N-Fuzzy Sets and Their Applications

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N-FUZZY SETS AND THEIR APPLICATIONS

HARIWAN Z. IBRAHIM

ABSTRACT. The purpose of this paper is to define n-Fuzzy sets and study their relationship with intuitionistic fuzzy sets, Pythagorean fuzzy sets and Fermatean fuzzy sets. The n-Fuzzy sets can deal with more uncertain situations than intuitionistic fuzzy sets, Pythagorean fuzzy sets and Fermatean fuzzy sets because of its larger range of describing the membership grades. The set operations, score function, accuracy function and Euclidean distance of n-Fuzzy sets will study. Finally, we study the Sanchez's approach for medical diagnosis and extend this concept with the notion of n-Fuzzy set.

1. Introduction

Many terms that we use randomly in everyday life usually have a fuzzy structure. Verbal or numerical expressions we use, while describing something, explaining an event, commanding and in many other cases include fuzziness. After the introduction of the concept of Fuzzy sets by Zadeh [17] several researches were conducted on the generalizations of the notion of fuzzy set. With the emergence of Fuzzy set theory, science and technology have made great improvement. It is seen to have many applications related to the Fuzzy set theory in both theoretical and practical studies from health sciences to computer science, from physical sciences to arts, and from engineering and humanities to life sciences. For theoretical study examples, refer to the [2, 15, 8, 18, 19, 20]. Fuzzy sets were further generalized to intuitionistic fuzzy sets by Atanassov [1]. The idea of intuitionistic fuzzy set seems to be resourceful in modeling many real-life situations like multi-criteria decision-making [3, 4, 5]. Applications of intuitionistic fuzzy set appear in various fields, including medical diagnosis, optimization problems, and decision-making. Yager [7] offered a new Fuzzy set called Pythagorean fuzzy sets, which is the generalization of intuitionistic fuzzy sets. The construct of Pythagorean fuzzy sets can be used to characterize uncertain information more sufficiently and accurately than intuitionistic fuzzy set. Garg [6] presented an improved score function for the ranking order of interval-valued Pythagorean fuzzy sets. Fermatean fuzzy sets introduced by Senapati et al. [9], and they also defined basic operations over the Fermatean fuzzy sets. The main advantage of Fermatean fuzzy sets is that it can describe more

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uncertainties than Pythagorean fuzzy sets, which can be applied in many decision-making problems. The relevant research can be referred to [13, 14]. The aim of this paper is to introduce n-Fuzzy sets and compare n-Fuzzy sets with intuitionistic fuzzy sets, Pythagorean fuzzy sets and Fermatean fuzzy sets. We focus on complement operator of n-Fuzzy sets. We find out the fundamental set of operations for the n-Fuzzy sets. We define score function and accuracy function for ranking of n-Fuzzy sets. In addition, we also study Euclidean distance between two n-Fuzzy sets. Finally, we propose an improved version of max-min-max composite relation for n-Fuzzy sets with application to medical diagnosis. In the future works, more applications of n-Fuzzy sets may be studied, and introduce n-Fuzzy topological spaces, n-Fuzzy nano topological spaces and n-Fuzzy soft topological spaces may be studied. And also operation on n-Fuzzy topological spaces may be studied same as in [10, 11, 16]. The motivation and objectives of this extended work are given step by step in the whole manuscript. The n-Fuzzy set is most generalized form and use to collect data at a large scale and applicable in medical, engineering, artificial intelligence, agriculture, and other daily life problems. In future, this work can be gone easily for other approaches and different types of fuzzy sets.

Definition 1.1. [1] The intuitionistic fuzzy sets are defined on a non-empty set X as objects having the form $I = \{\langle x, \alpha_I(x), \beta_I(x) \rangle : x \in X\}$, where $\alpha_I(x) : X \rightarrow [0, 1]$ and $\beta_I(x) : X \rightarrow [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set I , respectively, and $0 \leq \alpha_I(x) + \beta_I(x) \leq 1$, for all $x \in X$.

Definition 1.2. [7] The Pythagorean fuzzy sets are defined on a non-empty set X as objects having the form $P = \{\langle x, \alpha_P(x), \beta_P(x) \rangle : x \in X\}$, where $\alpha_P(x) : X \rightarrow [0, 1]$ and $\beta_P(x) : X \rightarrow [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set P , respectively, and $0 \leq (\alpha_P(x))^2 + (\beta_P(x))^2 \leq 1$, for all $x \in X$.

Definition 1.3. [9] The Fermatean fuzzy sets are defined on a non-empty set X as objects having the form $F = \{\langle x, \alpha_F(x), \beta_F(x) \rangle : x \in X\}$, where $\alpha_F(x) : X \rightarrow [0, 1]$ and $\beta_F(x) : X \rightarrow [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set F , respectively, and $0 \leq (\alpha_F(x))^3 + (\beta_F(x))^3 \leq 1$, for all $x \in X$.

2. n-Fuzzy Sets

In this section, we introduce the concept of n-Fuzzy sets and discuss its relationship with other types of fuzzy sets. We describe some of the basic set operations on n-Fuzzy sets.

Definition 2.1. Let X be a universal set. Then, the n-Fuzzy set (n-FS) N , which is a set of ordered pairs over X , is defined by the following:

$$N = \{\langle x, \delta_N(x), \theta_N(x) \rangle : x \in X\},$$

where the functions $\delta_N(x) : X \rightarrow [0, 1]$ and $\theta_N(x) : X \rightarrow [0, 1]$ denote the degree of membership and the degree of non-membership, respectively, of the element $x \in X$ to N , and including the condition

$$0 \leq (\delta_N(x))^n + (\theta_N(x))^n \leq 1,$$

for all $x \in X$ and for some natural numbers n . Then, there is a degree of indeterminacy of $x \in X$ to N defined by

$$\pi_N(x) = \sqrt[n]{1 - [(\delta_N(x))^n + (\theta_N(x))^n]} \text{ and } \pi_N(x) \in [0, 1].$$

In what follows, $(\delta_N(x))^n + (\theta_N(x))^n + (\pi_N(x))^n = 1$. Otherwise, $\pi_N(x) = 0$ whenever $(\delta_N(x))^n + (\theta_N(x))^n = 1$.

In the interest of simplicity, we shall mention the symbol $N = (\delta_N, \theta_N)$ for the n-FS $N = \{\langle x, \delta_N(x), \theta_N(x) \rangle : x \in X\}$.

Example 2.2. Suppose that $\delta_N(x) = 0.9$ and $\theta_N(x) = 0.7$ for $X = \{x\}$. Clearly, $0.9 + 0.7 = 1.6 > 1$, $(0.9)^2 + (0.7)^2 = 1.3 > 1$ and $(0.9)^3 + (0.7)^3 = 1.072 > 1$ but $(0.9)^4 + (0.7)^4 = 0.8962 \leq 1$. Thus, $(\pi_N(x))^4 = 0.1038$, and hence, $(\delta_N(x))^4 + (\theta_N(x))^4 + (\pi_N(x))^4 = 1$. Therefore, $N = (0.9, 0.7)$ is n-FS, for $n = 4, 5, 6, 7, 8, \dots$

Theorem 2.3. The set of n-Fuzzy membership grades ($n \geq 4$) is larger than the set of intuitionistic membership grades, Pythagorean membership grades and Fermatean membership grades.

Proof. Since for any two numbers $m, r \in [0, 1]$, we have

$$m^n \leq m^{n-1} \leq \dots \leq m^3 \leq m^2 \leq m$$

and

$$r^n \leq r^{n-1} \leq \dots \leq r^3 \leq r^2 \leq r.$$

Therefore, we get

$$\begin{aligned} m + r &\leq 1 \\ \Rightarrow m^2 + r^2 &\leq 1 \\ \Rightarrow m^3 + r^3 &\leq 1 \\ \Rightarrow m^4 + r^4 &\leq 1 \\ &\dots \\ &\dots \\ &\dots \\ \Rightarrow m^{n-1} + r^{n-1} &\leq 1 \\ \Rightarrow m^n + r^n &\leq 1. \end{aligned}$$

Then, the space of n-Fuzzy membership grades is larger than the space of intuitionistic membership grades, Pythagorean membership grades and Fermatean membership grades. This development can be evidently recognized in Figure 1. $\square$

Figure 1: Comparison of grades space of IFSs, FFSs, and n-FSSs

Remark 2.4. Table (*) explains the difference between intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PFSs), Fermatean fuzzy sets (FFSs) and n-Fuzzy Sets (n-FSSs).

Table (*) IFSs, PFSs, FFSs and n-FSs

IFSs or 1-FSs	PFSs or 2-FSs	FFSs or 3-FSs	n-FSs, for $n \geq 4$
			$\delta + \theta \leq 1$ or $\delta + \theta \geq 1$ or $\delta^2 + \theta^2 \leq 1$ or $\delta^2 + \theta^2 \geq 1$ or $\delta^3 + \theta^3 \leq 1$ or $\delta^3 + \theta^3 \geq 1$
$\delta + \theta \leq 1$	$\delta + \theta \leq 1$ or $\delta + \theta \geq 1$	$\delta + \theta \leq 1$ or $\delta + \theta \geq 1$ or $\delta^2 + \theta^2 \leq 1$ or $\delta^2 + \theta^2 \geq 1$	
$0 \leq \delta + \theta \leq 1$	$0 \leq \delta^2 + \theta^2 \leq 1$	$0 \leq \delta^3 + \theta^3 \leq 1$	$0 \leq \delta^n + \theta^n \leq 1$
$\pi = 1 - [\delta + \theta]$	$\pi = \sqrt{1 - [\delta^2 + \theta^2]}$	$\pi = \sqrt[3]{1 - [\delta^3 + \theta^3]}$	$\pi = \sqrt[n]{1 - [\delta^n + \theta^n]}$
$\delta + \theta + \pi = 1$	$\delta^2 + \theta^2 + \pi^2 = 1$	$\delta^3 + \theta^3 + \pi^3 = 1$	$\delta^n + \theta^n + \pi^n = 1$

Remark 2.5. From Definition 2.1, we have the following diagram:

Theorem 2.6. Let $X = \{x_i\}$ be a universal set, for $i = 1, \dots, k$ and N be 4-FS. If $\pi_N(x_i) = 0$, then the following hold:

- (1) $|\delta_N(x_i)| = \sqrt[4]{((\theta_N(x_i))^2 - 1)((\theta_N(x_i))^2 + 1)}$.
- (2) $|\theta_N(x_i)| = \sqrt[4]{((\delta_N(x_i))^2 - 1)((\delta_N(x_i))^2 + 1)}$.

Proof. Let $x_i \in X$ and N be 4-FS. Suppose that that $\pi_N(x_i) = 0$ for $x_i \in X$, then we have the following:

- (1) $(\delta_N(x_i))^4 + (\theta_N(x_i))^4 = 1 \Rightarrow -(\delta_N(x_i))^4 = (\theta_N(x_i))^4 - 1$
 $\Rightarrow -(\delta_N(x_i))^4 = ((\theta_N(x_i))^2)^2 - 1$
 $\Rightarrow -(\delta_N(x_i))^4 = ((\theta_N(x_i))^2 - 1)((\theta_N(x_i))^2 + 1)$
 $\Rightarrow |(\delta_N(x_i))^4| = |((\theta_N(x_i))^2 - 1)((\theta_N(x_i))^2 + 1)|$
 $\Rightarrow |\delta_N(x_i)|^4 = |((\theta_N(x_i))^2 - 1)((\theta_N(x_i))^2 + 1)|$
 $\Rightarrow |\delta_N(x_i)| = \sqrt[4]{|((\theta_N(x_i))^2 - 1)((\theta_N(x_i))^2 + 1)|}.$
- (2) The proof is similar to (1).

□

Example 2.7. Let N be 4-FS and $X = \{x\}$. If $\theta_N(x) = 0.9$, then
 $|\delta_N(x)| = \sqrt[4]{|((\theta_N(x))^2 - 1)((\theta_N(x))^2 + 1)|} = \sqrt[4]{|(-0.19)(1.81)|} = \sqrt[4]{0.3439}.$
 Therefore, $(\delta_N(x_i))^4 + (\theta_N(x_i))^4 = 0.3439 + 0.6561 = 1$ implies $\pi_N(x) = 0$.

Definition 2.8. Let $N = (\delta_N, \theta_N)$, $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be three n-FSs and ζ be a positive real number ($\zeta > 0$), then their operations are defined as follows:

- (1) $N_1 \cap N_2 = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}).$
- (2) $N_1 \cup N_2 = (\max\{\delta_{N_1}, \delta_{N_2}\}, \min\{\theta_{N_1}, \theta_{N_2}\}).$
- (3) $N^c = (\theta_N, \delta_N).$
- (4) $N_1 \oplus N_2 = \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n}, \theta_{N_1} \theta_{N_2} \right).$
- (5) $N_1 \otimes N_2 = \left(\delta_{N_1} \delta_{N_2}, \sqrt[n]{\theta_{N_1}^n + \theta_{N_2}^n - \theta_{N_1}^n \theta_{N_2}^n} \right).$
- (6) $\zeta N = \left(\sqrt[n]{1 - (1 - \delta_N^n)^\zeta}, \theta_N^\zeta \right).$
- (7) $N^\zeta = \left(\delta_N^\zeta, \sqrt[n]{1 - (1 - \theta_N^n)^\zeta} \right).$

Example 2.9. Assume that $N_1 = (\delta_{N_1} = 0.89, \theta_{N_1} = 0.84)$ and $N_2 = (\delta_{N_2} = 0.9, \theta_{N_2} = 0.83)$ are both 5-FSs for $X = \{x\}$. Then,

- (1) $N_1 \cap N_2 = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\})$
 $= (\min\{0.89, 0.9\}, \max\{0.84, 0.83\}) = (0.89, 0.84).$
- (2) $N_1 \cup N_2 = (\max\{\delta_{N_1}, \delta_{N_2}\}, \min\{\theta_{N_1}, \theta_{N_2}\})$
 $= (\max\{0.89, 0.9\}, \min\{0.84, 0.83\}) = (0.9, 0.83).$
- (3) $N_1^c = (0.84, 0.89).$
- (4) $N_1 \oplus N_2 = \left(\sqrt[5]{\delta_{N_1}^5 + \delta_{N_2}^5 - \delta_{N_1}^5 \delta_{N_2}^5}, \theta_{N_1} \theta_{N_2} \right)$
 $= \left(\sqrt[5]{0.89^5 + 0.9^5 - (0.89)^5(0.9)^5}, (0.84)(0.83) \right) \approx (0.9609, 0.6972).$
- (5) $N_1 \otimes N_2 = \left(\delta_{N_1} \delta_{N_2}, \sqrt[5]{\theta_{N_1}^5 + \theta_{N_2}^5 - \theta_{N_1}^5 \theta_{N_2}^5} \right)$
 $= \left((0.89)(0.9), \sqrt[5]{0.84^5 + 0.83^5 - (0.84)^5(0.83)^5} \right) \approx (0.801, 0.917).$
- (6) $\zeta N_1 = \left(\sqrt[5]{1 - (1 - \delta_{N_1}^5)^\zeta}, \theta_{N_1}^\zeta \right) = \left(\sqrt[5]{1 - (1 - 0.89^5)^2}, 0.84^2 \right) \approx (0.96, 0.71),$
 for $\zeta = 2$.

$$(7) \ N_1^\zeta = \left(\delta_{N_1}^\zeta, \sqrt[\zeta]{1 - (1 - \theta_{N_1}^n)^\zeta} \right) = \left(0.89^2, \sqrt[5]{1 - (1 - 0.84^5)^2} \right) \approx (0.79, 0.92),$$

for $\zeta = 2$.

Remark 2.10. It is noticed that $(N^c)^c = N$. This shows the validity of complementary law in n-FS.

Theorem 2.11. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n -FSs, then the following properties hold:

- (1) $N_1 \oplus N_2 = N_2 \oplus N_1$.
- (2) $N_1 \otimes N_2 = N_2 \otimes N_1$.
- (3) $N_1 \cap N_2 = N_2 \cap N_1$.
- (4) $N_1 \cup N_2 = N_2 \cup N_1$.

Proof. From Definition 2.8, we can obtain:

- (1) $N_1 \oplus N_2 = \left(\sqrt[\zeta]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n}, \theta_{N_1} \theta_{N_2} \right)$
 $\left(\sqrt[\zeta]{\delta_{N_2}^n + \delta_{N_1}^n - \delta_{N_2}^n \delta_{N_1}^n}, \theta_{N_2} \theta_{N_1} \right) = N_2 \oplus N_1$.
- (2) $N_1 \otimes N_2 = \left(\delta_{N_1} \delta_{N_2}, \sqrt[\zeta]{\theta_{N_1}^n + \theta_{N_2}^n - \theta_{N_1}^n \theta_{N_2}^n} \right)$
 $= \left(\delta_{N_2} \delta_{N_1}, \sqrt[\zeta]{\theta_{N_2}^n + \theta_{N_1}^n - \theta_{N_2}^n \theta_{N_1}^n} \right) = N_2 \otimes N_1$.
- (3) $N_1 \cap N_2 = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\})$
 $= (\min\{\delta_{N_2}, \delta_{N_1}\}, \max\{\theta_{N_2}, \theta_{N_1}\}) = N_2 \cap N_1$.
- (4) The proof is similar to (3).

□

Theorem 2.12. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n -FSs, then

- (1) $(N_1 \cap N_2) \cup N_2 = N_2$.
- (2) $(N_1 \cup N_2) \cap N_2 = N_2$.

Proof. From Definition 3.7, we can obtain:

- (1) $(N_1 \cap N_2) \cup N_2 = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \cup (\delta_{N_2}, \theta_{N_2})$
 $= (\max\{\min\{\delta_{N_1}, \delta_{N_2}\}, \delta_{N_2}\}, \min\{\max\{\theta_{N_1}, \theta_{N_2}\}, \theta_{N_2}\})$
 $= (\delta_{N_2}, \theta_{N_2}) = N_2$.
- (2) The proof is similar to (1).

□

In the below results we are claimed $A \oplus B$ and $A \otimes B$ are n-FSs for any two n-FSs A and B .

Theorem 2.13. Let $N = (\delta_N, \theta_N)$, $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be three n -FSs, then

- (1) $\zeta(N_1 \oplus N_2) = \zeta N_1 \oplus \zeta N_2$, for $\zeta > 0$.
- (2) $(\zeta_1 + \zeta_2)N = \zeta_1 F \oplus \zeta_2 F$, for $\zeta_1, \zeta_2 > 0$.
- (3) $(N_1 \otimes N_2)^\zeta = N_1^\zeta \otimes N_2^\zeta$, for $\zeta > 0$.
- (4) $N^{\zeta_1} \otimes N^{\zeta_2} = N^{(\zeta_1 + \zeta_2)}$, for $\zeta_1, \zeta_2 > 0$.

Proof. For the three n-FSs F, N_1 and N_2 , and $\zeta, \zeta_1, \zeta_2 > 0$, according to Definition 2.8, we can obtain:

$$\begin{aligned}
(1) \quad & \zeta(N_1 \oplus N_2) = \zeta \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n}, \theta_{N_1} \theta_{N_2} \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_{N_1}^n - \delta_{N_2}^n + \delta_{N_1}^n \delta_{N_2}^n)^\zeta}, (\theta_{N_1} \theta_{N_2})^\zeta \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_{N_1}^n)^\zeta (1 - \delta_{N_2}^n)^\zeta}, \theta_{N_1}^\zeta \theta_{N_2}^\zeta \right). \\
&\text{And } \zeta N_1 \oplus \zeta N_2 = \left(\sqrt[n]{1 - (1 - \delta_{N_1}^n)^\zeta}, \theta_{N_1}^\zeta \right) \oplus \left(\sqrt[n]{1 - (1 - \delta_{N_2}^n)^\zeta}, \theta_{N_2}^\zeta \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_{N_1}^n)^\zeta + 1 - (1 - \delta_{N_2}^n)^\zeta - (1 - (1 - \delta_{N_1}^n)^\zeta)(1 - (1 - \delta_{N_2}^n)^\zeta)}, \theta_{N_1}^\zeta \theta_{N_2}^\zeta \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_{N_1}^n)^\zeta (1 - \delta_{N_2}^n)^\zeta}, \theta_{N_1}^\zeta \theta_{N_2}^\zeta \right) = \zeta(N_1 \oplus N_2). \\
(2) \quad & (\zeta_1 + \zeta_2)N = (\zeta_1 + \zeta_2)(\delta_N, \theta_N) = \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_1 + \zeta_2}}, \theta_N^{\zeta_1 + \zeta_2} \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_1} (1 - \delta_N^n)^{\zeta_2}}, \theta_N^{\zeta_1 + \zeta_2} \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_1} + 1 - (1 - \delta_N^n)^{\zeta_2} - (1 - (1 - \delta_N^n)^{\zeta_1})(1 - (1 - \delta_N^n)^{\zeta_2})}, \theta_N^{\zeta_1} \theta_N^{\zeta_2} \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_1}}, \theta_N^{\zeta_1} \right) \oplus \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_2}}, \theta_N^{\zeta_2} \right) \\
&= \zeta_1 F \oplus \zeta_2 F. \\
(3) \quad & (N_1 \otimes N_2)^\zeta = \left(\delta_{N_1} \delta_{N_2}, \sqrt[n]{\theta_{N_1}^n + \theta_{N_2}^n - \theta_{N_1}^n \theta_{N_2}^n} \right)^\zeta \\
&= \left((\delta_{N_1} \delta_{N_2})^\zeta, \sqrt[n]{1 - (1 - \theta_{N_1}^n - \theta_{N_2}^n + \theta_{N_1}^n \theta_{N_2}^n)^\zeta} \right) \\
&= \left(\delta_{N_1}^\zeta \delta_{N_2}^\zeta, \sqrt[n]{1 - (1 - \theta_{N_1}^n)^\zeta (1 - \theta_{N_2}^n)^\zeta} \right) \\
&= \left(\delta_{N_1}^\zeta, \sqrt[n]{1 - (1 - \theta_{N_1}^n)^\zeta} \right) \otimes \left(\delta_{N_2}^\zeta, \sqrt[n]{1 - (1 - \theta_{N_2}^n)^\zeta} \right) \\
&= N_1^\zeta \otimes N_2^\zeta. \\
(4) \quad & N^{\zeta_1} \otimes N^{\zeta_2} = \left(\delta_N^{\zeta_1}, \sqrt[n]{1 - (1 - \theta_N^n)^{\zeta_1}} \right) \otimes \left(\delta_N^{\zeta_2}, \sqrt[n]{1 - (1 - \theta_N^n)^{\zeta_2}} \right) \\
&= \left(\delta_N^{\zeta_1 + \zeta_2}, \sqrt[n]{1 - (1 - \theta_N^n)^{\zeta_1 + \zeta_2}} \right) \\
&= N^{(\zeta_1 + \zeta_2)}.
\end{aligned}$$

□

Theorem 2.14. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$, $N_2 = (\delta_{N_2}, \theta_{N_2})$ and $N_3 = (\delta_{N_3}, \theta_{N_3})$ be three n-FSs and $\zeta > 0$, then

- (1) $N_1 \cap (N_2 \cap N_3) = (N_1 \cap N_2) \cap N_3$.
- (2) $N_1 \cup (N_2 \cup N_3) = (N_1 \cup N_2) \cup N_3$.
- (3) $\zeta(N_1 \cup N_2) = \zeta N_1 \cup \zeta N_2$.
- (4) $(N_1 \cup N_2)^\zeta = N_1^\zeta \cup N_2^\zeta$.

Proof. For the three n-FSs N_1, N_2 and N_3 , and $\zeta > 0$, according to Definition 2.8, we can obtain:

$$\begin{aligned}
(1) \quad & N_1 \cap (N_2 \cap N_3) = (\delta_{N_1}, \theta_{N_1}) \cap (\min\{\delta_{N_2}, \delta_{N_3}\}, \max\{\theta_{N_2}, \theta_{N_3}\}) \\
&= (\min\{\delta_{N_1}, \min\{\delta_{N_2}, \delta_{N_3}\}\}, \max\{\theta_{N_1}, \max\{\theta_{N_2}, \theta_{N_3}\}\}) \\
&= (\min\{\min\{\delta_{N_1}, \delta_{N_2}\}, \delta_{N_3}\}, \max\{\max\{\theta_{N_1}, \theta_{N_2}\}, \theta_{N_3}\})
\end{aligned}$$

$$\begin{aligned}
&= (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \cap (\delta_{N_3}, \theta_{N_3}) \\
&= (N_1 \cap N_2) \cap N_3.
\end{aligned}$$

(2) The proof is similar to (1).

$$\begin{aligned}
(3) \quad \zeta(N_1 \cup N_2) &= \zeta(\max\{\delta_{N_1}, \delta_{N_2}\}, \min\{\theta_{N_1}, \theta_{N_2}\}) \\
&= \left(\sqrt[n]{1 - (1 - \max\{\delta_{N_1}^n, \delta_{N_2}^n\})^\zeta}, \min\{\theta_{N_1}^\zeta, \theta_{N_2}^\zeta\} \right).
\end{aligned}$$

$$\begin{aligned}
\text{And } \zeta N_1 \cup \zeta N_2 &= \left(\sqrt[n]{1 - (1 - \delta_{N_1}^n)^\zeta}, \theta_{N_1}^\zeta \right) \cup \left(\sqrt[n]{1 - (1 - \delta_{N_2}^n)^\zeta}, \theta_{N_2}^\zeta \right) \\
&= \left(\max\left\{ \sqrt[n]{1 - (1 - \delta_{N_1}^n)^\zeta}, \sqrt[n]{1 - (1 - \delta_{N_2}^n)^\zeta} \right\}, \min\{\theta_{N_1}^\zeta, \theta_{N_2}^\zeta\} \right) \\
&= \left(\sqrt[n]{1 - (1 - \max\{\delta_{N_1}^n, \delta_{N_2}^n\})^\zeta}, \min\{\theta_{N_1}^\zeta, \theta_{N_2}^\zeta\} \right) = \zeta(N_1 \cup N_2).
\end{aligned}$$

(4) The proof is similar to (3).

□

Theorem 2.15. Let $N = (\delta_N, \theta_N)$, $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be three n -FSs, and $\zeta > 0$, then

- (1) $(N_1 \cap N_2)^c = N_1^c \cup N_2^c$.
- (2) $(N_1 \cup N_2)^c = N_1^c \cap N_2^c$.
- (3) $(N_1 \oplus N_2)^c = N_1^c \otimes N_2^c$.
- (4) $(N_1 \otimes N_2)^c = N_1^c \oplus N_2^c$.
- (5) $(N^c)^\zeta = (\zeta F)^c$.
- (6) $\zeta(N)^c = (N^\zeta)^c$.

Proof. For the three n -FSs F , N_1 and N_2 , and $\zeta > 0$, according to Definition 2.8, we can obtain:

$$\begin{aligned}
(1) \quad (N_1 \cap N_2)^c &= (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\})^c \\
&= (\max\{\theta_{N_1}, \theta_{N_2}\}, \min\{\delta_{N_1}, \delta_{N_2}\}) \\
&= (\theta_{N_1}, \delta_{N_1}) \cup (\theta_{N_2}, \delta_{N_2}) \\
&= N_1^c \cup N_2^c.
\end{aligned}$$

(2) The proof is similar to (1).

$$\begin{aligned}
(3) \quad (N_1 \oplus N_2)^c &= \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n}, \theta_{N_1} \theta_{N_2} \right)^c \\
&= \left(\theta_{N_1} \theta_{N_2}, \sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n} \right) \\
&= (\theta_{N_1}, \delta_{N_1}) \otimes (\theta_{N_2}, \delta_{N_2}) \\
&= N_1^c \otimes N_2^c.
\end{aligned}$$

(4) The proof is similar to (3).

$$\begin{aligned}
(5) \quad (N^c)^\zeta &= (\theta_N, \delta_N)^\zeta \\
&= \left(\theta_N^\zeta, \sqrt[n]{1 - (1 - \delta_N^n)^\zeta} \right) \\
&= \left(\sqrt[n]{1 - (1 - \delta_N^n)^\zeta}, \theta_N^\zeta \right)^c \\
&= (\zeta F)^c.
\end{aligned}$$

(6) The proof is similar to (5).

□

Theorem 2.16. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$, $N_2 = (\delta_{N_2}, \theta_{N_2})$ and $N_3 = (\delta_{N_3}, \theta_{N_3})$ be three n -FSs, then

- (1) $(N_1 \cap N_2) \oplus N_3 = (N_1 \oplus N_3) \cap (N_2 \oplus N_3).$
- (2) $(N_1 \cup N_2) \oplus N_3 = (N_1 \oplus N_3) \cup (N_2 \oplus N_3).$
- (3) $(N_1 \cap N_2) \otimes N_3 = (N_1 \otimes N_3) \cap (N_2 \otimes N_3).$
- (4) $(N_1 \cup N_2) \otimes N_3 = (N_1 \otimes N_3) \cup (N_2 \otimes N_3).$

Proof. By Definition 2.8, we can obtain:

$$\begin{aligned}
 (1) \quad & (N_1 \cap N_2) \oplus N_3 = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \oplus (\delta_{N_3}, \theta_{N_3}) \\
 & = \left(\sqrt[n]{\min\{\delta_{N_1}^n, \delta_{N_2}^n\} + \delta_{N_3}^n - \delta_{N_3}^n \min\{\delta_{N_1}^n, \delta_{N_2}^n\}}, \max\{\theta_{N_1}, \theta_{N_2}\} \theta_{N_3} \right) \\
 & = \left(\sqrt[n]{(1 - \delta_{N_3}^n) \min\{\delta_{N_1}^n, \delta_{N_2}^n\} + \delta_{N_3}^n}, \max\{\theta_{N_1} \theta_{N_3}, \theta_{N_2} \theta_{N_3}\} \right). \\
 & \text{And } (N_1 \oplus N_3) \cap (N_2 \oplus N_3) \\
 & = \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_3}^n - \delta_{N_1}^n \delta_{N_3}^n}, \theta_{N_1} \theta_{N_3} \right) \cap \left(\sqrt[n]{\delta_{N_2}^n + \delta_{N_3}^n - \delta_{N_2}^n \delta_{N_3}^n}, \theta_{N_2} \theta_{N_3} \right) \\
 & = \left(\min\left\{ \sqrt[n]{\delta_{N_1}^n + \delta_{N_3}^n - \delta_{N_1}^n \delta_{N_3}^n}, \sqrt[n]{\delta_{N_2}^n + \delta_{N_3}^n - \delta_{N_2}^n \delta_{N_3}^n} \right\}, \max\{\theta_{N_1} \theta_{N_3}, \theta_{N_2} \theta_{N_3}\} \right) \\
 & = \left(\min\left\{ \sqrt[n]{(1 - \delta_{N_3}^n) \delta_{N_1}^n + \delta_{N_3}^n}, \sqrt[n]{(1 - \delta_{N_3}^n) \delta_{N_2}^n + \delta_{N_3}^n} \right\}, \max\{\theta_{N_1} \theta_{N_3}, \theta_{N_2} \theta_{N_3}\} \right) \\
 & = \left(\sqrt[n]{(1 - \delta_{N_3}^n) \min\{\delta_{N_1}^n, \delta_{N_2}^n\} + \delta_{N_3}^n}, \max\{\theta_{N_1} \theta_{N_3}, \theta_{N_2} \theta_{N_3}\} \right).
 \end{aligned}$$

Thus, $(N_1 \cap N_2) \oplus N_3 = (N_1 \oplus N_3) \cap (N_2 \oplus N_3).$

(2) The proof is similar to (1).

$$\begin{aligned}
 (3) \quad & (N_1 \cap N_2) \otimes N_3 = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \otimes N_3 \\
 & = \left(\min\{\delta_{N_1}, \delta_{N_2}\} \delta_{N_3}, \sqrt[n]{\max\{\theta_{N_1}^n, \theta_{N_2}^n\} + \theta_{N_3}^n - \theta_{N_3}^n \max\{\theta_{N_1}^n, \theta_{N_2}^n\}} \right) \\
 & = \left(\min\{\delta_{N_1} \delta_{N_3}, \delta_{N_2} \delta_{N_3}\}, \sqrt[n]{(1 - \theta_{N_3}^n) \max\{\theta_{N_1}^n, \theta_{N_2}^n\} + \theta_{N_3}^n} \right).
 \end{aligned}$$

$$\begin{aligned}
 & \text{And } (N_1 \otimes N_3) \cap (N_2 \otimes N_3) = \left(\delta_{N_1} \delta_{N_3}, \sqrt[n]{\theta_{N_1}^n + \theta_{N_3}^n - \theta_{N_1}^n \theta_{N_3}^n} \right) \\
 & \cap \left(\delta_{N_2} \delta_{N_3}, \sqrt[n]{\theta_{N_2}^n + \theta_{N_3}^n - \theta_{N_2}^n \theta_{N_3}^n} \right) \\
 & = \left(\delta_{N_1} \delta_{N_3}, \sqrt[n]{(1 - \theta_{N_3}^n) \theta_{N_1}^n + \theta_{N_3}^n} \right) \cap \left(\delta_{N_2} \delta_{N_3}, \sqrt[n]{(1 - \theta_{N_3}^n) \theta_{N_2}^n + \theta_{N_3}^n} \right) \\
 & = \left(\min\{\delta_{N_1} \delta_{N_3}, \delta_{N_2} \delta_{N_3}\}, \max\left\{ \sqrt[n]{(1 - \theta_{N_3}^n) \theta_{N_1}^n + \theta_{N_3}^n}, \sqrt[n]{(1 - \theta_{N_3}^n) \theta_{N_2}^n + \theta_{N_3}^n} \right\} \right) \\
 & = \left(\min\{\delta_{N_1} \delta_{N_3}, \delta_{N_2} \delta_{N_3}\}, \sqrt[n]{(1 - \theta_{N_3}^n) \max\{\theta_{N_1}^n, \theta_{N_2}^n\} + \theta_{N_3}^n} \right).
 \end{aligned}$$

Thus, $(N_1 \cap N_2) \otimes N_3 = (N_1 \otimes N_3) \cap (N_2 \otimes N_3).$

(4) The proof is similar to (3).

□

Theorem 2.17. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$, $N_2 = (\delta_{N_2}, \theta_{N_2})$ and $N_3 = (\delta_{N_3}, \theta_{N_3})$ be three n -FSs, then

- (1) $N_1 \oplus N_2 \oplus N_3 = N_1 \oplus N_3 \oplus N_2.$
- (2) $N_1 \otimes N_2 \otimes N_3 = N_1 \otimes N_3 \otimes N_2.$

Proof.

$$\begin{aligned}
 (1) \quad & N_1 \oplus N_2 \oplus N_3 \\
 & = (\delta_{N_1}, \theta_{N_1}) \oplus (\delta_{N_2}, \theta_{N_2}) \oplus (\delta_{N_3}, \theta_{N_3}) \\
 & = \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n}, \theta_{N_1} \theta_{N_2} \right) \oplus (\delta_{N_3}, \theta_{N_3}) \\
 & = \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n + \delta_{N_3}^n - \delta_{N_3}^n (\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n)}, \theta_{N_1} \theta_{N_2} \theta_{N_3} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n + \delta_{N_3}^n - \delta_{N_1}^n \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_3}^n - \delta_{N_2}^n \delta_{N_3}^n + \delta_{N_1}^n \delta_{N_2}^n \delta_{N_3}^n}, \theta_{N_1} \theta_{N_2} \theta_{N_3} \right) \\
&= \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_3}^n - \delta_{N_1}^n \delta_{N_3}^n + \delta_{N_2}^n - \delta_{N_2}^n (\delta_{N_1}^n + \delta_{N_3}^n - \delta_{N_1}^n \delta_{N_3}^n)}, \theta_{N_1} \theta_{N_2} \theta_{N_3} \right) \\
&= \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_3}^n - \delta_{N_1}^n \delta_{N_3}^n}, \theta_{N_1} \theta_{N_3} \right) \oplus (\delta_{N_2}, \theta_{N_2}) \\
&= N_1 \oplus N_3 \oplus N_2.
\end{aligned}$$

(2) The proof is similar to (1). $\square$

Definition 2.18. Let $N = (\delta_N, \theta_N)$ be a n-FS, then score function of N can be represented as $s(N) = \delta_N^n - \theta_N^n$.

Theorem 2.19. For any n-FS $N = (\delta_N, \theta_N)$, the suggested score function $s(N) \in [-1, 1]$.

Proof. We know that for n-FS, $\delta_N^n + \theta_N^n \leq 1$. Thus, $\delta_N^n - \theta_N^n \leq \delta_N^n \leq 1$ and $\delta_N^n - \theta_N^n \geq -\theta_N^n \geq -1$. Therefore, $-1 \leq \delta_N^n - \theta_N^n \leq 1$, namely $s(N) \in [-1, 1]$. In particular, if $N = (0, 1)$, then $s(N) = -1$ and if $N = (1, 0)$, then $s(N) = 1$. $\square$

Definition 2.20. If $N = (\delta_N, \theta_N)$ is n-FS, then accuracy function of N can be represented as $a(N) = \delta_N^n + \theta_N^n$.

Remark 2.21. For any n-FS $N = (\delta_N, \theta_N)$, the suggested accuracy function $a(N) \in [0, 1]$.

Example 2.22. Let N be n-FS. If $s(N) = -0.4$ and $a(N) = 0.6$, then, it follows that $\delta_N(x) = \sqrt[n]{0.1}$, $\theta_N(x) = \sqrt[n]{0.5}$, and $\pi_N(x) = \sqrt[n]{0.4}$.

Theorem 2.23. Let $N = (\delta_N, \theta_N)$ be a n-FS. Then, $s(N) = 0$ if and only if $\delta_N(x) = \theta_N(x)$, for all $x \in X$.

Proof. Let $s(N) = 0$. Then, $(\delta_N(x))^n = (\theta_N(x))^n$ implies that $\delta_N(x) = \theta_N(x)$ for all $x \in X$.

Conversely, suppose that $\delta_N(x) = \theta_N(x)$. It follows immediately that, for all $x \in X$, $(\delta_N(x))^n = (\theta_N(x))^n$. Therefore, $(\delta_N(x))^n - (\theta_N(x))^n = 0$. Thus, $s(N) = 0$. $\square$

Theorem 2.24. Let $N = (\delta_N, \theta_N)$ be a n-FS, then the following statements hold for all $x \in X$:

- (1) $a(N) = 1$ if and only if $\pi_N(x) = 0$.
- (2) If $a(N) = 0$, then $|\delta_N(x)| = |\theta_N(x)|$.

Proof. (1) Let $a(N) = 1$. Then, $(\delta_N(x))^n + (\theta_N(x))^n = 1$. Since $\pi_N(x) = \sqrt[n]{1 - [(\delta_N(x))^n + (\theta_N(x))^n]}$, then $\pi_N(x) = 0$.

Conversely, suppose that $\pi_N(x) = 0$. Then, it follow that, $(\delta_N(x))^n + (\theta_N(x))^n = 1$ implies $a(N) = 1$.

- (2) Let $a(N) = 0$. Then, $(\delta_N(x))^n = -(\theta_N(x))^n$ implies that $|\delta_N(x)|^n = |\theta_N(x)|^n$ and hence $|\delta_N(x)| = |\theta_N(x)|$. $\square$

Definition 2.25. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n-FSSs, then

- (1) $N_1 = N_2$ if and only if $\delta_{N_1} = \delta_{N_2}$ and $\theta_{N_1} = \theta_{N_2}$.
- (2) $N_1 \geq N_2$ if and only if $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \theta_{N_2}$.

Theorem 2.26. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n-FSSs. If $s(N_1) = s(N_2)$, and

- (1) If $a(N_1) < a(N_2)$, then $\delta_{N_1} < \delta_{N_2}$.
- (2) If $a(N_1) > a(N_2)$, then $\delta_{N_1} > \delta_{N_2}$.
- (3) If $a(N_1) = a(N_2)$, then $N_1 = N_2$.

Proof. Straightforward. $\square$

Definition 2.27. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n-FSSs. Then, the Euclidean distance between N_1 and N_2 is:

$$d(N_1, N_2) = \sqrt{\frac{1}{2}[(\delta_{N_1}^n - \delta_{N_2}^n)^2 + (\theta_{N_1}^n - \theta_{N_2}^n)^2 + (\pi_{N_1}^n - \pi_{N_2}^n)^2]}.$$

Theorem 2.28. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n-FSSs, then

- (1) $d(N_1, N_2) = d(N_2, N_1)$.
- (2) $d(N_1, N_2) = 0$ if and only if $N_1 = N_2$.
- (3) $0 \leq d(N_1, N_2) \leq \sqrt{2}$.

Proof. From Definition 2.1, we can obtain $\delta_{N_1}, \theta_{N_1}, \delta_{N_2}, \theta_{N_2} \in [0, 1]$, $0 \leq \delta_{N_1}^n + \theta_{N_1}^n \leq 1$ and $0 \leq \delta_{N_2}^n + \theta_{N_2}^n \leq 1$, then

- (1) $d(N_1, N_2) = \sqrt{\frac{1}{2}[(\delta_{N_1}^n - \delta_{N_2}^n)^2 + (\theta_{N_1}^n - \theta_{N_2}^n)^2 + (\pi_{N_1}^n - \pi_{N_2}^n)^2]}$
 $= \sqrt{\frac{1}{2}[(\delta_{N_2}^n - \delta_{N_1}^n)^2 + (\theta_{N_2}^n - \theta_{N_1}^n)^2 + (\pi_{N_2}^n - \pi_{N_1}^n)^2]}$
 $= d(N_2, N_1)$.
- (2) Let $d(N_1, N_2) = \sqrt{\frac{1}{2}[(\delta_{N_1}^n - \delta_{N_2}^n)^2 + (\theta_{N_1}^n - \theta_{N_2}^n)^2 + (\pi_{N_1}^n - \pi_{N_2}^n)^2]} = 0$, then we get $(\delta_{N_1}^n - \delta_{N_2}^n) = 0$, $(\theta_{N_1}^n - \theta_{N_2}^n) = 0$ and $(\pi_{N_1}^n - \pi_{N_2}^n)^2 = 0$, and thus $\delta_{N_1} = \delta_{N_2}$, $\theta_{N_1} = \theta_{N_2}$, and $\pi_{N_1} = \pi_{N_2}$, that is, $N_1 = N_2$.
- (3) Obviously $0 \leq d(N_1, N_2)$. Now,
 $d(N_1, N_2) = \sqrt{\frac{1}{2}[(\delta_{N_1}^n - \delta_{N_2}^n)^2 + (\theta_{N_1}^n - \theta_{N_2}^n)^2 + (\pi_{N_1}^n - \pi_{N_2}^n)^2]}$
 $= \sqrt{\frac{1}{2}[(\delta_{N_1}^n - \delta_{N_2}^n)^2 + (\theta_{N_1}^n - \theta_{N_2}^n)^2 + [(\delta_{N_1}^n + \delta_{N_2}^n) + (\theta_{N_1}^n + \theta_{N_2}^n)]^2]}$
 $\leq \sqrt{\frac{1}{2}[2\delta_{N_1}^{2n} + 2\delta_{N_2}^{2n} + 2\theta_{N_1}^{2n} + 2\theta_{N_2}^{2n} + 4\delta_{N_1}^n \theta_{N_1}^n + 4\delta_{N_2}^n \theta_{N_2}^n]}$
 $= \sqrt{(\delta_{N_1}^n + \theta_{N_1}^n)^2 + (\delta_{N_2}^n + \theta_{N_2}^n)^2} \leq \sqrt{2}.$

$\square$

Definition 2.29. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n-FSSs, then

- (1) $N_1 \ominus N_2 = \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n}}, \frac{\theta_{N_1}}{\theta_{N_2}} \right)$ if $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min \left\{ \theta_{N_2}, \frac{\theta_{N_2} \pi_1}{\pi_2} \right\}$.

$$(2) \quad N_1 \odot N_2 = \left(\frac{\delta_{N_1}}{\delta_{N_2}}, \sqrt[n]{\frac{\theta_{N_1}^n - \theta_{N_2}^n}{1 - \theta_{N_2}^n}} \right) \text{ if } \delta_{N_1} \leq \min \left\{ \delta_{N_2}, \frac{\delta_{N_2} \pi_1}{\pi_2} \right\} \text{ and } \theta_{N_1} \geq \theta_{N_2}.$$

Theorem 2.30. *If $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ are two n -FSs, then*

- (1) $N_1 \ominus N_2$ and $\zeta(N_1 \ominus N_2)$ are n -FSs.
- (2) $N_1 \odot N_2$ is n -FS.

Proof. (1) Since $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min \left\{ \theta_{N_2}, \frac{\theta_{N_2} \pi_1}{\pi_2} \right\}$, then

$$\begin{aligned} & \theta_{N_1} \pi_2 \leq \theta_{N_2} \pi_1 \\ & \Rightarrow \theta_{N_1}^n \pi_2^n \leq \theta_{N_2}^n \pi_1^n \\ & \Rightarrow \theta_{N_1}^n \theta_{N_2}^n + \theta_{N_1}^n \pi_2^n \leq \theta_{N_1}^n \theta_{N_2}^n + \theta_{N_2}^n \pi_1^n \\ & \Rightarrow \theta_{N_1}^n (\theta_{N_2}^n + \pi_2^n) \leq \theta_{N_2}^n (\theta_{N_1}^n + \pi_1^n) \\ & \Rightarrow \theta_{N_1}^n (1 - \delta_{N_2}^n) \leq \theta_{N_2}^n (1 - \delta_{N_1}^n) \\ & \Rightarrow \left(\frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right)^\zeta \leq \left(\frac{1 - \delta_{N_1}^n}{1 - \delta_{N_2}^n} \right)^\zeta \text{ or } \left(\frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right) \leq \left(\frac{1 - \delta_{N_1}^n}{1 - \delta_{N_2}^n} \right) \\ & \Rightarrow 1 - \left(\frac{1 - \delta_{N_1}^n}{1 - \delta_{N_2}^n} \right)^\zeta + \left(\frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right)^\zeta \leq 1 \text{ or } 1 - \left(\frac{1 - \delta_{N_1}^n}{1 - \delta_{N_2}^n} \right) + \left(\frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right) \leq 1 \\ & \Rightarrow \left(\sqrt[n]{1 - \left(\frac{1 - \delta_{N_1}^n}{1 - \delta_{N_2}^n} \right)^\zeta} \right)^n + \left(\frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right)^\zeta \leq 1 \\ & \text{or } \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n}} \right)^n + \left(\frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right)^n \leq 1. \text{ Thus, } N_1 \ominus N_2 \text{ and } \zeta(N_1 \ominus N_2) \text{ are } n\text{-FSs.} \end{aligned}$$

- (2) The proof is similar to (2). □

Theorem 2.31. *Let $N = (\delta_N, \theta_N)$, $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be three n -FSs and $\zeta, \zeta_1, \zeta_2 > 0$, then*

- (1) $\zeta(N_1 \ominus N_2) = \zeta N_1 \ominus \zeta N_2$ if $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min \left\{ \theta_{N_2}, \frac{\theta_{N_2} \pi_1}{\pi_2} \right\}$.
- (2) $(N_1 \odot N_2)^\zeta = N_1^\zeta \odot N_2^\zeta$ if $\delta_{N_1} \leq \min \left\{ \delta_{N_2}, \frac{\delta_{N_2} \pi_1}{\pi_2} \right\}$ and $\theta_{N_1} \geq \theta_{N_2}$.
- (3) $\zeta_1 F \ominus \zeta_2 N = (\zeta_1 - \zeta_2) F$ if $\zeta_1 \geq \zeta_2$.
- (4) $N^{\zeta_1} \odot N^{\zeta_2} = N^{(\zeta_1 - \zeta_2)}$ if $\zeta_1 \geq \zeta_2$.

Proof. (1) Since $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min \left\{ \theta_{N_2}, \frac{\theta_{N_2} \pi_1}{\pi_2} \right\}$, then from Definitions 2.8 and 2.29, we get

$$\begin{aligned} \zeta(N_1 \ominus N_2) &= \zeta \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n}}, \frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right) \\ &= \left(\sqrt[n]{1 - \left(1 - \frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n} \right)^\zeta}, \left(\frac{\theta_{N_1}^n}{\theta_{N_2}^n} \right)^\zeta \right) \end{aligned}$$

$$= \left(\sqrt[n]{1 - \left(\frac{1 - \delta_{N_1}^n}{1 - \delta_{N_2}^n} \right)^\zeta}, \left(\frac{\theta_{N_1}^\zeta}{\theta_{N_2}^\zeta} \right) \right),$$

and

$$\begin{aligned} \zeta N_1 \ominus \zeta N_2 &= \left(\sqrt[n]{1 - (1 - \delta_{N_1}^n)^\zeta}, \theta_{N_1}^\zeta \right) \ominus \left(\sqrt[n]{1 - (1 - \delta_{N_2}^n)^\zeta}, \theta_{N_2}^\zeta \right) \\ &= \left(\sqrt[n]{\frac{1 - (1 - \delta_{N_1}^n)^\zeta - 1 + (1 - \delta_{N_2}^n)^\zeta}{1 - 1 + (1 - \delta_{N_2}^n)^\zeta}}, \frac{\theta_{N_1}^\zeta}{\theta_{N_2}^\zeta} \right) \\ &= \left(\sqrt[n]{\frac{(1 - \delta_{N_2}^n)^\zeta - (1 - \delta_{N_1}^n)^\zeta}{(1 - \delta_{N_2}^n)^\zeta}}, \frac{\theta_{N_1}^\zeta}{\theta_{N_2}^\zeta} \right) \\ &= \left(\sqrt[n]{1 - \left(\frac{1 - \delta_{N_1}^n}{1 - \delta_{N_2}^n} \right)^\zeta}, \frac{\theta_{N_1}^\zeta}{\theta_{N_2}^\zeta} \right). \end{aligned}$$

Hence, $\zeta(N_1 \ominus N_2) = \zeta N_1 \ominus \zeta N_2$.

(2) The proof is similar to (1).

$$\begin{aligned} (3) \quad \zeta_1 F \ominus \zeta_2 N &= \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_1}}, \theta_N^{\zeta_1} \right) \ominus \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_2}}, \theta_N^{\zeta_2} \right) \\ &= \left(\sqrt[n]{\frac{1 - (1 - \delta_N^n)^{\zeta_1} - 1 + (1 - \delta_N^n)^{\zeta_2}}{1 - 1 + (1 - \delta_N^n)^{\zeta_2}}}, \frac{\theta_N^{\zeta_1}}{\theta_N^{\zeta_2}} \right) \\ &= \left(\sqrt[n]{1 - (1 - \delta_N^n)^{\zeta_1 - \zeta_2}}, \theta_N^{\zeta_1 - \zeta_2} \right) \\ &= (\zeta_1 - \zeta_2) F. \end{aligned}$$

(4) The proof is similar to (3).

□

Theorem 2.32. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n -FSs, then

- (1) $(N_1 \ominus N_2)^c = N_1^c \oslash N_2^c$ if $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min \left\{ \theta_{N_2}, \frac{\theta_{N_2} \pi_1}{\pi_2} \right\}$.
- (2) $(N_1 \oslash N_2)^c = N_1^c \ominus N_2^c$ if $\delta_{N_1} \leq \min \left\{ \delta_{N_2}, \frac{\delta_{N_2} \pi_1}{\pi_2} \right\}$ and $\theta_{N_1} \geq \theta_{N_2}$.

Proof. From Definitions 2.8 and 2.29, we can get

$$\begin{aligned} (1) \quad (N_1 \ominus N_2)^c &= \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n}}, \frac{\theta_{N_1}}{\theta_{N_2}} \right)^c = \left(\frac{\theta_{N_1}}{\theta_{N_2}}, \sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n}} \right) \\ &= (\theta_{N_1}, \delta_{N_1}) \ominus (\theta_{N_2}, \delta_{N_2}) = N_1^c \oslash N_2^c. \end{aligned}$$

(2) The proof is similar to (1).

□

Theorem 2.33. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n -FSs, then

- (1) $(N_1 \cup N_2) \ominus (N_1 \cap N_2) = N_1 \ominus N_2$ if $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min \left\{ \theta_{N_2}, \frac{\theta_{N_2} \pi_1}{\pi_2} \right\}$.
- (2) $(N_1 \cup N_2) \oslash (N_1 \cap N_2) = N_1 \oslash N_2$ if $\delta_{N_1} \leq \min \left\{ \delta_{N_2}, \frac{\delta_{N_2} \pi_1}{\pi_2} \right\}$ and $\theta_{N_1} \geq \theta_{N_2}$.

Proof. From Definitions 2.8 and 2.29, we can get

$$\begin{aligned} (1) \quad (N_1 \cup N_2) \ominus (N_1 \cap N_2) &= (\max\{\delta_{N_1}, \delta_{N_2}\}, \min\{\theta_{N_1}, \theta_{N_2}\}) \ominus (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \end{aligned}$$

$$\begin{aligned}
&= \left(\sqrt[n]{\frac{(\max\{\delta_{N_1}, \delta_{N_2}\})^n - (\min\{\delta_{N_1}, \delta_{N_2}\})^n}{1 - \min\{\delta_{N_1}, \delta_{N_2}\}^n}}, \frac{\min\{\theta_{N_1}, \theta_{N_2}\}}{\max\{\theta_{N_1}, \theta_{N_2}\}} \right) \\
&= \left(\sqrt[n]{\frac{\max\{\delta_{N_1}^n, \delta_{N_2}^n\} - \min\{\delta_{N_1}^n, \delta_{N_2}^n\}}{1 - \min\{\delta_{N_1}^n, \delta_{N_2}^n\}}}, \frac{\theta_{N_1}}{\theta_{N_2}} \right) \\
&= \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n}}, \frac{\theta_{N_1}}{\theta_{N_2}} \right) \\
&= N_1 \ominus N_2.
\end{aligned}$$

(2) The proof is similar to (1).

□

Theorem 2.34. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n -FSs, then

- (1) $(N_1 \cap N_2) \ominus (N_1 \cup N_2) = N_2 \ominus N_1$ if $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min\left\{\theta_{N_2}, \frac{\theta_{N_2}\pi_1}{\pi_2}\right\}$.
- (2) $(N_1 \cap N_2) \otimes (N_1 \cup N_2) = N_1 \otimes N_2$ if $\delta_{N_1} \leq \min\left\{\delta_{N_2}, \frac{\delta_{N_2}\pi_1}{\pi_2}\right\}$ and $\theta_{N_1} \geq \theta_{N_2}$.

Proof. From Definitions 2.8 and 2.29, we can get

$$\begin{aligned}
(1) \quad &(N_1 \cap N_2) \ominus (N_1 \cup N_2) \\
&= (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \ominus (\max\{\delta_{N_1}, \delta_{N_2}\}, \min\{\theta_{N_1}, \theta_{N_2}\}) \\
&= \left(\sqrt[n]{\frac{(\min\{\delta_{N_1}, \delta_{N_2}\})^n - (\max\{\delta_{N_1}, \delta_{N_2}\})^n}{1 - \max\{\delta_{N_1}, \delta_{N_2}\}^n}}, \frac{\max\{\theta_{N_1}, \theta_{N_2}\}}{\min\{\theta_{N_1}, \theta_{N_2}\}} \right) \\
&= \left(\sqrt[n]{\frac{\min\{\delta_{N_1}^n, \delta_{N_2}^n\} - \max\{\delta_{N_1}^n, \delta_{N_2}^n\}}{1 - \max\{\delta_{N_1}^n, \delta_{N_2}^n\}}}, \frac{\theta_{N_2}}{\theta_{N_1}} \right) \\
&= \left(\sqrt[n]{\frac{\delta_{N_2}^n - \delta_{N_1}^n}{1 - \delta_{N_1}^n}}, \frac{\theta_{N_2}}{\theta_{N_1}} \right) \\
&= N_2 \ominus N_1.
\end{aligned}$$

(2) The proof is similar to (1).

□

Theorem 2.35. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n -FSs, then

- (1) $(N_1 \ominus N_2) \oplus N_2 = N_1$ if $\delta_{N_1} \geq \delta_{N_2}$ and $\theta_{N_1} \leq \min\left\{\theta_{N_2}, \frac{\theta_{N_2}\pi_1}{\pi_2}\right\}$.
- (2) $(N_1 \otimes N_2) \otimes N_2 = N_1$ if $\delta_{N_1} \leq \min\left\{\delta_{N_2}, \frac{\delta_{N_2}\pi_1}{\pi_2}\right\}$ and $\theta_{N_1} \geq \theta_{N_2}$.

Proof. From Definitions 2.8 and 2.29, we can get

$$\begin{aligned}
(1) \quad &(N_1 \ominus N_2) \oplus N_2 = \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n}}, \frac{\theta_{N_1}}{\theta_{N_2}} \right) \oplus (\delta_{N_2}, \theta_{N_2}) \\
&= \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n} + \delta_{N_2}^n - \frac{\delta_{N_1}^n - \delta_{N_2}^n}{1 - \delta_{N_2}^n} \delta_{N_2}^n}, \frac{\theta_{N_1} \theta_{N_2}}{\theta_{N_2}} \right) \\
&= \left(\sqrt[n]{\frac{\delta_{N_1}^n - \delta_{N_2}^n + \delta_{N_2}^n - \delta_{N_2}^{2n} - \delta_{N_1}^n \delta_{N_2}^n + \delta_{N_2}^{2n}}{1 - \delta_{N_2}^n}}, \theta_{N_1} \right) \\
&= N_1
\end{aligned}$$

$$= \left(\sqrt[n]{\frac{\delta_{N_1}^n (1 - \delta_{N_2}^n)}{1 - \delta_{N_2}^n}}, \theta_{N_1} \right) \\ = (\delta_{N_1}, \theta_{N_1}) = N_1.$$

(2) The proof is similar to (1). □

Definition 2.36. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$ and $N_2 = (\delta_{N_2}, \theta_{N_2})$ be two n-FSSs, then average operator of N_1 and N_2 is defined as

$$N_1 \textcircled{a} N_2 = \left(\frac{\delta_{N_1}^n + \delta_{N_2}^n}{2}, \frac{\theta_{N_1}^n + \theta_{N_2}^n}{2} \right).$$

Theorem 2.37. Let $N_1 = (\delta_{N_1}, \theta_{N_1})$, $N_2 = (\delta_{N_2}, \theta_{N_2})$ and $N_3 = (\delta_{N_3}, \theta_{N_3})$ be three n-FSSs, then

- (1) $(N_1 \cap N_2) \textcircled{a} N_3 = (N_1 \textcircled{a} N_3) \cap (N_2 \textcircled{a} N_3).$
- (2) $(N_1 \cup N_2) \textcircled{a} N_3 = (N_1 \textcircled{a} N_3) \cup (N_2 \textcircled{a} N_3).$
- (3) $(N_1 \textcircled{a} N_2)^c = N_1^c \textcircled{a} N_2^c.$
- (4) $((N_1 \cap N_2) \oplus (N_1 \cup N_2)) \textcircled{a} ((N_1 \cap N_2) \otimes (N_1 \cup N_2)) = N_1 \textcircled{a} N_2.$

Proof. From Definitions 2.8 and 2.37, we can get

- (1) $(N_1 \cap N_2) \textcircled{a} N_3 = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \textcircled{a} (\delta_{N_3}, \theta_{N_3})$
 $= \left(\frac{\min\{\delta_{N_1}^n, \delta_{N_2}^n\} + \delta_{N_3}^n}{2}, \frac{\max\{\theta_{N_1}^n, \theta_{N_2}^n\} + \theta_{N_3}^n}{2} \right)$
 $= \left(\min \left\{ \frac{\delta_{N_1}^n + \delta_{N_3}^n}{2}, \frac{\delta_{N_2}^n + \delta_{N_3}^n}{2} \right\}, \max \left\{ \frac{\theta_{N_1}^n + \theta_{N_3}^n}{2}, \frac{\theta_{N_2}^n + \theta_{N_3}^n}{2} \right\} \right)$
 $= \left(\frac{\delta_{N_1}^n + \delta_{N_3}^n}{2}, \frac{\theta_{N_1}^n + \theta_{N_3}^n}{2} \right) \cap \left(\frac{\delta_{N_2}^n + \delta_{N_3}^n}{2}, \frac{\theta_{N_2}^n + \theta_{N_3}^n}{2} \right)$
 $= (N_1 \textcircled{a} N_3) \cap (N_2 \textcircled{a} N_3).$
- (2) The proof is similar to (1).
- (3) $(N_1 \textcircled{a} N_2)^c = \left(\frac{\delta_{N_1}^n + \delta_{N_2}^n}{2}, \frac{\theta_{N_1}^n + \theta_{N_2}^n}{2} \right)^c = \left(\frac{\theta_{N_1}^n + \theta_{N_2}^n}{2}, \frac{\delta_{N_1}^n + \delta_{N_2}^n}{2} \right)$
 $= (\theta_{N_1}, \delta_{N_1}) \textcircled{a} (\theta_{N_2}, \delta_{N_2}) = N_1^c \textcircled{a} N_2^c.$
- (4) Since for any two real numbers c and d , we have
 $\min(c, d) + \max(c, d) = c + d$, and
 $\min(c, d) \cdot \max(c, d) = c \cdot d.$

Now,

$$(N_1 \cap N_2) \oplus (N_1 \cup N_2) \\ = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \oplus (\max\{\delta_{N_1}, \delta_{N_2}\}, \min\{\theta_{N_1}, \theta_{N_2}\}) \\ = (\sqrt[n]{\min\{\delta_{N_1}^n, \delta_{N_2}^n\} + \max\{\delta_{N_1}^n, \delta_{N_2}^n\} - \min\{\delta_{N_1}^n, \delta_{N_2}^n\} \max\{\delta_{N_1}^n, \delta_{N_2}^n\}}, \\ \max\{\theta_{N_1}, \theta_{N_2}\} \min\{\theta_{N_1}, \theta_{N_2}\}) = (\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n}, \theta_{N_1} \theta_{N_2}).$$

And

$$(N_1 \cap N_2) \otimes (N_1 \cup N_2) \\ = (\min\{\delta_{N_1}, \delta_{N_2}\}, \max\{\theta_{N_1}, \theta_{N_2}\}) \otimes (\max\{\delta_{N_1}, \delta_{N_2}\}, \min\{\theta_{N_1}, \theta_{N_2}\}) \\ = (\min\{\delta_{N_1}, \delta_{N_2}\} \max\{\delta_{N_1}, \delta_{N_2}\}, \\ \sqrt[n]{\max\{\theta_{N_1}^n, \theta_{N_2}^n\} + \min\{\theta_{N_1}^n, \theta_{N_2}^n\} - \max\{\theta_{N_1}^n, \theta_{N_2}^n\} \min\{\theta_{N_1}^n, \theta_{N_2}^n\}}) \\ = (\delta_{N_1} \delta_{N_2}, \sqrt[n]{\theta_{N_1}^n + \theta_{N_2}^n - \theta_{N_1}^n \theta_{N_2}^n}).$$

Thus,

$$\begin{aligned}
 & ((N_1 \cap N_2) \oplus (N_1 \cup N_2)) \textcircled{a} ((N_1 \cap N_2) \otimes (N_1 \cup N_2)) \\
 &= \left(\sqrt[n]{\delta_{N_1}^n + \delta_{N_2}^n - \delta_{N_1}^n \delta_{N_2}^n}, \theta_{N_1} \theta_{N_2} \right) \textcircled{a} \left(\delta_{N_1} \delta_{N_2}, \sqrt[n]{\theta_{N_1}^n + \theta_{N_2}^n - \theta_{N_1}^n \theta_{N_2}^n} \right) \\
 &= \left(\frac{\delta_{N_1}^n + \delta_{N_2}^n}{2}, \frac{\theta_{N_1}^n + \theta_{N_2}^n}{2} \right) \\
 &= N_1 \textcircled{a} N_2.
 \end{aligned}$$

□

3. Application of n-Fuzzy Sets ($n > 3$)

In this section, we present an application of n-Fuzzy set in Sanchez's approach [12] for medical diagnosis. Analogous to Sanchez's notion of Medical Knowledge we define n-Fuzzy Medical Knowledge as a n-Fuzzy relation R from the set of symptoms to the set of diagnoses which reveals the degree of association and the degree of non-association between symptoms and diagnosis.

Now, let X and Y be two sets. A n-Fuzzy relation (n-FR) R from X to Y is a n-FS of $X \times Y$ characterized by the degree of membership function δ_R and degree of non-membership function θ_R . A n-FR R from X to Y will be denoted by $R(X \rightarrow Y)$. If N is a n-FS of X , then the max-min-max composition of the n-FR $R(X \rightarrow Y)$ with N is a n-FS B of Y denoted by $B = RoN$, and is defined by the membership function

$$\delta_{RoN}(y) = \bigvee_x [\delta_N(x) \wedge \delta_R(x, y)],$$

and the non-membership function

$$\theta_{RoN}(y) = \bigwedge_x [\theta_N(x) \vee \theta_R(x, y)],$$

for all $y \in Y$, where $\bigvee = \max$ and $\bigwedge = \min$.

Let $Q(X \rightarrow Y)$ and $R(Y \rightarrow Z)$ be two n-FRs. The max-min-max composition RoQ is the n-Fuzzy relation from X to Z defined by the membership function

$$\delta_{RoQ}(x, z) = \bigvee_y [\delta_Q(x, y) \wedge \delta_R(y, z)],$$

and the non-membership function

$$\theta_{RoQ}(x, z) = \bigwedge_y [\theta_Q(x, y) \vee \theta_R(y, z)],$$

$\forall (x, z) \in X \times Z$ and $\forall y \in Y$.

Now, let us discuss n-Fuzzy medical diagnosis. The methodology involves mainly the following three jobs:

- (1) Determination of symptoms.
- (2) Formulation of medical knowledge based on n-Fuzzy relations.
- (3) Determination of diagnosis on the basis of composition of n-Fuzzy relations.

To see the application of the method, let us make a hypothetical case study below:

Application example:

Assume there are four patients exist, namely Ali, Bewar, Reber and Ziyad.

Their symptoms are Burning eyes, head pain, chest-pain, Back pain and High fever. Let $P = \{Ali, Bewar, Reber, Ziyad\}$ be the set of patients and $S = \{burning\ eyes, head-pain, chest-pain, back-pain, high\ fever\}$ be the set of symptoms. Additionally, five diagnosis, Eye allergies, migraine, Heart problem, Spinal problem and Typhoid are defined, represented as $D = \{Eye\ allergies, Migraine, Heart\ problem, Spinal\ problem, Typhoid\}$.

Algorithm:

Step 1: The 4-Fuzzy relation $Q(P \rightarrow S)$ and the 4-Fuzzy relation $R(S \rightarrow D)$ are given as in (hypothetical) Table 1 and Table 2, respectively.

Step 2: Compute the composition RoQ as in Table 3.

Step 3: Calculate $S_R = \delta_{RoQ} - \theta_{RoQ} \cdot \pi_{RoQ}$ as in Table 4.

Step 4: From Table 4, we present the decision making. Decisions are made based on the greatest value of relation between patients and diseases.

Table 1

$Q(P \rightarrow S)$	burning eyes	head-pain	chest-pain	back-pain	high fever
<i>Ali</i>	(0.90, 0.70)	(0.99, 0.40)	(0.50, 0.96)	(0.99, 0.40)	(0.40, 0.99)
<i>Bewar</i>	(0.63, 0.91)	(0.82, 0.82)	(0.99, 0.40)	(0.65, 0.90)	(0.70, 0.90)
<i>Reber</i>	(0.90, 0.70)	(0.90, 0.70)	(0.50, 0.97)	(0.67, 0.89)	(0.50, 0.96)
<i>Ziyad</i>	(0.92, 0.61)	(0.87, 0.80)	(0.85, 0.73)	(0.89, 0.67)	(0.85, 0.73)

Table 2

$R(S \rightarrow D)$	Eye allergies	Migraine	Heart problem	Spinal problem	Typhoid
<i>burning eyes</i>	(0.81, 0.79)	(0.87, 0.79)	(0.80, 0.80)	(0.65, 0.90)	(0.70, 0.90)
<i>head-pain</i>	(0.85, 0.76)	(0.44, 0.99)	(0.99, 0.40)	(0.84, 0.75)	(0.65, 0.90)
<i>chest-pain</i>	(0.66, 0.90)	(0.39, 0.98)	(0.67, 0.89)	(0.90, 0.65)	(0.72, 0.90)
<i>back-pain</i>	(0.73, 0.85)	(0.87, 0.79)	(0.44, 0.99)	(0.67, 0.89)	(0.72, 0.90)
<i>high fever</i>	(0.66, 0.90)	(0.70, 0.90)	(0.41, 0.98)	(0.67, 0.89)	(0.90, 0.70)

Table 3

R o Q	Eye allergies	Migraine	Heart problem	Spinal problem	Typhoid
<i>Ali</i>	(0.85, 0.76)	(0.87, 0.79)	(0.99, 0.40)	(0.84, 0.75)	(0.72, 0.90)
<i>Bewar</i>	(0.82, 0.82)	(0.70, 0.90)	(0.82, 0.82)	(0.90, 0.65)	(0.72, 0.90)
<i>Reber</i>	(0.85, 0.76)	(0.87, 0.79)	(0.90, 0.70)	(0.84, 0.75)	(0.70, 0.90)
<i>Ziyad</i>	(0.85, 0.79)	(0.87, 0.79)	(0.87, 0.80)	(0.85, 0.73)	(0.85, 0.73)

Table 4

S_R	Eye allergies	Migraine	Heart problem	Spinal problem	Typhoid
<i>Ali</i>	0.38	0.52	0.85	0.35	0.25
<i>Bewar</i>	0.36	0.19	0.36	0.49	0.25
<i>Reber</i>	0.38	0.52	0.50	0.35	0.19
<i>Ziyad</i>	0.42	0.52	0.58	0.37	0.37

Decisions on the patients medical conditions:

This approach is with respect to the patient against diseases. From the horizontal view of Table 4, we see that:

- (1) Ali is suffering from Heart problem (0.85) with some elements of Migraine (0.52).
- (2) Bewar is suffering from Spinal problem (0.49).
- (3) Reber is suffering from Migraine (0.52) with some elements of Heart problem (0.50).
- (4) Ziyad is suffering from Heart problem (0.58) with some element of Migraine (0.52) and Eye allergies (0.42), respectively.

4. Conclusions

For express fuzzy information, n-Fuzzy is a good tool. It has a parameter n , so it holds a wider range of fuzzy information than IFS and PFS. In this paper, we have introduced a new generalized intuitionistic fuzzy set is called n-Fuzzy sets and studied their relationship with intuitionistic fuzzy, Pythagorean fuzzy and Fermatean fuzzy sets. In addition, some operators on n-Fuzzy sets are defined and their relationship have been proved. Finally, an application of the improved composite relation for 4-Fuzzy sets was carried out in medical diagnosis case.

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Compliance with ethical standards

Conflict of interest: Hariwan Z. Ibrahim (author) declares that he has no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by of the author.

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