071 \\ -0.7071 & -0.7071 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} -1 & -1 \\ 0 & 6 \end{pmatrix}.$$

$$U_3 = \begin{pmatrix} -0.7415 & -0.6710 \\ 0.6710 & -0.7415 \end{pmatrix}, U_3^T = \begin{pmatrix} -0.7415 & 0.6710 \\ -0.6710 & -0.7415 \end{pmatrix} \text{ and } R_3 = \begin{pmatrix} -1.5249 & 0 \\ 0 & 8.5249 \end{pmatrix}.$$

$$U_4 = \begin{pmatrix} -0.7593 & -0.6508 \\ 0.6508 & -0.7593 \end{pmatrix}, U_4^T = \begin{pmatrix} -0.7593 & 0.6508 \\ -0.6508 & -0.7593 \end{pmatrix} \text{ and } R_4 = \begin{pmatrix} -2 & 1 \\ 0 & 11 \end{pmatrix}.$$

$$V_1 = \begin{pmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{pmatrix}, V_1^T = \begin{pmatrix} 0.4472 & 0.8944 \\ -0.8944 & 0.4472 \end{pmatrix} \text{ and } S_1 = \begin{pmatrix} 3 & -3 \\ 0 & -1 \end{pmatrix}.$$

$$V_2 = \begin{pmatrix} -0.6611 & -0.7503 \\ 0.7503 & -0.6611 \end{pmatrix}, V_2^T = \begin{pmatrix} -0.6611 & 0.7503 \\ -0.7503 & -0.6611 \end{pmatrix} \text{ and } S_2 = \begin{pmatrix} -1.4051 & -2 \\ 0 & 6.4051 \end{pmatrix}.$$

$$V_3 = \begin{pmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{pmatrix}, V_3^T = \begin{pmatrix} -0.7071 & 0.7071 \\ -0.7071 & -0.7071 \end{pmatrix} \text{ and } S_3 = \begin{pmatrix} -1 & -1 \\ 0 & 10 \end{pmatrix}.$$

$$V_4 = \begin{pmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{pmatrix}, V_4^T = \begin{pmatrix} -0.7071 & 0.7071 \\ -0.7071 & -0.7071 \end{pmatrix} \text{ and } S_4 = \begin{pmatrix} -1 & -2 \\ 0 & 13 \end{pmatrix}.$$

And,

$$\begin{aligned} P_1 &= I \otimes R_1 + S_1^T \otimes I \\ P_2 &= I \otimes R_2 + S_2^T \otimes I \\ P_3 &= I \otimes R_3 + S_3^T \otimes I \\ P_4 &= I \otimes R_4 + S_4^T \otimes I \end{aligned}$$

Kronecker sum Definition 2.10 can be applied to obtain P_1, P_2, P_3 and P_4 as follows:

$$P_1 = \begin{pmatrix} 6 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & -1.9 \end{pmatrix}, P_2 = \begin{pmatrix} -2.4 & 0 & -1 & 0 \\ -2 & 5.4 & 0 & -1 \\ 0 & 0 & 4.6 & 0 \\ 0 & 0 & -2 & 12.4 \end{pmatrix}, P_3 = \begin{pmatrix} -2.5 & 0 & 0 & 0 \\ -1 & 8.5 & 0 & 0 \\ 0 & 0 & 7.5 & 0 \\ 0 & 0 & -1 & 18.5 \end{pmatrix}$$

$$\text{and } P_4 = \begin{pmatrix} -3 & 0 & 1 & 0 \\ -2 & 11 & 0 & 1 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & -2 & 24 \end{pmatrix}.$$

In order to calculate w_1, w_2, w_3 and w_4 , we need to find D_1, D_2, D_3 and D_4 as follows:

$$D_1 = U_1^T c_{ij}^{(1)} V_1 = \begin{pmatrix} 24.6658 & -16.1276 \\ 4.4272 & -4.1109 \end{pmatrix}$$

$$D_2 = U_2^T c_{ij}^{(2)} V_2 = \begin{pmatrix} -0.8598 & -18.0904 \\ 2.5554 & 86.3335 \end{pmatrix}$$

$$D_3 = U_3^T c_{ij}^{(3)} V_3 = \begin{pmatrix} -0.1258 & -4.2926 \\ 7.5156 & 184.9434 \end{pmatrix}$$

$$D_4 = U_4^T c_{ij}^{(4)} V_4 = \begin{pmatrix} -1.1504 & 5.5988 \\ 4.6018 & 300.9571 \end{pmatrix}$$

We can find d_1 and d_2 by applying Definition 2.11 on D_1, D_2, D_3 and D_4 as follow:

$$d_1 = \text{vec}(D_1) = \begin{pmatrix} 24.6658 \\ -16.1276 \\ 4.4272 \\ -4.1109 \end{pmatrix}, d_2 = \text{vec}(D_2) = \begin{pmatrix} -0.8598 \\ -18.0904 \\ 2.5554 \\ 86.3335 \end{pmatrix}, d_3 = \text{vec}(D_3) = \begin{pmatrix} -0.1258 \\ -4.2926 \\ 7.5156 \\ 184.9434 \end{pmatrix}$$

$$\text{and, } d_4 = \text{vec}(D_4) = \begin{pmatrix} -1.1504 \\ 5.5988 \\ 4.6018 \\ 300.9571 \end{pmatrix}.$$

$$\text{Since } W_1 = \begin{pmatrix} w_{11}^{(1)} & w_{12}^{(1)} \\ w_{21}^{(1)} & w_{22}^{(1)} \end{pmatrix}, W_2 = \begin{pmatrix} w_{11}^{(2)} & w_{12}^{(2)} \\ w_{21}^{(2)} & w_{22}^{(2)} \end{pmatrix}, W_3 = \begin{pmatrix} w_{11}^{(3)} & w_{12}^{(3)} \\ w_{21}^{(3)} & w_{22}^{(3)} \end{pmatrix} \text{ and } W_4 = \begin{pmatrix} w_{11}^{(4)} & w_{12}^{(4)} \\ w_{21}^{(4)} & w_{22}^{(4)} \end{pmatrix}$$

then,

$$w_1 = \text{vec}(W_1) = \begin{pmatrix} w_{11}^{(1)} \\ w_{12}^{(1)} \\ w_{21}^{(1)} \\ w_{22}^{(1)} \end{pmatrix}, w_2 = \text{vec}(W_2) = \begin{pmatrix} w_{11}^{(2)} \\ w_{12}^{(2)} \\ w_{21}^{(2)} \\ w_{22}^{(2)} \end{pmatrix}, w_3 = \text{vec}(W_3) = \begin{pmatrix} w_{11}^{(3)} \\ w_{12}^{(3)} \\ w_{21}^{(3)} \\ w_{22}^{(3)} \end{pmatrix} \text{ and,}$$

$$w_4 = \text{vec}(W_4) = \begin{pmatrix} w_{11}^{(4)} \\ w_{12}^{(4)} \\ w_{21}^{(4)} \\ w_{22}^{(4)} \end{pmatrix}.$$

Now we can solve for w_1, w_2, w_3 and w_4 as follows:

$$\begin{aligned} P_1 w_1 &= d_1 \\ P_2 w_2 &= d_2 \\ P_3 w_3 &= d_3 \\ P_4 w_4 &= d_4 \end{aligned}$$

$$\begin{pmatrix} 6 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & -1.9 \end{pmatrix} \begin{pmatrix} w_{11}^{(1)} \\ w_{12}^{(1)} \\ w_{21}^{(1)} \\ w_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 24.6658 \\ -16.1276 \\ 4.4272 \\ -4.1109 \end{pmatrix}$$

$$\begin{pmatrix} -2.4 & 0 & -1 & 0 \\ -2 & 5.4 & 0 & -1 \\ 0 & 0 & 4.6 & 0 \\ 0 & 0 & -2 & 12.4 \end{pmatrix} \begin{pmatrix} w_{11}^{(2)} \\ w_{12}^{(2)} \\ w_{21}^{(2)} \\ w_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} -0.8598 \\ -18.0904 \\ 2.5554 \\ 86.3335 \end{pmatrix}$$

$$\begin{pmatrix} -2.5 & 0 & 0 & 0 \\ -1 & 8.5 & 0 & 0 \\ 0 & 0 & 7.5 & 0 \\ 0 & 0 & -1 & 18.5 \end{pmatrix} \begin{pmatrix} w_{11}^{(2)} \\ w_{12}^{(2)} \\ w_{21}^{(2)} \\ w_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} -0.1258 \\ -4.2926 \\ 7.5156 \\ 184.9434 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 1 & 0 \\ -2 & 11 & 0 & 1 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & -2 & 24 \end{pmatrix} \begin{pmatrix} w_{11}^{(2)} \\ w_{12}^{(2)} \\ w_{21}^{(2)} \\ w_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} -1.1504 \\ 5.5988 \\ 4.6018 \\ 300.9571 \end{pmatrix}$$

Gaussian elimination and back substitution are applied to obtain w_1 , w_2 , w_3 and w_4 and therefore,

$$w_1 = \begin{pmatrix} 4.111 \\ -1.897 \\ 2.2136 \\ -1.265 \end{pmatrix}, w_2 = \begin{pmatrix} 0.126 \\ -1.996 \\ 0.556 \\ 7.049 \end{pmatrix}, w_3 = \begin{pmatrix} 0.049 \\ -0.501 \\ 0.998 \\ 10.037 \end{pmatrix} \text{ and } w_4 = \begin{pmatrix} 0.536 \\ -0.536 \\ 0.460 \\ 12.578 \end{pmatrix}.$$

Step 2: We compute x_{ij} and y_{ij} as follows:

$$x_{ij} = U_1 W_1 V_1^T = \begin{pmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{pmatrix} \begin{pmatrix} 4.111 & -1.897 \\ 2.2136 & -1.265 \end{pmatrix} \begin{pmatrix} 0.4472 & 0.8944 \\ -0.8944 & 0.4472 \end{pmatrix}. \text{ Which can be}$$

$$\text{simplified to:}$$

$$x_{ij} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}.$$

$$y_{ij} = U_2 W_2 V_2^T = \begin{pmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{pmatrix} \begin{pmatrix} 0.126 & -1.996 \\ 0.556 & 7.049 \end{pmatrix} \begin{pmatrix} -0.6611 & 0.7503 \\ -0.7503 & -0.6611 \end{pmatrix}. \text{ Which can be}$$

$$\text{simplified to:}$$

$$y_{ij} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}.$$

$$z_{ij} = U_3 W_3 V_3^T = \begin{pmatrix} -0.7415 & -0.6710 \\ 0.6710 & -0.7415 \end{pmatrix} \begin{pmatrix} 0.049 & -0.501 \\ 0.998 & 10.037 \end{pmatrix} \begin{pmatrix} -0.7071 & 0.7071 \\ -0.7071 & -0.7071 \end{pmatrix}. \text{ Which can be}$$

$$\text{simplified to:}$$

$$z_{ij} = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}.$$

$$q_{ij} = U_4 W_4 V_4^T = \begin{pmatrix} -0.7593 & -0.6508 \\ 0.6508 & -0.7593 \end{pmatrix} \begin{pmatrix} 0.536 & -0.536 \\ 0.460 & 12.578 \end{pmatrix} \begin{pmatrix} -0.7071 & 0.7071 \\ -0.7071 & -0.7071 \end{pmatrix}. \text{ Which can be}$$

$$\text{simplified to:}$$

$$q_{ij} = \begin{pmatrix} 6 & 5 \\ 7 & 7 \end{pmatrix}.$$

Step 3: Combining $x_{ij}^{(1)}$, $x_{ij}^{(2)}$, $x_{ij}^{(3)}$ and $x_{ij}^{(4)}$ which obtained in step 2. The solution of TrFFSME is represented by:

$$\tilde{X} = \begin{pmatrix} (1,3,5,6) & (1,2,4,5) \\ (4,5,6,7) & (3,4,5,7) \end{pmatrix}.$$

Step 4: Feasibility of the solution

Since,

$$\text{I) } x_{ij}^{(1)} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} > 0, \forall \{1 \leq i, j \leq 2\}.$$

$$\text{II) } x_{ij}^{(2)} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} > 0, \forall \{1 \leq i, j \leq 2\}.$$

$$\text{III) } x_{ij}^{(3)} = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix} > 0, \forall \{1 \leq i, j \leq 2\}.$$

$$\text{IV) } x_{ij}^{(4)} = \begin{pmatrix} 6 & 5 \\ 7 & 7 \end{pmatrix} > 0, \forall \{1 \leq i, j \leq 2\}.$$

$$\text{V) } x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall \{1 \leq i, j \leq 2\}.$$

The positive fuzzy solution $\tilde{X} = \begin{pmatrix} (4, 5, 2, 1) & (4, 5, 3, 1) \\ (3, 4, 2, 2) & (3, 4, 2, 1) \end{pmatrix}$, is feasible.

Figure 2 shows the positive fuzzy solution \tilde{X} .

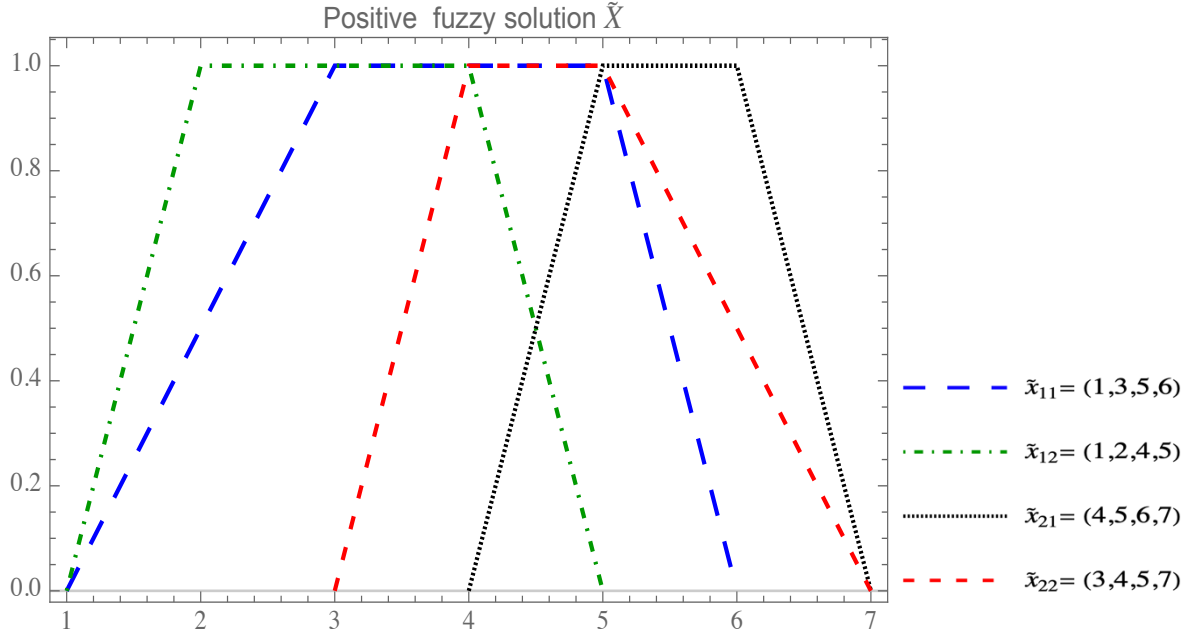


Fig. 2 The positive fuzzy solution \tilde{X} of Example 4.2

Remark 4.1: Extension of the proposed method to other fuzzy numbers and systems.

To the best of our knowledge, the proposed methods are the first approaches that can be applied to different fuzzy systems with arbitrary coefficients without any amendments. For example, it can be applied to TrFFSME in the form $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ with TFNs whenever, the mean values in the TrFNs used are equal. In addition, it can also be applied to the FFME in the form $\tilde{A}\tilde{X} = \tilde{C}$ with arbitrary coefficients if we allow $\tilde{B} = 0$, in $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$. Therefore, the proposed method is able to solve the following fuzzy systems: FFSME with TFNs and TrFNs and FFME with arbitrary coefficients.

5 Conclusion

In this paper, a new methods are developed to solve the TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ with restricted and unrestricted coefficient matrices, based on new fuzzy arithmetic operation between TrFNs. Different linear systems are developed corresponding to the TrFFSME based on the sign of the coefficients, where the solution can be obtained directly by MATLAB or Mathematica. In addition, the proposed methods are applicable for arbitrary FFSME with TFNs and TrFNs and arbitrary FFME with TFNs and TrFNs. As future work the proposed method will be applied to TrFFSME with trapezoidal bipolar fuzzy numbers.

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Compliance with Ethical Standards

Conflict of interest The authors declare no conflict of interest, financial or otherwise.

Human and animals rights This article does not contain any studies with human participants or animals performed by any of the authors.

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