(f, g)- DERIVATION IN RESIDUATED MULTILATTICES

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Abstract.

This paper's primary goal, is to extend the study of (f, g)-derivation to residuated multilattices with some examples and to study the first properties. We study ideal (f, g)- derivation and investigate several of its properties.
Keywords: Residuated multilattices, derivation, (f, g)-derivation, ideal (f, g)-derivation.

Introduction

A lattice is a poset where every pair of elements has a least upper bound and a greatest lower bound. In order to generalize the theory of lattices, some hyperstructures were introduced. In particular, multilattices are structure where the property of being a complete lattices, namely the existence of least upper bound(resp greatest lower bound) for every subset, is weakened to the existence of minimal upper(resp maximal lower) bound.[4].

Residuation plays a prominent role in the algebraic study of logical systems, which are usually modeled as partially ordered sets with some operations reflecting the properties of connective. Focusing on the link between the theory of multilattices and residuation, I.P. Cabrera et al. introduces residuated multilattices [4].

The notion of derivation, which comes from mathematical analysis, is also useful to study some structural properties of various kinds of algebra. The concept of derivation has been introduced in the commutative rings 1957[3]; BCI- algebra 2004[7], lattice 2001[1], MV-algebra 2010[5], BL-algebra 2017[1], residuated lattice 2016[6] and recently greater interest has been developped in residuated multilattice, introduced by Maffeu et al. [8]. The notion was further explored in the form of f-derivations on residuated multilattice by the same author. In this paper, we extend the study of (f, g)-derivation to

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residuated multilattices and find some examples and first properties. We study ideal (f, g)- derivation and investigate several of its properties.

This paper is organized as follows: in section 1, we recall some basic notions and properties of residuated multilattice, in section 2, we introduce the notions of (f, g)-derivation on residuated multilattice with some useful examples and investigate some properties.

1. Definitions and Preliminaries

In this section, we recall some definitions and essential properties used in this work.

Let $(M, \leq)$ be a poset, the set of upper bounds of $x \in M$ will be denoted $\uparrow_M x = \{a \in M, a \geq x\}$ and dually $\downarrow_M x = \{a \in M : a \leq x\}$ denotes the set of lower bounds. For $X \subseteq M$ the upper closure of X is $\uparrow_M X = \bigcup_{x \in X} \uparrow_M x$ and the lower closure of X is $\downarrow_M X = \bigcup_{x \in X} \downarrow_M x$. A **Multi-supremun** of X is a minimal element of the set of upper bounds of X and a **Multi-infimun** is a maxinal element of the set of lower bounds of X. The set of Multi-suprema of X is denoted by **Multisup(X)** and the set of Multi-infima of X is denoted by **Multiinf(X)** according to ([4],[8]).

Notation 1.1. [4]

- *We will write $a \sqcup b$ to denote $Multisup\{a, b\}$ and $a \sqcap b$ to denote $Multiinf\{a, b\}$*
- *When $a \sqcup b$ (resp. $a \sqcap b$) is a singleton $\{x\}$ (resp$\{y\}$), we write $a \vee b = x$ (resp $a \wedge b = y$)*

Definition 1.2. [8] *A poset $(M, \leq)$ is an **ordered multillattice** if and only if it satisfies that: for all $a, b, c \in M$, $a \leq c$ and $b \leq c$ implies that there exists $x \in a \sqcup b$ such that $x \leq c$ and its dual version for $a \sqcap b$.*

According to [4] a multillatice is said to be full if $a \sqcup b \neq \emptyset$ and $a \sqcap b \neq \emptyset$ for all $a, b \in M$.

An example of bounded full multilattices which is not a lattice is the multillatice with the following Hasse diagram:

Definition 1.3. [4] *A map $h : M \longrightarrow M'$ between multillatices is said to be a **homomorphism** if $h(a \sqcup b) \subseteq h(a) \sqcup h(b)$ and $h(a \sqcap b) \subseteq h(a) \sqcap h(b)$ for all $a, b \in M$.*

Definition 1.4. *([4],[8])* ***A pocrim*** *is a poset $(M, \leq)$ with the maximun element $\top$ and two binary operation $\odot, \rightarrow$ such that:*

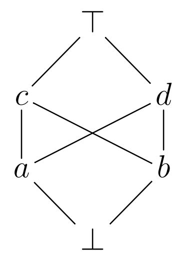


FIGURE 1. ML_6

1) $(M; \odot, \top)$ *is a commutative monoid,*
2) $a \odot b \leq c$ *if and only if* $b \leq a \to c$ *for all* $a, b, c \in M$.

A pocrim is said to be bounded if it has a least element.

According to [4], for a bounded pocrim $(M; \leq, \odot, \to, \top)$ with a least element $\bot$, we define $x^{'} = x \to \bot$ and $X^{'} = \{x^{'}, x \in X\}$, $x^0 = \top$, $x^n = x^{n-1} \odot x$ for every $x \in M$.
Now we recall the properties of pocrim that will be used.

Proposition 1.5. *([4],[8]) Let* $(M; \leq, \odot, \to, \top)$ *be a pocrim, for any* $a, b, c \in A$*, we have the following properties:*

(P_1) $a \odot b \leq a, b,$
(P_2) $a \odot (a \to b) \leq a \leq b \to (a \odot b)$ *and* $a \odot (a \to b) \leq b \leq a \to (a \odot b)$,
(P_3) $a \leq b$ *implies* $a \odot c \leq b \odot c, c \to a \leq c \to b \quad b \to c \leq a \to c,$
(P_4) $a \to (b \to c) = b \to (a \to c),$
(P_5) $a \to b \leq (a \odot c) \to (b \odot c),$
(P_6) $(a \to b) \odot (b \to c) \leq a \to c,$
(P_7) $(a \to b) \leq (c \to a) \to (c \to b)$ *and* $(a \to b) \leq (b \to c) \to (a \to c).$

Definition 1.6. *([8],[4]) A* ***residuated multilattice*** $\mathcal{M} := (M; \odot, \to, \top)$ *is a pocrim* $(M; \leq, \odot, \to, \top)$ *whose underlying poset*$(M, \leq)$ *is an ordered multilattice.*

Remark 1.7. [4]

- *A residuated multilattice is called bounded if it has a bottom element* $\bot$*,*
- *Notice that every residuated multillatice is full.*

The example below is the smallest residuated multilattice that is not a residuated lattice. According to [10], it has seven elements and will be denoted by RML_7.

Example 1.8. *Let $(M, \leq)$ be the multilattice decribed in the following figure:*

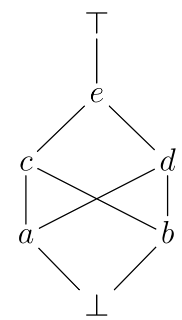


FIGURE 2.

The operations $\odot$ and $\to$ on M are defined by the following tables:

$\to$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	e	$\top$	e	$\top$	$\top$	$\top$	$\top$
b	a	a	$\top$	$\top$	$\top$	$\top$	$\top$
c	a	a	e	$\top$	e	$\top$	$\top$
d	a	a	e	e	$\top$	$\top$	$\top$
e	a	a	e	e	e	$\top$	$\top$
$\top$	$\bot$	a	b	c	d	e	$\top$

$\odot$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	b	b	b	b
c	$\bot$	$\bot$	b	b	b	b	c
d	$\bot$	$\bot$	b	b	b	b	d
e	$\bot$	$\bot$	b	b	b	b	d
$\top$	$\bot$	a	b	c	d	e	$\top$

Then $\mathcal{M} := (M; \odot, \to, \top)$ is a residuated multilattices denoted by RML_7.

From now, we give some useful properties and definitions in the rest of the paper. Every residuated multilattice $\mathcal{M} := (M; \odot, \to, \top)$ will be denoted just by $\mathcal{M}$ to simplify notation.

Proposition 1.9. *([4],[8]) Let $\mathcal{M}$ be a residuated multilattice, for any $a, b, c \in M$, we have the following properties:*

(M_1) $a \odot b, a \odot (a \to b) \in\downarrow_M (a \sqcap b)$,
(M_2) $(a \odot b) \sqcup (a \odot c) \subseteq a \odot (b \sqcup c)$,
(M_3) $a \odot (b \sqcap c) \subseteq\downarrow_M [(a \odot b) \sqcap (a \odot c)]$,
(M_4) $a \odot (b \sqcup c) \subseteq\uparrow_M [(a \odot b) \sqcup (a \odot c)]$,
(M_5) $(a \sqcap b) \to c \subseteq\uparrow_M [(a \to c) \sqcup (b \to c)]$,
(M_6) $(a \sqcup b) \to c \subseteq\downarrow_M [(a \to c) \sqcap (b \to c)]$,
(M_7) $(a \to c) \sqcap (b \to c) \subseteq a \sqcup b \to c$,
(M_8) $a \to b = max\{(a \sqcup b) \to b\} = max\{a \to (a \sqcap b)\}$,

(M_9) $a \leq a^{''}$, $\quad a^{'} = a^{'''}, a^{''} \to b^{''} = b^{'} \to a^{'}, (a \odot b)^{'} = a \to b^{'}$,
(M_{10}) $a \odot a^{'} = 0$ *and* $a \leq b^{'}$ *implies* $a \odot b = 0$,
(M_{11}) $(a \sqcap b)^{'} \subseteq \uparrow_M (a^{'} \sqcup b^{'})$,
(M_{12}) $(a \sqcup b)^{'} \subseteq \downarrow_M (a^{'} \sqcap b^{'})$,
(M_{13}) $(a^{'} \sqcap b^{'}) \subseteq (a^{'} \sqcup b^{'})$.

Definition 1.10. [8] *Let $h : \mathcal{M} \longrightarrow \mathcal{M}^{'}$ be a map betweeen residuated multillatice, h is said to be a* ***homomorphism*** *if h is a multilattice homomorphism and also $h(a \to b) = h(a) \to h(b), h(a \odot b) = h(a) \odot h(b)$ for all $a, b \in M$.*

Remark 1.11. [8] *For all endomorphism h between residuated multilattices, one can observe that $h(\top) = \top$*

Here we summarize the main results on the set of complemented elements needed throughout this work. Let $\mathcal{M}$ be a bounded residuated multillatices. According to [9] an element $c \in M$ is called complemented if there is an element c^* such that $\top \in c \sqcup c^*$ and $\bot \in c \sqcap c^*$. We call $c^{'}$ complement of c in M. We denote by $C(M)$ the set of all complemented elements of M.

Proposition 1.12. [9] *Let $\mathcal{M}$ be a residuated multilattice and $c \in C(M)$ an element which has a complement $c^* \in M$.*

(i) *If c^{**} is another complement of c then $c^* = c^{**}$,*
(ii) $c = c^{'*}, c^{'} = c^*$ *and* $c = c^{''}$,
(iii) $c \odot c = c$,
(iv) $e \in C(\mathcal{M})$ *and* $x \in M, e \odot x \in e \sqcap x$, *in particular* $e \wedge x$ *exists in* $\mathcal{M}$ *and* $e \wedge x = e \odot x$,
(v) *for every $e, f \in C(M), e \wedge f, e \vee f$ exist and belong to $C(M)$. Moreover, $e \wedge f = e \odot f \in C(M)$ and $e \vee f = e^{'} \to f$,*
(vi) *for every $e \in M, e \in C(M)$ if and only if $e \vee e^{'} = \top$.*

2. (f,g)-DERIVATIONS ON RESIDUATED MULTILATTICES

This section, introduces (f,g)-derivation on residuated multilattice and investigate some fundamental properties.

Definition 2.1. *Let $\mathcal{M}$ be a residuated multilattice. The map $d : \mathcal{M} \to \mathcal{M}$ is called a (f,g)-derivation on $\mathcal{M}$ for the given two homorphisms $f, g : \mathcal{M} \to \mathcal{M}$ if $d(x \odot y) \in (d(x) \odot f(y)) \sqcup (g(x) \odot d(y))$ for all $x, y \in M$.*
If d satisfies the identity $d(x \odot y) \in (d(x) \odot g(y)) \sqcup (f(x) \odot d(y))$ for all $x, y \in M$, then d is called (g,f)-derivation on $\mathcal{M}$. Moreover, if d is both an (f,g) and (g,f)-derivation, it is called a $\{f,g\}$-derivation on $\mathcal{M}$.

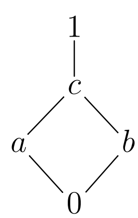
Remark 2.2.
- *It is obvious that, (f, f)-derivation coincide with f-derivation defined by Maffeu in [9],*
- *If we choose the homomorphism f and g identity id_M, then we obtain multiplicative derivation defined by Maffeu in[8].*

Example 2.3. *Let $\mathcal{M}$ be a residuated multillatice. We define the homomorphism $f$and g by $f(x) = \top$ and $g(x) = \top$ for all $x \in M$.*

- *The map $d : \mathcal{M} \to \mathcal{M}$ defined by $d(x) = \bot$ for all $x \in M$, then d is a (f, g)-derivation on $\mathcal{M}$ which is called a bottom (f, g)-derivation,*
- *The map $d_{id_M} : \mathcal{M} \to \mathcal{M}$ defined by $d_{id_M}(x) = x$ for all $x \in M$, then d_{id_M} is a (f, g)-derivation on $\mathcal{M}$, wich is called an identity (f, g)-derivation.*

We apply the ordinal sum to construct a new example of a residuated multilattice using the procedure define by Maffeu in [10]

Example 2.4. *We construct a new residuated multilattice as the ordinal sum of RML_7 and the residuated lattice with underlying set $\{L, 0, a, b, c, 1\}$ such that the order on L is given by the following Hasse diagram:*



The operations $\odot$ and $\to$ on L are defined by the following tables:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

We rename the elements in the ordinal sum as: $\bot, a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, \top$. we obtain the residuated multilattice where the order are define as follows: $\bot \leq a_0, a_1 \leq a_2, a_3 \leq a_4 \leq b_0 \leq b_1, b_2 \leq b_3 \leq \top$.

The operations $\odot$ and $\to$ on M are defined by:

$$x \odot y = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \text{ or } (x = a_0 \text{ and } y \in A) \text{ or } (y = a_0 \text{ and } x \in A) \\ a_1 & \text{if } x \in A \backslash \{a_0\} \text{ and } y \in A \backslash \{a_4\} \\ x & \text{if } (x \in A \text{ and } y \in B) \text{ or } y = 1 \text{ or } (x \in \{b_1; b_2\} \text{ and } y \in \{x; b_3\}) \\ & \text{or } (x = b_0 \text{ and } y \in B) \text{ or } x = y = b_3 \\ y & \text{if } (x \in B \text{ and } y \in B) \text{ or } x = 1 \text{ or } (y \in \{b_1; b_2\} \text{ and } x \in \{y; b_3\}) \\ & \text{or } (y = b_0 \text{ and } x \in B) \\ b_0 & \text{if } (x = b_1 \text{ and } y = b_2) \text{ or } (x = b_2 \text{ and } y = b_1) \end{cases}$$

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x \in B \text{ and } A \backslash \{a_0\} \text{ or } x = 1 \text{ or } (x = b_3 \text{ and } x \in B \backslash \{b_3\}) \\ a_0 & \text{if } x \in A \backslash \{a_0\} \text{ and } y \in \{0; a_0\} \\ b_2 & \text{if } x \in b_1 \text{ and } y \in \{b_0; b_2\} \\ b_1 & \text{if } x \in b_2 \text{ and } y \in \{b_0; b_1\} \\ a_4 & \text{otherwise} \end{cases}$$

We define the homomorphism f, g and the map d on M by:

$$f(x) = \begin{cases} b_1 & \text{if } x = b_2 \\ b_2 & \text{if } x = b_1 \\ x & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} b_1 & \text{if } x = b_2 \\ b_2 & \text{if } x = b_1 \\ \top & \text{if } x = b_3 \\ x & \text{otherwise} \end{cases}$$

$$d_1(x) = \begin{cases} a_1 & \text{if } x \in \{a_1, a_2, a_4\} \\ \bot & \text{otherwise} \end{cases} \qquad d_2(x) = \begin{cases} a_0 & \text{if } x \in \{b_0, b_2\} \\ a_1 & \text{if } x \in \{a_1, a_2, a_4\} \\ \bot & \text{otherwise} \end{cases} .$$

It is easy to verify that d_1 and d_2 are both (f,g) and (g,f)-derivation on $\mathcal{M}$, So d_1 and d_2 are $\{f,g\}$-derivations.

Example 2.5. *We construct a new residuated multilattice as the ordinal sum of RML_7 and the residuated lattice with underlying set $\{L, 0, a, b, c, 1\}$ is the chain. The operations $\odot$ and $\to$ on L are defined by the following tables:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	a	a	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	a	c	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

We rename the elements in the ordinal sum as: $0, a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, 1$. we obtain the residuated multilattice such the order are define as follows: $\bot \leq a_0 \leq a_1 \leq a_2 \leq a_3 \leq b_0, b_1 \leq b_2, b_3 \leq b_4 \leq \top$.

The operations $\odot$ and $\to$ on M are defined by :

$$x \odot y = \begin{cases} \bot & \text{if } x = \bot \text{ or } y = \bot \\ x & \text{if } x \in A \text{ and } y \in B \cup \{a_3\} \text{ or } y = \top \\ y & \text{if } x \in B \cup \{a_3\} \text{ and } y \in A \backslash \{a_3\} \text{ or } x = \top \\ b_1 & \text{if } x \in B \backslash \{b_0\} \text{ and } y \in B \backslash \{b_0\} \\ a_3 & \text{if } x \in B \text{ and } y \in \{a_3; b_0\} \text{ or } y = b_0 \text{and } y \in B \\ a_0 & \text{if } x \in \{a_0; a_1\} \text{ and } y \in \{a_0; a_1; a_2\} \text{ or } x = a_2 \text{and } y \in \{a_0; a_1\} \\ a_2 & x = y = a_2 \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \leq y \\ \bot & \text{if } y = \bot \\ y & \text{if } x \in B \cup \{a_1; a_2\} \text{ and } y \in \{a_0; a_1; a_2\} \text{ or } x = \top \\ b_0 & \text{if } x \in B \text{ and } y \in \{0_3; b_0\} \\ a_2 & \text{if } x = a_1 \text{ and } y = a_0 \\ b_4 & \text{otherwise} \end{cases}.$$

Now, we define the homomorphism f, g and the map d on M by:

$$f(x) = \begin{cases} \top & \text{if } x \in \{\top, a_3\} \cup \{b_i, 0 \leq i \leq 4\} \\ a_0 & \text{if } x \in \{a_0, a_1, a_2\} \\ \bot & \text{if } x = \bot \end{cases}$$

$$g(x) = \begin{cases} \bot & \text{if } x = \bot \\ a_0 & \text{if } x \in \{a_0, a_1\} \\ a_2 & \text{if } x = a_2 \\ \top & \text{if } x \in \{\top, a_3\} \cup \{b_i, 0 \leq i \leq 4\} \end{cases}$$

$$d(x) = \begin{cases} \bot & \text{if } x = \bot \\ a_0 & \text{if } x = a_0 \\ a_2 & \text{if } x \in \{a_1, a_2, a_3\} \\ a_1 & \text{if } x \in \{\top\} \cup \{b_i, 0 \leq i \leq 4\} \end{cases}.$$

d is a (f, g)-derivation, since $d(a_1 \odot a_2) = d(a_0) = a_0$ and $(d(a_1) \odot f(a_2)) \sqcup (g(a_1) \odot d(a_2)) = (a_2 \odot a_0) \sqcup (a_0 \odot a_2) = a_0 \sqcup a_0 = a_0$.
But d is not a (g, f)-derivation, since $(d(a_1) \odot g(a_2)) \sqcup (f(a_1) \odot d(a_2)) = (a_2 \odot a_2) \sqcup (a_0 \odot a_2) = a_2 \sqcup a_0 = a_2 \neq a_0$.

Example 2.6. *Let f and g be the homomorphisms define in Example 2.4 and $d : \mathcal{M} \to \mathcal{M}$ by $d(x) = \top$ for all $x \in M$. d is not a (f, g)-derivation, since $d(a_0 \odot a_1) = \top$ but $(d(a_0) \odot f(a_1)) \sqcup (g(a_0) \odot d(a_1)) = (\top \odot a_1) \sqcup (a_0 \odot \top) = a_1 \sqcup a_0 = a_1$.*
d is not a (g, f)-derivation, since $d(a_0 \odot a_1) = \top$ but $(d(a_0) \odot g(a_1)) \sqcup (f(a_0) \odot d(a_1)) = (\top \odot a_1) \sqcup (a_0 \odot \top) = a_1 \sqcup a_0 = a_1$.

Proposition 2.7. *Let $\mathcal{M}$ be a residuated multilattice and d be a (f, g)-derivation on $\mathcal{M}$. For all $x, y \in M$, we have the following statements:*

1) $d(x) \geq g(x) \odot d(\top)$,
2) *If* $d(\bot) = \bot$ *and* $x \leq y^{'}$, *then* $d(y) \leq (g(x))^{'}$ *and* $d(x) \leq (f(y))^{'}$,
3) *If* $d(\bot) = \bot$, *then* $d(x^{'}) \leq (f(x))^{'}$ *and* $d(x) \leq (f(x^{'}))^{'}$,
4) *If* $f(\bot) = \bot$, *then* $d(x) \leq (f(x))^{''}$.

Proof.

1) We have $d(x) = d(x \odot \top) \in (d(x) \odot f(\top)) \sqcup (g(x) \odot d(\top)) = d(x) \sqcup (g(x) \odot d(\top))$because $f(\top) = \top$, hence $d(x) \geq g(x) \odot d(\top)$.
2) We have $x \leq y^{'}$ implies $x \odot y = \bot$, thus $\bot = d(\bot) = d(x \odot y) \in (d(x) \odot f(y)) \sqcup (g(x) \odot d(y))$, that is $d(x) \odot f(y) = g(x) \odot d(y) = \bot$, hence $d(y) \leq (g(x))^{'}$ and $d(x) \leq (f(y))^{'}$.
3) $x \odot x^{'} = \bot$, thus $\bot = d(\bot) = d(x \odot x^{'}) \in (d(x) \odot f(x^{'})) \sqcup (g(x) \odot d(x^{'}))$, that is $d(x) \odot f(x^{'}) = g(x) \odot d(x^{'}) = \bot$, hence $d(x^{'}) \leq (g(x))^{'}$ and $d(x) \leq (f(x^{'}))^{'}$.
4) Since $f(\bot) = \bot$, then $f(x^{'}) = f(x \to \bot) = f(x) \to f(\bot) = f(x) \to \bot = (f(x))^{'}$. Therefore by 4) we have $d(x) \leq (f(x))^{''}$.

□

Proposition 2.8. *Let* $\mathcal{M}$ *be a residuated multilattice and* d *be a* $\{f,g\}$-*derivation on* $\mathcal{M}$. *We have* $d(\bot) \in d(\bot) \odot (f(\bot) \sqcup g(\bot))$, *futhermore if* $f(\bot) = \bot$ *and* $g(\bot) = \bot$, *then* $d(\bot) = \bot$.

Proof. We have $d(\bot \odot \bot) = d(\bot)$, futhermore $d(\bot \odot \bot) \in (d(\bot) \odot f(\bot)) \sqcup (g(\bot) \odot d(\bot)) = (d(\bot) \odot f(\bot)) \sqcup (d(\bot) \odot g(\bot)) \subseteq d(\bot) \odot (f(\bot) \sqcup g(\bot))$ by proposition 1.9 part (2), thus $d(\bot) = d(\bot \odot \bot) \in d(\bot) \odot (f(\bot) \sqcup g(\bot))$, hence $d(\bot) \in d(\bot) \odot f(\bot) \sqcup g(\bot)$. Moreover if $f(\bot) = \bot$ and $g(\bot) = \bot$ we conclude that $d(\bot) = \bot$. □

Definition 2.9. *Let* $\mathcal{M}$ *be a residuated multilattice and* d *be an* (f,g)-*derivation on* $\mathcal{M}$.

1) d *is a* f-*contractive* (f,g)-*derivation if* $d(x) \leq f(x)$ *for all* $x \in M$ *and* g-*contractive* (f,g)-*derivation if* $d(x) \leq g(x)$ *for all* $x \in M$. d *is called contractive* (f,g)-*derivation if* d *is both* f-*contractive and* g-*contractive.*
2) d *is an ideal* (f,g)-*derivation if* d *is both monotone and contractive.*

In particular, if d *is both monotone and* f-*contractive, we call* d *an* f-*ideal* (f,g)-*derivation.*

Example 2.10. *Let* $\mathcal{M}$ *be a residuated multilattice depicted in Example 2.4. We can oberve that* d_1 *and* d_2 *are* (f,g)-*contractive.*

Example 2.11. *Let $\mathcal{M}$ be a residuated multilattice depicted in Example 2.5. We can oberve that d is not f-contractive since $d(a_2) = a_2$ and $f(a_2) = a_0$ and $a_2 \nleq a_0$ but d is g-contractive.*

Example 2.12. *Let $\mathcal{M}$ be a residuated multilattice depicted in Example 2.4. We can oberve that a bottom (f,g)-derivation and identity (f,g)-derivation are ideal (f,g)-derivation. Furthermore it can be easy to verify that d_1 and d_2 are not ideal (f,g)-derivation on $\mathcal{M}$, because d_1 and d_2 are not monotone.*

Proposition 2.13. *Let $\mathcal{M}$ be a residuated multilattice and d be a monotone (f,g)-derivation on $\mathcal{M}$. We have the following statements:*

1) *If $z \leq x \to y$ then $f(z) \leq d(x) \to d(y)$ and $g(x) \leq d(z) \to d(y)$ for all $x, y, z \in M$,*
2) *$f(x \to y) \leq d(x) \to d(y), d(x \to y) \leq g(x) \to d(y)$ for all $x, y, z \in M$.*

Proof.

1) Let $x, y, z \in M, z \leq x \to y$ implies $x \odot z \leq y$ by the monotonicity of d, we have $d(x \odot z) \leq d(y)$ since $d(x \odot z) \in (d(x) \odot f(z)) \sqcup (g(x) \odot d(z))$, we obtain that $d(x) \odot f(z), g(x) \odot d(z) \leq d(x \odot z) \leq d(y)$. Hence by the adjointness conditions we have $f(z) \leq d(x) \to d(y)$ and $g(x) \leq d(z) \to d(y)$.
2) Let $x, y, z \in M$. Using the inequality $x \odot (x \to y) \leq y$, we obtain $f(x \to y) \leq d(x) \to d(y), d(x \to y) \leq g(x) \to d(y)$.

□

Proposition 2.14. *Let $\mathcal{M}$ be a residuated multilattice and d be a contractive (f,g)-derivation on $\mathcal{M}$. We have $d(x) \odot d(y) \leq d(x \odot y)$ for all $x, y \in M$.*

Proof. Let $x, y \in M$, conbining the fact that d is contractive and by using Proposition 1.5 (P_3), we have $d(x) \odot d(y) \leq d(x) \odot f(y)$ and $d(x) \odot d(y) \leq g(x) \odot d(y)$, thus $d(x) \odot d(y) \leq a$ with $a \in (d(x) \odot f(y)) \sqcup (g(x) \odot d(y))$, hence $d(x) \odot d(y) \leq d(x \odot y)$. □

Proposition 2.15. *Let $\mathcal{M}$ be a residuated multilattice and d be a (f,g)-derivation on $\mathcal{M}$. We have the following statements:*

1) *Let $x, y \in M$, for all $b \in d(x) \sqcup d(y)$, there exist $a \in (d(x) \odot f(y)) \sqcup (g(x) \odot d(y))$ such that $a \leq b$,*
2) *If d is monotone and f-contractive, then $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to f(y)$ for all $x, y \in M$,*
3) *If for all $x \in M$, $f(x) \odot d(\top) = d(\top)$, then $d(x) \leq d(\top)$.*

Proof.

1) Let $x, y \in M$ and $b \in d(x) \sqcup d(y)$, we have $d(x) \odot f(y) \leq d(x)$ and $g(x) \odot d(y) \leq d(y)$, therefore there exist $a \in (d(x) \odot f(y)) \sqcup (g(x) \odot d(y))$ such that $a \leq b$.
2) We have $x \odot (x \to y) \leq y$ by P_2 from Proposition 1.5, since d is monotone we have $d(x \odot (x \to y)) \leq d(y)$, hence by the adjointness conditions we have $d(x \to y) \leq d(x) \to d(y)$. Futhermore we have $d(y) \leq f(y)$ because d is $(f, g)-$ contractive and by P_3 we have $d(x) \to d(y) \leq d(x) \to f(y)$.
3) We have $d(x) = d(x \odot \top) \in (d(x) \odot f(\top)) \sqcup (g(x) \odot d(\top)) = (d(x) \odot f(\top)) \sqcup d(\top) = d(x) \odot d(\top)$ because $f(\top) = \top$ and $g(x) \odot d(\top) = d(\top)$.

□

Proposition 2.16. *Let $\mathcal{M}$ be a residuated multilattice, d be an f-contractive (f, g)-derivation on $\mathcal{M}$. We have the following properties:*

1) *Let $x, y \in M$ if $y \leq x$, $FIX_d^{f,g}(M) := \{x \in M : d(x) = f(x) = g(x)\}$ and there exists $u \in M$ such that $x \odot u = y$, then $d(y) = f(y)$,*
2) *$FIX_d^{f,g}(M) := \{x \in M : d(x) = f(x) = g(x)\}$ is closed under $\odot$,*
3) *$d(\top) = \top$ if and only if $FIX_d^{f,-}(M) = M$.*

Proof.

1) Let $x, y \in M$, we have $y \leq x$ and $u \in M$ such that $x \odot u = y$. By the Definition of (f, g)-derivation, we have $d(y) = d(x \odot u) \in (d(x) \odot f(u)) \sqcup (g(x) \odot d(u)) = (f(x) \odot f(u)) \sqcup (d(x) \odot d(u)) = f(x) \odot f(u)$ because d is f-contractive, $d(x) = f(x); d(x) = g(x)$; thus $d(y) \in f(x) \odot f(u) = f(x \odot u) = f(y)$.
2) Let $x, y \in FIX_d^{f,g}(M)$, By the definition of (f, g)-derivation, we have $d(x \odot y) \in (d(x) \odot f(y)) \sqcup (g(x) \odot d(y)) = (f(x) \odot f(y)) \sqcup (g(x) \odot f(y)) = \{f(x) \odot f(y)\} = \{f(x \odot y)\}$, hence $FIX_d^{f,g}(M)$ is closed under $\odot$,
3) Let $x \in M$, we have $d(x) = d(\top \odot x) \in (d(\top) \odot f(x)) \sqcup (g(\top) \odot d(x)) = d(x) \sqcup d(x) = \{d(x)\}$ because $d(\top) = \top, g(\top) = \top$ and $d(x) = f(x)$, hence $FIX_d^{f,-}(M) = M$. It is obvious that if $FIX_d^{f,-}(M) = M$ implies $d(\top) = \top$.

□

Remark 2.17.

1) $FIX_d^{f,g}(M) = FIX_d^{f,-}(M) \cap FIX_d^{g,-}(M)$
2) $d(\top) = \top$ *if and only if* $FIX_d^{g,-}(M) = M$.

Remark 2.18.

1) *Let $\mathcal{M}$ be a residuated multilattice depicted in Example 2.5. We can observe that $d(a_1 \odot a_3) = d(a_1) = a_2$, $f(a_1 \odot a_3) = f(a_1) = a_0$, thus $a_2 \neq a_0$. Hence $FIX_d^{f,-}(M) := \{x \in M : d(x) = f(x)\}$ is not closed under $\odot$.*
2) *Let $\mathcal{M}$ be a residuated multilattice depicted in Example 2.4. We can observe that $FIX_{d_1}^{f,-}(M) = FIX_{d_1}^{g,-}(M) = \{\perp, a_1\}$*
3) *Let $\mathcal{M}$ be a residuated multilattice depicted in Example 2.5. we can observe that $FIX_d^{f,-}(M) \neq FIX_d^{g,-}(M)$ since $FIX_d^{f,-}(M) = \{\perp, a_0\}$ and $FIX_d^{g,-}(M) = \{\perp, a_0, a_2\}$.*

Definition 2.19. [2] *Let L be a lattice. A map $d : L \to L$ is called generalized (f, g)−derivation on L; for the given two homomorphisms $f, g : L \to L$ if*

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (g(x) \wedge d(y)) \text{ for all } x, y \in L.$$

Proposition 2.20. *Let $\mathcal{M}$ be a residuated multillatice, d be an (f, g)-derivation on $C(M)$ and either $(f(x) \geq g(x)$ or $g(x) \geq f(x))$ for all $x \in M$. We have the following properties:*

1) *$d(x) = d(x) \odot f(x)$ or $d(x) = d(x) \odot g(x)$ for all $x \in C(M)$*
2) *If $d(\top) = \top$, then the definition of (f, g)-derivation in the set of all complemented elements $C(M)$ of a residuated multilattice $\mathcal{M}$ coincides with the definition of generalized (f, g)-derivation in lattice.*

Proof.

1) By Proposition 1.12 *iii*) we have $x \odot x = x$. By definition of (f, g)-derivation and Proposition 1.9 (M_2) we have $d(x) \in (d(x) \odot f(x)) \sqcup (g(x) \odot d(x)) = (d(x) \odot f(x)) \sqcup (d(x) \odot g(x)) \subseteq d(x) \odot (f(x) \sqcup g(x))$. Then $d(x) \in d(x) \odot (f(x) \sqcup g(x))$ and by the hypothesis we obtain $f(x) \sqcup g(x) = f(x)$ or $f(x) \sqcup g(x) = g(x)$, hence $d(x) = d(x) \odot f(x)$ or $d(x) = d(x) \odot g(x)$ for all $x \in \mathcal{M}$,
2) In fact we prove that either $f(C(M)) \subseteq C(M)$ or $g(C(M)) \subseteq C(M)$. By *vi* of Proposition 1.12 we need only to prove that either $f(x) \sqcup f(x)^* = \top$ or $g(x) \sqcup g(x)^* = \top$. We have $f(x) \sqcup f(x)^* \subseteq f(x \sqcup x^*) = f(\top) = \top$. The other equality is obtained similarly.
   Since $d(\top) = \top$, $x \odot \perp = \perp \odot x$ and the fact that d is a (f, g)-derivation, we have $d(x) \in d(x) \sqcup f(x)$ and $d(x) \in d(x) \sqcup g(x)$, thus $f(x) \leq d(x)$ and $g(x) \leq d(x)$, furthermore we have $d(x) = d(x) \odot f(x)$ or $d(x) = d(x) \odot g(x)$ by 1), that is $d(x) \leq f(x)$ and $d(x) \leq g(x)$, hence $d(x) = f(x) \in C(\mathcal{M})$ and $d(x) = g(x) \in C(\mathcal{M})$.

$d(x\odot y)=(d(x)\odot f(y))\sqcup(g(x)\odot d(y))=(d(x)\wedge f(y))\sqcup(g(x)\wedge d(y))=(d(x)\wedge f(y))\vee(g(x)\wedge d(y))=d(x\wedge y)$.

□

Proposition 2.21. *Let $\mathcal{M}$ be a residuated multilattice and d be an f-contractive (f,g)-derivation on $\mathcal{M}$ with $d(x)\in C(M)$ for all $x\in M$. The following assertions are equivalent:*

1) *d is an ideal (f,g)-derivation,*
2) *$d(x)\leq d(\top)$ for all $x\in M$,*
3) *For all $x\in M, d(x)=f(x)\odot d(\top)$,*
4) *For all $x,y\in M$, if $a\in x\sqcup y$, then there exists $z\in d(x)\sqcup d(y)$ such that $d(a)\leq z$.*

Proof. 1) $\Rightarrow$ 2) Let $x\in M$, we have $x\leq\top$ by the hypothesis d is monotone, thus $d(x)\leq d(\top)$,
2) $\Rightarrow$ 3) let $x\in M$, by the hypothesis we have $d(x)\leq d(\top)$, thus $d(x)=d(x)\wedge d(\top)=d(x)\odot d(\top)$ by Proposition 1.12 iii), since d is f-contractive and $\odot$ is monotone, we have $d(x)\leq f(x)$, thus $d(x)\odot d(\top)\leq f(x)\odot d(\top)$ that is $d(x)\leq f(x)\odot d(\top)$. Furthermore $d(x)\geq d(\top)\odot f(x)$ by Proposition 2.7 2), hence $d(x)=f(x)\odot d(\top)$.
3) $\Rightarrow$ 4) Let $x,y\in M$ and $a\in x\sqcap y$; from 3) we have $d(a)=d(\top)\odot f(a)$, furthermore $d(\top)\odot f(a)\in d(\top)\odot[f(x)\sqcap f(y)]\subseteq\downarrow_M[(d(\top)\odot f(x))\sqcap(d(\top)\odot f(y))]=\downarrow_M[d(x)\sqcap d(y)]$, so there exists $z\in d(x)\sqcap d(y)$ such that $d(1)\odot f(a)=d(a)\leq z$.
4) $\Rightarrow$ 1) Let $x,y\in M$ such that $x\leq y$. We know that $x\leq x$, thus $x\in x\sqcap y$, by the hypothesis there exists $z\in d(x)\sqcap d(y)$ such that $d(x)\leq z$, hence $d(x)\leq d(y)$. □

Proposition 2.22. *Let $\mathcal{M}$ be a residuated multilattice and d be an contractive (f,g)-derivation on $\mathcal{M}$ with $d(x)\in C(M)$. The following assertions are equivalent:*

1) *d is a monotone (f,g)-derivation,*
2) *For all $x,y\in M$, $d(x\odot y)=d(x)\odot d(y)=d(x)\odot f(y)=g(x)\odot d(y)$.*

Proof. 1) $\Rightarrow$ 2) Let $x,y\in M$ by Proposition 2.15 1) we have $d(x)\odot d(y)\leq d(x\odot y)$, moreover we know that $x\odot y\leq x,y$ using the hypothesis we have $d(x\odot y)\leq d(x),d(y)$, by the definition of multilattice there exists $z\in d(x)\sqcap d(y)$ such that $d(x\odot y)\leq z$. As $d(x),d(y)\in C(M)$ by Proposition 1.12 (iii), we obtain $d(x)\sqcap d(y)=d(x)\wedge d(y)=d(x)\odot d(y)$ that is $d(x\odot y)\leq d(x)\odot d(y)$, hence $d(x\odot y)=d(x)\odot d(y)$.
Furthermore $d(x\odot y)=d(x)\odot d(y)\in(d(x)\odot f(y))\sqcup(g(x)\odot d(y))$,

thus $d(x) \odot f(y), g(x) \odot d(y) \leq d(x) \odot d(y)$. Since d is (f, g)-contractive we have $d(x) \leq g(x)$ and $d(y) \leq f(y)$. since d is monotone we have $d(x) \odot d(y) \leq d(x) \odot f(y)$ and $d(x) \odot d(y) \leq g(x) \odot d(y)$. we conclude that $d(x \odot y) = d(x) \odot d(y) = d(x) \odot f(y) = g(x) \odot d(y)$.
2) $\Rightarrow$ 1) Let $x, y \in M$ such that $x \leq y$. we know that $x \leq x$, by the definition of multilattice there exists $a \in x \sqcap y$ such that $a \leq x$. By the hypothesis 2) we have $d(x \odot y) = d(x) \odot d(y) = d(x) \odot f(y) = g(x) \odot f(y)$, for all $x, y \in M$, thus $d(a) = d(\top) \odot f(a) \in d(\top) \odot (f(x) \sqcap f(y))$. Furthermore $d(\top) \odot (f(x) \sqcap f(y)) \subseteq\downarrow_M [(d(\top) \odot f(x)) \sqcap (d(\top) \odot f(y))] =\downarrow_M [d(x) \sqcap d(y)]$, so there exists $z \in d(x) \sqcap d(y)$ such that $d(a) = d(\top) \odot f(a) \leq z$. In particular, for $x \leq y$ we have $d(x) = d(\top) \odot f(x) = d(\top) \odot g(x) \leq z$ where $z \in d(x) \sqcap d(y)$, hence $d(x) \leq d(y)$. □

3. CONCLUSION

In this paper, we introduced the concept of (f, g)-derivation on residuated multilattice and investigated some fundamnental properties. We observed that (f, f)-derivation are exatly f-derivation defined in [8] by Maffeu.

Declaration

This article does not use any particular data, or human participant! Indeed, the results obtained have been established from the articles cited in the references! However, we remain ready to transmit any information useful for a good understanding of our article.

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