A theory of viscoelastic crack growth-Revisited (Revised)

Richard A Schapery (✉ schapery@gmail.com)

University of Texas at Austin

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A theory of viscoelastic crack growth-Revisited

R.A. Schapery

Professor Emeritus, Department of Aerospace Engineering and Engineering Mechanics The

University of Texas at Austin

schapery@gmail.com

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Abstract

A theory of viscoelastic crack growth developed nearly five decades ago is generalized to express traction in the so-called fracture process zone or failure zone as a function of the crack opening displacement (COD). In earlier work, except for a minor exception, traction was specified. The current model leads to a nonlinear double integral that has to be solved for the COD before crack growth can be predicted. First, a closed-form, accurate approximation is found for a linear elastic body. We then show that this COD may be easily and accurately extended to linear viscoelasticity using a realistic, broad spectrum creep compliance. An analytical relationship between stress intensity factor and crack speed then follows. Consistent with earlier work, it is defined almost entirely by creep compliance. Five different failure zone tractions are employed; their differences are shown to have little effect on the crack growth other than through a speed shift factor. The Appendix discusses initiation of growth.

1.0 Introduction

The subject of crack growth in linear viscoelastic media has received considerable attention, both theoretically and experimentally. This may be seen in some recent and earlier reviews and analyses of the subject (Greenwood 2004; Kaminsky 2014; Knauss 2015; Rodriguez et al. 2020). A widely used idealization of the material that separates as the crack grows is that of a thin layer next to the crack tip called the *fracture process zone* or *failure zone*. This zone is where the continuum material begins to come apart and eventually separates entirely. In all cases known to the author, except for one, in which the failure zone is modeled as a continuum,

the traction is specified as a function of location. The exception is that in (Greenwood 2007), but it is limited to a 3-element model and uses a direct numerical method to solve the elasticity problem. A realistic, broad spectrum compliance is used for all the traction models developed in this paper.

It has been argued (Ciavarella 2020) that a realistic model should express the traction as a function of the opening displacement, as in Greenwood (2007). The author agrees, unless, of course, the material fails by yielding at a constant stress (the so-called Dugdale (1960) model,). As a result, the author re-examined his early work (Schapery 1973, 1975a,b,c), as well as that of (Greenwood and Johnson 1981) and was able to construct an explicit and simple solution to the problem using realistic models. We should add that the author is aware of one publication (Kaminsky 2014) in which the problem of predicting the failure-zone traction was solved by dividing the zone into several individual filaments, thus requiring the solution of a large complex set of equations.

The basic mathematical problem for a continuum failure zone involves solving a nonlinear single integral equation for an elastic body; and then a nonlinear double integral equation for the viscoelastic body. The solution is accomplished by an approximate method, but it is shown to be very accurate.

First, we consider only elastic behavior in Section 2. The results are used to construct the viscoelastic solution in Section 3 for continuous crack growth. A so-called modified power law is employed to characterize the viscoelasticity. The models used for failure zone traction and viscoelasticity are realistic, but the analysis method is not limited to them. The analysis leads to a simple closed-form relationship connecting crack speed to the stress intensity factor. The initiation of growth is briefly discussed in the Appendix.

2.0 Elasticity

2.1 Crack tip model

The idealized crack tip region is shown in Fig.1, in which the crack tip is defined to be the point P . The failure zone is idealized as a thin layer where all nonlinearity and bond breakage exist and may be elastic or viscoelastic. Outside of this zone the material is assumed to be

linearly elastic or viscoelastic and isotropic. Extension to an orthotropic body (Brockway and Schapery 1978) is readily accomplished if its material axes are aligned with those in Fig. 1.

For a linear elastic material, after removing the stress singularity using Barenblatt's (1962) condition, the crack opening displacement (COD) given in Schapery (1975a) is,

$$v = (C_g / 2\pi) \int_0^\alpha \sigma_f[v(x')] F(x'/x) dx' \quad (2.1)$$

where

$$F(x'/x) = 2 \sqrt{\frac{1}{(x'/x)}} - \ln \left[\frac{\sqrt{x'/x} + 1}{\sqrt{x'/x} - 1} \right] \quad (2.2)$$

and C_g is the compliance for a locally plane strain state,

$$C_g = 4(1 - \nu_g^2) / E_g \quad (2.3)$$

defined in terms of Young's modulus E_g and Poisson's ratio ν_g . For later work on viscoelasticity it is helpful to arbitrarily define the elastic properties as those in the short time *glassy* state, which is denoted with subscript g .

The singularity was removed by selecting the cohesive (or adhesive) traction σ_f to satisfy a condition in terms of the stress intensity factor K and the length of the failure zone α :

$$K = (2 / \pi)^{1/2} \int_0^\alpha [\sigma_f(x) / x^{1/2}] dx \quad (2.4)$$

The traction σ_f in this paper will be expressed in terms of the opening displacement v , which will then be found as a function of the distance x between a generic location in the failure zone and the crack tip, P , Fig.1. Eq.(2.4) is valid for both elasticity *and* viscoelasticity.

The specific form of the traction used throughout this paper is

$$\sigma_f = \sigma_m (1 + v / v_0)^{-q} \quad (2.5)$$

Greenwood and Johnson (1981) used this form with $q=3$ to model the effect of intermolecular adhesion. Here we shall use $q=0, 1, 2$, and 3 in example predictions of the theory; the case $q=0$ gives the so-called Dugdale traction. The quantity v_0 serves to nondimensionalise the displacement; it is a material constant. The nondimensional displacement $\bar{v} \equiv v/v_0$ must increase to satisfy condition Eq.(2.4) as K increases up to the critical value, K_c for crack growth. Its limiting value at the onset of crack growth is $\bar{v}_c$, which is assumed constant along with σ_m , for the purpose of simplifying the discussion; however, they *may vary with crack speed without changing the analysis*. These tractions for $q>0$ are shown in Fig.2 for $\bar{v}_c=1$, and for one case, $\bar{v}_c=3$.

The corresponding work input to a given column of material (or to the intermolecular force of adhesion) in the failure zone (above the crack plane and with unit cross-section) when it breaks (i.e. when traction at $x=\alpha$ vanishes) is the fracture energy,

$$\Gamma = \sigma_m v_0 \int_0^{\bar{v}_c} d\eta / (1+\eta)^q = \sigma_m v_0 \frac{1 - (\bar{v}_c + 1)^{1-q}}{q-1} \quad (2.6)$$

which is obviously independent of crack speed if σ_m and $\bar{v}_c$ are constant.

It should be noted that the COD and fracture energy in this paper refer to only those above the crack plane, and thus are one-half of what is sometimes used by others in cohesive crack growth.

To find $\bar{v}$ we must solve the integral equation, Eq. (2.1), except for $q=0$. The writer is not aware of any existing analytical method for this. While a totally numerical method could be used, this method is expected to be extremely tedious and time consuming, especially for the viscoelastic case. Here we use an approximate, but accurate, method that is as simple for viscoelastic media as for elastic media.

It is helpful to introduce additional dimensionless quantities. They, along with $\bar{v} \equiv v/v_0$, will be called *normalized* variables,

$$k_g \equiv C_g \sigma_m / 2\pi, \quad \bar{x} \equiv k_g x / v_0, \quad \bar{x}_1 \equiv k_g \alpha / v_0, \quad \bar{\Gamma} \equiv \Gamma / \sigma_m v_0, \quad \bar{\sigma} = \sigma_f / \sigma_m \quad (2.7)$$

Expressing Eq. (2.1) using these normalized quantities,

$$\bar{v} = \int_0^{\bar{x}_1} (1 + \bar{v}(\bar{x}'))^{-q} F(\bar{x}' / \bar{x}) d\bar{x}' \quad (2.8)$$

Similarly, Eq.(2.4) becomes

$$K^2 = \frac{2}{\pi} I_1^2 \sigma_m^2 \alpha \quad (2.9)$$

where

$$I_1 \equiv \int_0^1 (1 + \bar{v}(\eta \bar{x}_1))^{-q} \frac{d\eta}{\sqrt{\eta}} \quad (2.10)$$

which was originally defined in (Schapery (1975a)). It has its maximum value of 2 when $q=0$.

The energy release rate , ERR , and K are related through the equation (Anderson 2017)

$$ERR = C_g K^2 / 8 \quad (2.11)$$

At the start of crack growth ERR is equal to the fracture energy. Thus, equating ERR to Γ ,

and using Eq.(2.9), we find at the critical point,

$$\bar{x}_1 I_1^2 = 2\bar{\Gamma} \quad (2.12)$$

In addition to the requirement that the critical solution satisfies Eq.(2.12), it must also equal the critical displacement at $\bar{x}_1$,

$$\int_0^{\bar{x}_1} (1 + \bar{v}(\bar{x}'))^{-q} F(\bar{x}' / \bar{x}_1) d\bar{x}' = \bar{v}_c \quad (2.13).$$

We are now able to construct approximate solutions using the following form for all cases studied,

$$\begin{aligned} \bar{v}_A &= c_0 \bar{x}^{1.5} \text{ for } \bar{x} \leq \bar{x}_0 \\ \bar{v}_A &= c_1 \bar{x}^{c_2} \text{ for } \bar{x}_0 \leq \bar{x} \leq \bar{x}_1 \end{aligned} \quad (2.14)$$

The 1.5 exponent is required for small $\bar{x}$, according to the theory in (Barenblatt 1962). The solution for $\bar{x} \leq \bar{x}_1$ enables the prediction for $\bar{x} > \bar{x}_1$ using Eq.(2.8). The value of x_0 is found by invoking continuity at this point,

$$\bar{x}_0 = \left(\frac{c_1}{c_0} \right)^{\frac{1}{1.5-c_2}} \quad (2.15)$$

while Eq.(2.14) also provides the critical normalized failure zone length,

$$\bar{x}_1 = \left(\frac{\bar{v}_c}{c_1} \right)^{\frac{1}{c_2}} \quad (2.16)$$

Thus, there are three free constants, c_0, c_1, c_2 , while there are only two constraints, Eq.(2.12) and Eq.(2.13). Although one could use a minimum square error as the third condition, it would take considerable computational time in a search method. Instead, we manually selected c_2 to produce the smallest maximum error. For $q=1$ there is only one free constant (c_0 , later designated as c_1), which was found by making $\bar{v} = \bar{v}_c$ at $\bar{x} = \bar{x}_0$ (later designated as $\bar{x}_1$); condition Eq.(2.12) was then found to be closely satisfied. The “Find” program in Mathcad was used in all cases, which took less than one second per case on a laptop computer.

The *trial* solution Eq.(2.14) is used in Eq.(2.8) to predict the solution. If they agree, then the result is the exact solution. If they are very close we consider Eq.(2.14) an acceptable representation. Fig.3 for $\bar{v}_c = 1$ shows that Eq.(2.14) is indeed an acceptable solution; the result for $q = 2$, $\bar{v}_c = 3$ is similar to that for $q=3$. Table 1 lists the constants; for completeness, fitting constants for $q=0$ are included. Note that the exponent c_2 decreases as q and $\bar{v}_c$ increase. The result for $q=3$ agrees graphically with that in Greenwood and Johnson (1981).

3.0 Viscoelasticity.

The viscoelastic opening displacement is from (Schapery 1975a),

$$v_v = (1/2\pi) \int_{t_0}^t C_v(t-\tau) \left[\partial \left[\int_0^{\alpha} \sigma_f[v_v(x', \tau)] F(x'/x(X, \tau)) dx' \right] / \partial \tau \right] d\tau \quad (3.1)$$

where $t \geq t_0$ and t_0 is the time when the crack tip reaches the generic point X . Additionally,

$$x = x(X, \tau) = a(\tau) - X \quad (3.2)$$

3.1 Continuous crack growth

In this section we address only the problem of continuous growth, not the transient initiation of growth. The latter is covered in the Appendix. The prediction is simplified by the realistic assumption that the crack speed $\dot{a}$, is essentially constant during the time it takes for growth equal to the failure zone length α . Considering how small α is assumed to be (i.e. inside the singularity zone), this is not a serious restriction. Over the entire amount of growth, the speed $\dot{a}$ may vary widely. Without loss in generality for a locally homogeneous body there is no dependence on X , and we can set $t_0 = 0$ for notational simplicity. Thus, in Eq.(3.1)

$$x(X, \tau) = x(\tau) = \dot{a}\tau \quad (3.3)$$

A realistic creep compliance $C_v(t)$ will be used in the form of a modified power law,

$$\bar{C}(\bar{t}) = 1 + C_n \left(\frac{\bar{t}}{1 + \bar{t}} \right)^n \quad (3.4)$$

in which normalized quantities have been introduced,

$$\bar{C} \equiv C_v / C_g, \quad C_n \equiv \frac{C_e - C_g}{C_g}, \quad \bar{t} \equiv t / t_e \quad (3.5)$$

where C_g and C_e are the so-called glassy and equilibrium(elastic) compliances, respectively. The time t_e determines when the equilibrium compliance is approached and accounts for environmental effects, such as temperature and moisture, if the material is *rheologically simple*; these effects are assumed constant during the time taken for the crack to grow the current length of the failure zone α .

Elastic ($n=0$), viscous ($n=1$) and power law behavior ($\bar{t} \gg 1$) appear as special cases. The creep compliance for the polyurethane rubber in (Mueller and Knauss (1971) and in Schapery

1975c) is very well characterized with the values $n=0.5$, $C_n=145.7$, $t_e=0.84s$ ($0^\circ c$) . It is drawn in Fig 4 along with the other two compliances used in the examples in this paper. However, the theory will be developed using the general form Eq.(3.4).

After using normalizations, Eq.(3.1) becomes,

$$\bar{v}_v = \int_0^{\bar{t}} \left[1 + C_n \left(\frac{\bar{t} - \tau}{1 + \bar{t} - \tau} \right)^n \right] \left[\partial \left[\int_0^{\bar{x}_1} (1 + \bar{v}_v(\bar{y}))^{-q} F(\bar{y} / \bar{a} \tau) d\bar{y} \right] / \partial \tau d\tau \right] \quad (3.6)$$

Let us now define a normalized crack speed, $\bar{a}$,

$$\bar{a} \equiv k_g t_e \dot{a} / v_0 \quad (3.7)$$

which changes Eq.(3.3) to,

$$\bar{x} = \bar{a} \bar{t} \quad (3.8)$$

Eq.(3.6) becomes,

$$\bar{v}_v = \int_0^{\bar{x}} \left[1 + C_n \left(\frac{\bar{x} - z}{\bar{a} + \bar{x} - z} \right)^n \right] \left[\partial \left[\int_0^{\bar{x}_1} (1 + \bar{v}_v(y))^{-q} F(y / z) dy \right] / \partial z dz \right] \quad (3.9)$$

The problem is to find the $\bar{v}_v$ that satisfies the double integral Eq.(3.9) when $\bar{x} \leq \bar{x}_1$.

This can be done by first defining the *auxiliary displacement*,

$$v_w \equiv \int_0^{\bar{x}_1} (1 + \bar{v}_v(x'))^{-q} F(x' / \bar{x}) dx' \quad (3.10)$$

and introducing an effective compliance,

$$C_{eff}(\bar{x}, \bar{a}) \equiv 1 + C_n \frac{\int_0^{\bar{x}} \left[\left(\frac{\bar{x} - z}{\bar{a} + \bar{x} - z} \right)^n \right] [\partial v_w / \partial z] dz}{v_w} \quad (3.11)$$

Rewrite Eq.(3.9) using the effective compliance,

$$\bar{v}_v(\bar{x}, \bar{a}) = C_{eff}(\bar{x}, \bar{a}) \int_0^{\bar{x}_1} [1 + \bar{v}_v(x', \bar{a})]^{-q} F(x'/\bar{x}) dx' \quad (3.12)$$

In order to motivate the development of a solution $\bar{v}_v(\bar{x}, \bar{a})$ to Eq.(3.12), change variables,

$$y = C_{eff}(\bar{x}, \bar{a})\bar{x}', \quad y_1 = C_{eff}(\bar{x}, \bar{a})\bar{x}_1 \quad (3.13)$$

Thus,

$$\bar{v}_v(\bar{x}, \bar{a}) = \int_0^{y_1} [1 + \bar{v}_v(y, \bar{a})]^{-q} F(y/C_{eff}(\bar{x}, \bar{a})\bar{x}) dy \quad (3.14)$$

The solution to Eq.(3.14) would be the elastic solution with argument $\bar{x}$ replaced by $C_{eff}(\bar{x}, \bar{a})\bar{x}$ if the upper limit were $y_1 = C_{eff}(\bar{x}_1, \bar{a})\bar{x}_1$. Nevertheless, it motivates us to propose the solution,

$$\bar{v}_v = f(\bar{a})\bar{v}_A(C_{eff}(\bar{x}, \bar{a})\bar{x}) \quad (3.15)$$

where $f(0) = f(\infty) = 1$, and $\bar{v}_A$ is the elastic displacement, Eq(2.14). The coefficient $f(\bar{a})$ is found by using Eq.(3.12) to match $\bar{v}_v$ inside the integral (the *input*) to $\bar{v}_v$ outside (the *output*) at $\bar{x}_1$; this matching is done simultaneously with the determination of C_{eff} . That Eq.(3.15) is an accurate viscoelastic solution for all three q will be shown later, after f and C_{eff} are found.

We next observe that the displacement for a propagating crack is needed only at $\bar{x} = \bar{x}_1$. Let the viscoelastic solution at $\bar{x}_1$ be written as,

$$\bar{v}_{1v} = f(\bar{a})\bar{v}_A(C_{eff}(\bar{x}_1, \bar{a})\bar{x}_1) \quad (3.16)$$

Using approximate Eq.(2.14), the viscoelastic solution is simply,

$$\bar{v}_{1v} = f(\bar{a})c_1[C_{eff}(\bar{x}_{1v}, \bar{a})\bar{x}_{1v}]^{c_2} \quad (3.17)$$

where $x_{1v} = x_{1v}(\bar{a})$ is the root of the equation,

$$\bar{v}_c - f(\bar{a})c_1[C_{eff}(\bar{x}, \bar{a})\bar{x}]^{c_2} = 0 \quad (3.18)$$

which selects $\bar{x}_{lv}$ to be at the end of the failure zone as the crack propagates at speed $\bar{a}$ with $\bar{v} = \bar{v}_c$.

Observe that $f(\bar{a})=1$ for the Dugdale model because Eq.(3.14) must reduce to $\bar{v}_{lv} = \bar{v}_A(C_{eff}(\bar{x}_1, \bar{a})\bar{x}_1)$ when $q=0$.

3.2 Prediction of the effective compliance.

The problem has been reduced to finding $C_{eff}(\bar{x}, \bar{a})$ and $f(\bar{a})$. We will show that C_{eff} is well-approximated by the creep compliance itself, but shifted by a constant along the $\log(\bar{a})$ axis; this simplicity greatly aids the determination of $f(\bar{a})$ and C_{eff} simultaneously. The process is further simplified by neglecting the low speed portion of the compliance *for now*, using

$$\bar{C} = 1 + C_n \left(\frac{\bar{x}}{\bar{a}} \right)^n \quad (3.19)$$

As the first case, consider the Dugdale traction, for which $q=0$. The auxiliary displacement Eq.(3.10) is the elastic displacement, Eq.(2.8), although dependent on crack speed through $\bar{x}_1$. Fig.5 shows it is well-approximated by a simple power law, with $p=2$, for all speeds,

$$v_w = c\bar{x}^p \quad (3.20)$$

in which the speed-dependent coefficient obviously has no effect on C_{eff} . Upon using Eq.(3.20) in Eq.(3.11) we find,

$$\bar{C}_{eff} = 1 + C_n \left(\frac{\bar{x}}{s_f \bar{a}} \right)^n \quad (3.21)$$

where s_f is a shift factor on the $\log(\bar{a})$ axis. It can be expressed in terms of gamma functions,

$$s_f = \left[\frac{\Gamma(p+n+1)}{\Gamma(p+1)\Gamma(n+1)} \right]^{1/n} \quad (3.22)$$

This is the same shift factor reported in (Schapery 1975, Eq.28). Fig. 6 covers the range of all exponents, p , n and q in this paper. Fig.5 was plotted using $n=0.5$ and $\bar{a}=1$; but all three n and all $\bar{a}$ gave essentially the same agreement. The difference between the two curves in Fig.5 produces only a 1% difference in $C_{eff}(\bar{x}_1)$.

Let us now predict C_{eff} at $\bar{x}_1$ for $q=1, 2$ and 3. We will show that the auxiliary function is also well-approximated by Eq.(3.20) in which p is independent of crack speed. For this, Eq.(3.15) is substituted into Eq.(3.10), using Eq.(3.21) as a trial input, with p and c selected such that Eq.(3.20) and Eq.(3.10) agree closely in shape and match at $\bar{x}_1$, respectively. At the same time $f(\bar{a})$ is found by matching the associated input and output $\bar{v}_v$ at $\bar{x}_{1v}$ for a set of $\bar{a}$ spaced one-decade apart, and then fitting it to a sixth order polynomial.

Figs.7 and 8 show this factor and input-output pairs for selected cases. Results for all other cases provide even closer agreement between input and output, and therefore are not shown. It is found for $q=1$ and $q=2$ ($v_c=1$) that $p=1.5$ and 1.3, respectively; the deviation of $f(\bar{a})$ and I_R (Eq.(3.33) from unity was less than that in Fig. 7. The speed chosen for Fig.8 is close to the maximum of the curves in Fig.7, which produces the largest input-output difference, although barely graphically distinguishable.

. We have found that for all $\bar{a}$ and n , the difference between input Eq.(3.20) and output Eq.(3.10) is essentially negligible because of the insensitivity of the shape of $\bar{v}_w(\bar{x})$ to these quantities and the fact that c in Eq.(3.20) does not affect C_{eff} .

The complete effective compliance will be used in predicting crack growth so that the long-time equilibrium value is approached smoothly,

$$\bar{C}_{eff} = 1 + C_n \left(\frac{\bar{x}}{s_f \bar{a} + \bar{x}} \right)^n \quad (3.23)$$

which may be expressed more concisely as

$$\bar{C}_{eff} = \bar{C} \left(\frac{\bar{x}_{lv}}{s_f \bar{a}} \right) \quad (3.24)$$

where $\bar{C}$ is the normalized creep compliance, Eq.(3.4).

Although s_f was found without the low speed term, we shall now show that the approximate effective compliance Eq.(3.23) is quite good for all $\bar{a}$, compared to the use of the complete creep compliance, Eq.(3.4), in deriving C_{eff} . It is sufficient here to express the latter using the single variable $\bar{t}$:

$$c\bar{C}_{eff}(\bar{t}) \equiv 1 + C_n \frac{\int_0^{\bar{t}} \left[\frac{\bar{t} - \tau}{1 + \bar{t} - \tau} \right]^n \left[\frac{dv_w(\tau)}{d\tau} \right] d\tau}{v_w(\bar{t})} \quad (3.25)$$

where v_w is in Eq.(3.20), but with τ replacing $\bar{x}$. The value of p is affected only slightly at low speeds when Eq.(3.25) is used in Eq.(3.10).

Fig.9 shows, in the approach to equilibrium, the ratio R_c of Eq.(3.23) to Eq. (3.25) for the minimum and maximum values of p and three values of n . In this near-equilibrium period the difference is quite small, especially for the common range $n \leq 0.5$. The case $n=1$ only characterizes a viscous fluid, which has no equilibrium compliance; but it is used here to complete the upper end of the thermodynamically allowable range, $0 \leq n \leq 1$. Eq.(3.24) is used in the remainder of this paper.

Finally, we find from Eqs.(2.2) and (3.12) that

$$v_v \propto C_{eff}(\bar{x}, \bar{a}) \sqrt{\bar{x}} \text{ for } \bar{x} \gg \bar{x}_{lv}. \quad (3.26)$$

Thus, the displacement for large $\bar{x}$ does not behave as the elastic singular displacement, $\sqrt{\bar{x}}$. When the compliance is a power law,

$$v_v \propto \bar{x}^{n+0.5} \text{ for } \bar{x} \gg \bar{x}_{lv}. \quad (3.27)$$

3.3 Prediction of crack growth.

Eqs.(2.9) and (3.18) enable the prediction of crack speed as a function of the stress intensity factor. First we note that Eqs.(2.9) and (2.10) are valid for elastic and viscoelastic elastic materials. In the former case one uses Eq.(2.16) for $\bar{x}_1$ while in the latter case $\bar{x}_1 = \bar{x}_{1v}$, where $\bar{x}_{1v}$ is found from Eq.(3.18) . Let us define

$$\bar{\alpha} \equiv \bar{x}_{1v} = \frac{k_g}{v_0} \alpha \quad (3.28)$$

and introduce the stress intensity factor using Eq.(2.9) and fracture energy Eq.(2.12). Finally,

$$\bar{\alpha} = \frac{K^2 I_{1g}^2 C_g}{8 I_1^2 \Gamma} \left(\frac{\bar{v}_c}{c_1} \right)^{\frac{1}{c_2}} \quad (3.29)$$

where

$$I_1 \equiv \int_0^1 \left[1 + \bar{v}(\eta \bar{\alpha}) \right]^{-q} \frac{d\eta}{\sqrt{\eta}} \quad (3.30)$$

and I_{1g} is for the elastic state,

$$I_{1g} \equiv \int_0^1 \left[1 + \bar{v} \left(\eta \left(\frac{\bar{v}_c}{c_1} \right)^{\frac{1}{c_2}} \right) \right]^{-q} \frac{d\eta}{\sqrt{\eta}} \quad (3.31)$$

whose values are those of I_1 in Table 1.

Next substitute Eq.(3.29) into Eq.(3.18) and find,

$$\Gamma = \frac{1}{8} \left(\frac{K}{I_R} \right)^2 C_v \left(\frac{\bar{\alpha}}{s_f \bar{a}} \right) \quad (3.32)$$

where,

$$I_R \equiv \frac{I_1}{I_{1g} f^{\frac{1}{2c_2}}} \quad (3.33)$$

Observe that

$$\frac{\bar{\alpha}}{\bar{a}} = \frac{\alpha}{t_e \dot{a}} \quad (3.34)$$

is the normalized time for the crack to propagate the length of the failure zone. The shift factor s_f decreases this time to an *effective time for creep softening*. For an elastic body $I_R = 1$, which reduces the right side of Eq.(3.32) to the energy release rate.

Introduce a normalized stress intensity factor,

$$\bar{K} \equiv \sqrt{\frac{C_g}{\Gamma}} K \quad (3.35)$$

which reduces Eq.(3.32) to

$$\frac{1}{8} \left(\frac{\bar{K}}{I_R} \right)^2 \bar{C} \left(\frac{\bar{\alpha}}{s_f \bar{a}} \right) = 1 \quad (3.36)$$

Next, use Eqs.(3.33) and (3.35) in Eq.(3.29) to express $\bar{\alpha}$ in terms of quantities in Eq.(3.36),

$$\bar{\alpha} = \frac{1}{s_q f^{c_2}} \left(\frac{\bar{K}}{I_R} \right)^2 \quad (3.37)$$

where

$$s_q \equiv 8 \left(\frac{c_1}{\bar{v}_c} \right)^{\frac{1}{c_2}} \quad (3.38)$$

Values of this shift factor are in Table 1. Eq.(3.37) can be simplified to

$$\bar{\alpha} = \frac{1}{s_q} \left(\frac{I_{1g}}{I_1} \right)^2 \bar{K}^2 \quad (3.39)$$

which is drawn in Fig.10. The dotted lines are for $I_R = f = 1$, which shows these functions have only a small effect on $\bar{\alpha}$. Note that $\bar{\alpha} \rightarrow 8/s_q$ as $\bar{a} \rightarrow \infty$ in view of Eq.(3.39) and Eq.(3.44).

Finally, the crack growth equation becomes,

$$\frac{1}{8}(\bar{K}_R)^2 \bar{C} \left(\frac{\bar{K}_R^2}{s_q s_f A} \right) = 1 \quad (3.40)$$

where

$$\bar{K}_R \equiv \frac{\bar{K}}{I_R}, \quad A \equiv f^{\frac{1}{c_2}} \bar{a} \quad (3.41)$$

For the Dugdale model $I_R = 1$ because $I_1 = 2$ and $f(\bar{a}) = 1$, regardless of crack speed.

Observe that Eq.(3.40) provides a relationship between K_R and $s_q s_f A$ that is independent of the failure zone properties and depends only on the normalized creep compliance $\bar{C}(\bar{t})$. It may be solved explicitly for A , given the normalized compliance in Eq.(3.4). Thus,

$$A = \frac{K_R^2}{s_q s_f} \frac{1-S}{S} \quad (3.42)$$

where,

$$S \equiv \left(\frac{8}{C_n K_R^2} - \frac{1}{C_n} \right)^{1/n} \quad (3.43)$$

Let *glassy* $\bar{K}_g$ and *equilibrium*(elastic) $\bar{K}_e$ define the normalized stress intensity factors corresponding to the high ($S=0$) and low($S=1$) speed limits, respectively. Since $I_R = 1$ at these limits, we find,

$$\bar{K}_g = \sqrt{8} \quad \text{and} \quad \bar{K}_e = \sqrt{\frac{8}{C_n + 1}} \quad (3.44)$$

Except for these limits, I_R and A are rather involved function of crack speed. Taking this behavior into account, we can easily solve Eq.(3.40) numerically for $\bar{K}$, given $\bar{a}$.

Eq.(3.36) is plotted in Fig.11 with solid lines. Eq.(3.42) is plotted using dotted lines; the dotted lines, of course, can be interpreted as plots of $\log(\bar{K})$ vs. $\log(\bar{a})$ when $I_R = f = 1$.

Because $I_R = f = 1$ and $s_q = 16$ for the Dugdale model, the dotted lines in Fig.11a are also the solution for this model except for a small difference in s_f and s_q .

In an intermediate speed range Eq.(3.42) obeys the power law,

$$K_R = \left(\frac{8s_q^n s_f^n}{C_n} \right)^{\frac{1}{2(1+n)}} A^{\frac{n}{2(1+n)}} \quad (3.45)$$

which appears as a light straight line in Fig.11. The slope of this line for $n=0.5, 1/6$, is in agreement with the experimental data in Schapery (1975c) (neglecting the small effect of the speed dependence of I_1 and f). which was obtained by Mueller and Knauss (1971). As Fig. 11 shows, the effect of this speed dependence is quite small in a realistic range of n for polymeric solids.

Except for $q=2, \bar{v}_c=3$, there is very little difference in the $\bar{a}$ position for each q because all are shifted on the log scale about the same amount relative to the creep compliance. In the former case the relative shift is noticeable because of the small value of s_q in Table 1.

It should be added that if we set $f=1$, the change in $\log(\bar{K})$ vs. $\log(\bar{a})$ for $n=0.5$ and $n=0.25$ is practically negligible as a result of near-cancellation of its effect through Eqs.(3.33) and A, Eq.(3.41).

4.0 Conclusions

We have shown that crack growth in a linear viscoelastic body is defined almost entirely by the creep compliance when the failure zone traction is constant or a monotonically decreasing function of the crack opening displacement. The different failure zone tractions had only a small effect on $\bar{K}$ vs. $\bar{a}$, except through the shift factor s_q for the $\bar{v}_c = 3$ case. Because the shift factor is combined with failure zone properties, it cannot uniquely establish the form of the underlying function $\sigma_f(u)$ from only experimental data on crack growth. We have also examined an unreported case in which the traction vs. COD smoothly decreases and then increases slightly as the critical value is reached (cf. Fig.1), using a partial sine function; the methods used in this paper were found to work well.

All examples utilized a creep compliance that was characterized in an intermediate time period by a single power law. For some materials the intermediate period is very broad, covering many decades, with a gradually changing log-log slope. As shown in Schapery (1975b), the local log-log slope of the creep compliance can be used in place of a single, constant exponent, n to predict the effective compliance because it has a narrow-band weight function.

The crack growth Eq.(3.32) is identical to that in Schapery (1975b) if I_1 is constant and $f=1$. Here, the model enabled prediction of the crack speed dependence of I_1 and f .

The present model predicts unstable crack growth only when the glassy compliance is reached, or if one or more of the failure zone properties decrease with speed at some point. Instabilities at speeds lower than glassy speed in filled elastomers have been reported and explained by continuum nonlinearities (e.g. Morishita M et al. 2016). As shown in Schapery (1984), instabilities at these lower speeds are also predicted by embedding a failure zone inside a thin *process* zone with viscoelastic properties (different from the continuum) and using the so-called pseudo J -integral as the crack driving force; the craze zone in some plastics could be a candidate fitting this description. This J -integral and crack growth theory for the shear-mode has been used to predict rubber friction with Schallamach waves (Schapery 2020a,b).

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6.0 Appendix- Initiation of crack growth.

We continue to use Eq.(3.1), but now have a transient problem. Referring to Fig.1, the crack tip in the *initial* unloaded state is located at the coordinate system's origin, $X=0$. Thus, the left end of the failure zone in the initiation phase is at the point,

$$x_1(t) = \alpha(t) \quad (5.1)$$

Other points in the failure zone are located at,

$$x(X,t) = \alpha(t) - X > 0. \quad (5.2)$$

With this notation Eq.(3.1) becomes,

$$v(X,t) = (1/2\pi) \int_0^t C_v(t-\tau) [\partial v_w(X,\tau) / \partial \tau] d\tau \quad (5.3)$$

where v_w is defined as an auxiliary displacement,

$$v_w(X,\tau) \equiv \left[\int_0^{\alpha(\tau)} \sigma_f[v(X',\tau)] F \left[\frac{\alpha(\tau) - X'}{\alpha(t) - X} \right] dX' \right] \quad (5.4)$$

The problem of predicting the time at which crack propagation starts is analogous to that of continuous growth. However, it is greatly simplified if the so-called quasi-elastic approximation (Schapery 1965,1967) or if the Dugdale model is applicable. The former requires that both the

creep compliance and $v_w(X, t)$ (or their logarithms) exhibit small curvature on the logarithmic time scale. This approximation for the present problem reduces Eq.(5.3) to

$$v(X, t) = \frac{C_v(t)}{2\pi} v_w(X, t) \quad (5.5)$$

which is the elastic solution, but with creep compliance in place of the elastic compliance. Thus, the fracture initiation time t_c is the root of

$$\Gamma_t - \frac{C_v(t_c)}{8} K(t_c)^2 = 0, \quad (5.6)$$

where Γ_t is not necessarily the elastic fracture energy.

The Dugdale model, under the assumption that $K(t)$ is nondecreasing, provides a simple, exact solution for initiation time t_c (Schapery 1975b), which is the root of

$$\Gamma_t - \frac{C_{ief}(t_c)}{8} K(t_c)^2 = 0 \quad (5.7)$$

where

$$C_{ief}(t_c) = \frac{1}{K(t_c)^2} \int_0^{t_c} C_v(t_c - \tau) \frac{dK(\tau)^2}{d\tau} d\tau \quad (5.8)$$

is an effective (or secant) compliance. When $K(t)$ is a step function of time applied at $t=0$ this result obviously reduces to Eq.(5.6), which does not require Dugdale specialization.

We can now estimate the time t_c relative to the time taken for the crack to propagate the length of the failure zone, $\frac{\alpha}{\dot{a}}$, immediately after growth initiates. Comparing Eqs(3.32) and (5.6), assuming K is constant after it is applied stepwise, and that $\Gamma_t \approx \Gamma_e$ and $I_R \approx 1$,

$$t_c \approx \frac{\alpha}{s_f \dot{a}} \quad (5.9)$$

Referring to Fig.6, after growth begins it takes 3–4 times the initiation time for the crack to grow the length of the failure zone for the Dugdale model, and slightly less for the other models. Thus, time for the start of continuous growth may be negligible in many cases.

Table 1. Constants for elastic solutions (except for s_q).

q	v_c	c_0	c_1	c_2	I_1	s_q
0	1	2.10	4.56	2.19	2	16.0
1	1	0	2.88	1.5	1.67	16.2
2	1	3.59	2.25	1.23	1.39	15.5
2	3	3.00	1.61	0.819	0.836	3.74
3	1	4.11	1.77	1.00	1.15	14.2

Fig.1 Normal stresses acting along crack plane in the neighborhood of the crack tip **P**.

Fig. 2 Normalized failure zone tractions, $\bar{\sigma} = \sigma_f / \sigma_n$, vs. opening displacement.

Fig.3 Comparison of input (solid line) and output (dots) elastic CODs for $v_c = 1$.

Fig. 4 Creep compliances.

Fig.5. Predicted (dots) and power law ($p=2$, solid line) auxiliary displacements for the Dugdale model ($q=0$).

Fig. 6 Shift factor

Fig.7 The factor f and modified traction integral I_R vs. logarithmic crack speed..

Fig.8 Comparison of input (solid line) and output (dots) for auxiliary displacement and COD using $n=1$ and $\bar{a} = 0.1$.

Fig.9. Ratio of approximate-to-complete effective compliances in the transition to long-time elastic behavior.

Fig. 10. Failure zone length vs. $\log(\text{crack speed})$.

Fig.11. Log(stress intensity factor) vs. log(crack speed).