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Modified Van der Pol-Helmohltz oscillator equation with exact harmonic solutions

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Abstract

We present in this paper an exceptional Lienard equation consisting of a modified Van der Pol-Helmholtz oscillator equation. The equation, a frequency-dependent damping oscillator, does not satisfy the usual existence theorems but, nevertheless, has an isochronous centre at the origin. We exhibit the exact and explicit general harmonic and isochronous solutions by using the first integral approach. The numerical results match very well analytical solutions.

Keywords: Lienard equation, Van der Pol-Helmholtz oscillator, frequency-dependent damping oscillator, exact harmonic and isochronous solution, existence theorems.

Introduction

Many problems in applied mathematics and physics are often described in terms of nonlinear differential equations. One of the most investigated second-order nonlinear differential equation is the Lienard equation [1-3]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1}$$

where overdot means differentiation with respect to time, and $f(x)$ and $g(x)$ are functions of x. This general class of equations has been intensively studied in the literature from the view-point of theorem for the existence of periodic solutions [1-3]. A famous example of equations of type (1) is the Van der Pol oscillator known to have a unique limit cycle [1, 3]. It is also known since the work performed by Sabatini [2] that the equation (1) can have isochronous centre at the origin. However, it was in 2005 that such a property has explicitly been proved in [4]. Recently, Monsia and his coworkers presented by choosing adequately the functions $f(x)$ and $g(x)$, several equations of type (1) that have harmonic and isochronous periodic solutions [5-7]. The equations of type (1) are very interesting from physical point of view since dynamical systems are, in real world, subject to dissipative term. Thus, the problem of finding periodic or isochronous solutions to equations of the form (1) has become an attractive research field for the pure and applied mathematics. In this context, consider the generalized equation

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$$\ddot{x} + \beta\left(\alpha - \gamma x^2\right)^p \dot{x} + g(x) = 0 \tag{2}$$

An interesting family of equations of type (2) is the Van der Pol-Helmholtz equation.

$$\ddot{x} + \beta\left(\alpha - \gamma x^2\right)^p \dot{x} + a_1 x + a_2 x^2 + a_3 = 0 \tag{3}$$

where α, β, γ, a_1, a_2, a_3 and p, are arbitrary constants, $f(x) = \beta\left(\alpha - \gamma x^2\right)^p$, and $g(x) = a_1 x + a_2 x^2 + a_3$. When $p = 1$, $a_2 = a_3 = 0$, the equation (3) is said to be the generalized Van der Pol equation

$$\ddot{x} + \beta\left(\alpha - \gamma x^2\right)\dot{x} + a_1 x = 0 \tag{4}$$

When $\beta = 0$, the equation (3) reduces to

$$\ddot{x} + a_1 x + a_2 x^2 + a_3 = 0 \tag{5}$$

known as the conservative quadratic nonlinear Helmholtz oscillator or quadratic anharmonic oscillator [8-10]. When $\gamma = 0$ and $p = 1$, the equation (3) becomes the damped anharmonic Helmholtz oscillator

$$\ddot{x} + \lambda \dot{x} + a_1 x + a_2 x^2 + a_3 = 0 \tag{6}$$

where $\lambda = \beta\alpha$, which has been treated in numerous works [8, 11, 12]. With such a damping, the general solutions of the equation (6) are rather non-periodic [11, 12] contrary to the periodic solutions of the equation (5). Now, let $p = \frac{1}{2}$. Then, the generalized Van der Pol-Helmholtz equation (3) becomes

$$\ddot{x} + \beta\sqrt{\alpha - \gamma x^2}\,\dot{x} + a_1 x + a_2 x^2 + a_3 = 0 \tag{7}$$

Thus, an interesting mathematical problem to investigate is to ask whether the equation (7) can exhibit periodic solutions. This problem becomes more interesting when the question is to find general harmonic and isochronous periodic solutions like the solution of the linear harmonic oscillator. To our best knowledge, such a question has not been previously solved for the modified Van der Pol-Helmholtz equation of the form (7) in the literature. In this situation, the objective in this paper is to prove explicitly that the equation (7) can have isochronous centre at the origin for an appropriate choice of system parameters, contrary to the existence theorem predictions. To attain this objective, we analyze the equation (7) at the light of usual existence theorems and phase plane (section 2) and calculate explicitly the general harmonic periodic solutions that prove the existence of an isochronous centre at the origin (section 3). A conclusion is sketched finally for the work.

2. Existence theorem and phase plane analysis

The Lienard equation (1) has been deeply investigated in [1] from the perspective of theorem for the existence of a centre at the origin (See Theorem 11.3, page 390 of [1]). This theorem is in agreement with the theorems for the existence of a centre at the origin formulated in [2, 3]. Then, according to [1-3] the Lienard equation (1) has a centre at the origin when $f(x)$ and $g(x)$ are continuous functions, $f(x)$ and $g(x)$ are odd, and $g(x) > 0$, for $x > 0$, that is $g(0) = 0$. In the case of equation (7), $f(x) = \beta\sqrt{\alpha - \gamma x^2}$, and $g(x) = a_1 x + a_2 x^2 + a_3$. As observed, $f(x)$ is neither odd nor even. $g(x)$ is not also odd and $g(0) \neq 0$, since $a_3 \neq 0$. These results are sufficient to conclude that, according to [1-3], isochronous centre at the origin is excluded for the equation (7) for any arbitrary non-zero value of a_1, a_2, a_3, α, β and γ. Now, consider the dynamical system

$$\dot{x} = y \ , \ \ \dot{y} = -\beta\sqrt{\alpha - \gamma x^2}\, y - a_1 x - a_2 x^2 - a_3 \tag{8}$$

equivalent to the equation (7). The equilibrium points are given by $y = 0$, and the quadratic equation

$$a_2 x^2 + a_1 x + a_3 = 0 \tag{9}$$

The discriminant of the equation (9) reads

$$\Delta = a_1^2 - 4a_2 a_3 \tag{10}$$

and the solutions are written as

$$x_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_3}}{2a_2} \tag{11}$$

and

$$x_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_3}}{2a_2} \tag{12}$$

The origin is a single equilibrium point for the equation (7) when $x_1 = x_2 = 0$, that is $a_2 a_3 = 0$. This involves that $a_2 = 0$, or $a_3 = 0$. As these coefficients must be different from zero, then the equation (7) cannot have a centre at the origin. However, in the next section, we exhibit harmonic and isochronous periodic solutions to the equation (7) in order to prove explicitly the existence of an isochronous centre at the origin.

3. Harmonic and isochronous solutions

We state the conditions of integrability in terms of exact and explicit general harmonic and isochronous solutions of the equation (7) and existence of a centre in this section.

3.1 Conditions of integrability of equation (7) and existence of a centre

To explicitly integrate the equation (7), consider the general class of velocity-dependent Lienard equations

$$\ddot{x} \pm \frac{1}{2}\frac{u'(x)}{u^2(x)}\sqrt{u(x)[b - a\vartheta(x)]}\,\dot{x} + \frac{a}{2}\frac{\vartheta'(x)}{u(x)} = 0 \qquad (13)$$

which can be formulated from the theory performed in [9, 13], where $u(x)$ and $\vartheta(x)$ are functions of x, a and b are constants. Putting $u(x) = e^{2\beta x}$, and $\vartheta(x) = (cx^2 - q)e^{2\beta x}$, leads to the equations

$$\ddot{x} \pm \beta\sqrt{be^{-2\beta x} - a(cx^2 - q)}\,\dot{x} + acx + ac\beta x^2 - aq\beta = 0 \qquad (14)$$

where c and q are free constants, which can reduce to

$$\ddot{x} \pm \beta\sqrt{aq - acx^2}\,\dot{x} + acx + ac\beta x^2 - aq\beta = 0 \qquad (15)$$

when $b = 0$. As can be observed, the equations (15) are frequency-dependent damping systems and a first integral of the equations (13) can read

$$b = u(x)\dot{x}^2 + a\vartheta(x) \qquad (16)$$

Substituting the previous expressions of $u(x)$ and $\vartheta(x)$ into the equation (16) yields a first integral of the equations (15) in the form

$$aq = \dot{x}^2 + acx^2 \qquad (17)$$

In this perspective, the equations (15) become completely integrable. Since the equation (7) is identical with the equation (15) for $+\beta$, $\alpha = aq$, $\gamma = ac$, $a_1 = ac$, $a_2 = ac\beta$, and $a_3 = -aq\beta$, then it becomes also completely integrable under these conditions. As such, the equations (15) of interest do not satisfy the existence theorem for an isochronous centre at the origin mentioned in the above. However, the time-independent first integral (17) can be interpreted as the Hamiltonian of the systems (15) under the form

$$H(x, y) = \frac{1}{2}aq = \frac{1}{2}y^2 + \frac{1}{2}ac\,x^2 \qquad (18)$$

where $aq > 0$, $ac > 0$ and $y = \dot{x}$. This Hamiltonian (18), as well known, corresponds to closed trajectories in the (x, y) phase plane with a centre at the origin. This shows that the equations (15) of interest have a centre at the origin, contrary to the predictions of usual existence theorems. Now, we can formulate the exact and isochronous solutions of the equations (15).

3.2 General harmonic and isochronous solutions

From the equation (17) or (18), one can write

$$\frac{dx}{\sqrt{aq - acx^2}} = \pm dt \tag{19}$$

which can be integrated to get the harmonic solution of the equations (15) in the form

$$x(t) = \sqrt{\frac{q}{c}} \sin\left[\pm \sqrt{ac}\left(t + K\right)\right] \tag{20}$$

This equation (20), which characterizes amplitude-dependent frequency oscillator, becomes isochronous when $c = 1$, that is

$$x(t) = \sqrt{q} \sin\left[\sqrt{a}\,\left(t + K\right)\right] \tag{21}$$

where K is a constant of integration. It is worth to notice that the solution (21) is also the solution of the linear harmonic oscillator

$$\ddot{x} + ax = 0 \tag{22}$$

where the amplitude of oscillations is kept at $\sqrt{q}$, $a > 0$, and $q > 0$. The equation (22) is equivalent with the equations (15) when $c = 1$. The solution (20) or (21) does not depend on β. Thus, for $\pm\beta > 0$, corresponding to positive damping or for $\pm\beta < 0$, corresponding to negative damping, the solution remains harmonic. In this regard, numerical examples can be given to illustrate the theory.

4. Numerical applications

The illustration of the analytical theory is shown in this part by comparison with numerical results. In this way, consider the general initial conditions $x(0) = x_0$, and $\dot{x}(0) = \vartheta_0$. Thus, substituting these conditions into the general solution (21), yields the system of algebraic equations

$$\begin{cases} x(0) = x_0 = \sqrt{q}\sin\left(\sqrt{a}K\right) \\ \dot{x}(0) = \vartheta_0 = \sqrt{aq}\cos\left(\sqrt{a}K\right) \end{cases} \tag{23}$$

from which, one can obtain

$$K = \frac{\sqrt{a}}{a} arc\cot an\left(\frac{\vartheta_0\sqrt{a}}{x_0 a}\right) \tag{24}$$

So with that, the general isochronous solution (21) takes the form

$$x(t) = \sqrt{q}\sin\left[\sqrt{a}\,t + arc\cot an\left(\frac{\vartheta_0\sqrt{a}}{x_0 a}\right)\right] \tag{25}$$

The comparison of the solution (25) with the result obtained from numerical integration is shown graphically on the Figure 1 under the conditions that $+\beta = 0.0025$, $a = 0.5$, $c = 1$, $q = 0.25$, $x_0 = 0.5$, and $\vartheta_0 = 0.001$.

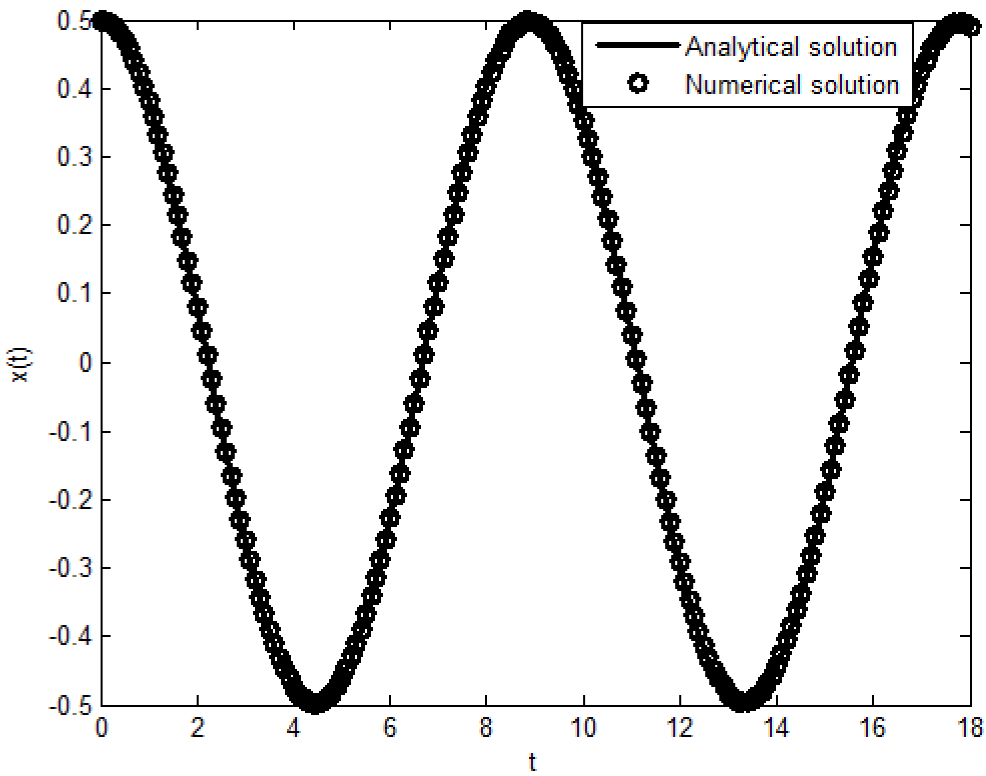

Figure 1: Comparison of solution (25) in solid line with numerical solution of equation (15) in circles line. Typical values are: $+\beta = 0.0025$, $a = 0.5$, $c = 1$, $q = 0.25$, $x_0 = 0.5$, and $\vartheta_0 = 0.001$.

As seen, there is an excellent agreement between numerical and exact analytical solutions. Now, we can address a conclusion for the work

Conclusion

We have presented in this paper a velocity-dependent Lienard equation consisting of a generalized Van der Pol-Helmholtz equation. The conditions of integrability in terms of periodic solutions are established. Thus, we have been able to exhibit the exact and general harmonic and isochronous solutions of the equation of interest, contrary to the predictions of usual existence theorems.

Ethics

Authors declare no conflict of interest.

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