A theory of viscoelastic crack growth-Revisited (Revised)

Richard A Schapery (✉ schapery@gmail.com)

University of Texas at Austin

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A theory of viscoelastic crack growth-Revisited (Revised)

R.A. Schapery

Professor Emeritus, Department of Aerospace Engineering and Engineering Mechanics

The University of Texas at Austin

schapery@gmail.com

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Abstract

A theory of viscoelastic crack growth developed nearly five decades ago is generalized to express traction in the so-called fracture process zone or failure zone as a function of the crack opening displacement (COD). In earlier work, except for minor exceptions, traction was specified as a function of location. The new model leads to a nonlinear double integral that has to be solved for the COD before crack growth can be predicted. First, a closed-form, accurate approximation is found for a linear elastic body. We then show that this COD may be easily and accurately extended to linear viscoelasticity using a realistic, broad spectrum creep compliance. An analytical relationship connecting the stress intensity factor to crack speed then follows. Consistent with earlier work, it is defined almost entirely by the creep compliance. Five different failure zone tractions are employed; their differences are shown to have little effect on crack growth other than through a speed shift factor. The Appendix discusses initiation of growth.

1.0 Introduction

The subject of crack growth in linear viscoelastic media has received considerable attention, both theoretically and experimentally. This may be seen in some recent and earlier reviews and analyses of the subject (Greenwood 2004; Kaminsky 2014; Knauss 2015; Rodriguez et al. 2020). A widely used idealization of the material that separates as the crack grows is that of a thin layer next to the crack tip called the *fracture process zone* or *failure zone* (FZ). This zone is where the continuum material begins to come apart and eventually separates entirely. In all

cases known to the author, except for two, in which the FZ is modeled as a continuum, the traction is specified as a function of location. One exception is in Greenwood (2007), but it is limited to a 3-element model and uses an iteration method of solution; the other is in Schapery (1975b, Appendix) but only a highly approximate analytical solution is found. In this paper accurate analytical solutions are obtained using a realistic, broad spectrum compliance for five traction models..

It has been argued (Ciavarella 2020) that a realistic model should express FZ traction as a function of the opening displacement, as in Greenwood (2007). The author agrees, unless, of course, the material fails by yielding at a constant stress (the so-called Dugdale (1960) model,). As a result, the author re-examined his early work (Schapery 1973, 1975a,b,c), as well as that of (Greenwood and Johnson 1981) and was able to construct an explicit and simple solution to the problem using realistic models. We should add that the author is aware of one publication (Kaminsky 2014) in which the problem of predicting the failure-zone traction was solved by dividing the zone into several individual filaments, thus requiring the solution of a large complex set of equations.

The basic mathematical problem for a continuum FZ involves solving a nonlinear single integral equation for an elastic body and then a nonlinear double integral equation for the viscoelastic body. An approximate analytical solution is found, but it is shown to be quite accurate.

First, we consider only elastic behavior in Section 2. The results are used to construct the viscoelastic solution in Section 3 for continuous crack growth. A so-called modified power law is employed to characterize the viscoelasticity. The models used for FZ traction and viscoelasticity are believed to be realistic, but the analysis method is not limited to them. The analysis leads to explicit relationships connecting the crack speed to the stress intensity factor and to the crack opening displacement. . The initiation of growth is briefly discussed in the Appendix. *Mathcad* is used for all calculations.

2.0 Elasticity

2.1 Crack tip model

The idealized crack tip region is shown in Fig.1, in which the crack tip is defined to be point P . The FZ is idealized as a thin layer of length α where all nonlinearity and bond breakage exist and may be elastic or a general rate-dependent material. Outside of this zone, the material is assumed to be linearly elastic or viscoelastic and isotropic. Extension to an orthotropic body (Brockway and Schapery 1978) is readily accomplished if its material axes are aligned with those in Fig. 1.

For a linear elastic material, after removing the stress singularity using Barenblatt's (1962) condition, the crack opening displacement (COD) given in Schapery (1975a) is

$$v = (C_g / 2\pi) \int_0^\alpha \sigma_f[v(x')] F(x'/x) dx' \quad (2.1)$$

where

$$F(x'/x) = 2\sqrt{\frac{1}{(x'/x)}} - \ln \left[\frac{\sqrt{x'/x} + 1}{\sqrt{x'/x} - 1} \right] \quad (2.2)$$

and C_g is the compliance for a locally plane strain state,

$$C_g = 4(1 - \nu_g^2) / E_g \quad (2.3)$$

defined in terms of Young's modulus E_g and Poisson's ratio ν_g . For later work on viscoelasticity, it is helpful to arbitrarily define the elastic properties as those in the short time *glassy* state, which is denoted with subscript g .

The singularity was removed by selecting the cohesive (or adhesive) traction σ_f to satisfy a condition in terms of the stress intensity factor K and the length of the FZ, α :

$$K = (2 / \pi)^{1/2} \int_0^\alpha [\sigma_f(x) / x^{1/2}] dx \quad (2.4)$$

The traction σ_f in this paper will be expressed in terms of the opening displacement v , which will then be found as a function of the distance x between a generic location in the FZ and the crack tip, P , Fig.1. Eq.(2.4) is valid for both elasticity and viscoelasticity.

The specific form of the traction used throughout this paper is

$$\sigma_f = \sigma_m (1 + v / v_0)^{-q} \quad (2.5)$$

Greenwood and Johnson (1981) used this form with $q = 3$ to model the effect of intermolecular adhesion. Here, we shall use $q=0, 1, 2$, and 3 in example predictions of the theory; the case $q=0$ gives the so-called Dugdale traction. The quantity v_0 serves to nondimensionalise the displacement; it is a material constant. The nondimensional displacement $\bar{v} \equiv v / v_0$ must increase to satisfy condition Eq.(2.4) as K increases up to the *critical* value, K_c , for the start of crack growth. Its limiting value at the onset of crack growth is $\bar{v}_c$, which is assumed to be constant along with σ_m , for the purpose of simplifying the discussion; however, for both elastic and viscoelastic global behavior, they *may vary with crack speed* without changing the method of analysis. These tractions for $q>0$ are shown in Fig.2 for $\bar{v}_c=1$, and for one case, $\bar{v}_c=3$.

The corresponding work input to a given column of material (or to the intermolecular force of adhesion) in the FZ ,above the crack plane and with unit cross-section, when it breaks (i.e. when traction at $x=\alpha$ vanishes) is the fracture energy,

$$\begin{aligned} \Gamma &= \sigma_m v_0 \int_0^{\bar{v}_c} d\bar{v} / (1 + \bar{v})^q = \sigma_m v_0 \frac{1 - (\bar{v}_c + 1)^{1-q}}{q-1} \quad \text{if } q \neq 1 \\ \Gamma &= \sigma_m v_0 \log(\bar{v}_c + 1) \quad \text{if } q = 1 \end{aligned} \quad (2.6)$$

which is obviously independent of crack speed if σ_m and $\bar{v}_c$ are constant. With *Mathcad* it is not necessary to use the logarithm when $q=1$.

It should be noted that the COD and fracture energy in this paper refer to only those above the crack plane, and thus are one-half of what is sometimes used by others in cohesive crack growth.

To find $\bar{v}$ we must solve the integral equation, Eq. (2.1), except for $q=0$. The writer is not aware of any existing analytical method for this. While an iterative or other numerical method could be used, as in Greenwood (2007), this method would greatly complicate the viscoelastic solution. Here, we use an approximate, but accurate, method that is nearly as simple for viscoelastic media as for elastic media.

It is helpful to introduce additional dimensionless quantities. They, along with $\bar{v} \equiv v / v_0$, will be called *normalized* variables,

$$k_g \equiv C_g \sigma_m / 2\pi, \quad \bar{x} \equiv k_g x / v_0, \quad \bar{x}_1 \equiv \bar{\alpha} \equiv k_g \alpha / v_0, \quad \bar{\Gamma} \equiv \Gamma / \sigma_m v_0, \quad \bar{\sigma} = \sigma_f / \sigma_m \quad (2.7)$$

Expressing Eq. (2.1) using these normalized quantities,

$$\bar{v} = \int_0^{\bar{x}_1} (1 + \bar{v}(\bar{x}'))^{-q} F(\bar{x}' / \bar{x}) d\bar{x}' \quad (2.8)$$

Partial normalization of Eq.(2.4) yields,

$$K^2 = \frac{2}{\pi} I_1^2 \sigma_m^2 x_1 \quad (2.9)$$

where

$$I_1 \equiv \int_0^1 (1 + \bar{v}(\eta \bar{x}_1))^{-q} \frac{d\eta}{\sqrt{\eta}} \quad (2.10)$$

which was originally defined in (Schapery (1975a)). It has its maximum value of 2 when $q=0$.

The energy release rate, ERR , and K are related through the equation (e.g. Anderson 2017)

$$ERR = C_g K^2 / 8 \quad (2.11)$$

At the start of crack growth ERR is equal to the fracture energy. Thus, equating ERR to Γ ,

and using Eq.(2.9), we find at the critical point,

$$\bar{x}_1 I_1^2 = 2\bar{\Gamma} \quad (2.12)$$

In addition to the requirement that the critical solution satisfies Eq.(2.12), it must also equal the critical displacement at $\bar{x}_1$,

$$\int_0^{\bar{x}_1} (1 + \bar{v}(\bar{x}'))^{-q} F(\bar{x}' / \bar{x}_1) d\bar{x}' = \bar{v}_c \quad (2.13).$$

We are now able to construct approximate solutions using the following form for all cases studied:

$$\begin{aligned}\bar{v}_A &= c_0 \bar{x}^{1.5} \quad \text{for } \bar{x} \leq \bar{x}_0 \\ \bar{v}_A &= c_1 \bar{x}^{c_2} \quad \text{for } \bar{x}_0 \leq \bar{x} \leq \bar{x}_1\end{aligned}\tag{2.14}$$

The exponent 1.5 is required for small $\bar{x}$, according to the theory in Barenblatt (1962). The solution for $\bar{x} \leq \bar{x}_1$ enables the prediction for $\bar{x} > \bar{x}_1$ using Eq.(2.8). The value of $\bar{x}_0$ is found by invoking continuity at this point,

$$\bar{x}_0 = \left(\frac{c_1}{c_0} \right)^{\frac{1}{1.5-c_2}}\tag{2.15}$$

while Eq.(2.14) provides the critical normalized FZ length,

$$\bar{x}_1 = \left(\frac{\bar{v}_c}{c_1} \right)^{\frac{1}{c_2}}\tag{2.16}$$

Thus, there are three free constants, c_0, c_1, c_2 , while there are only two constraints, Eq.(2.12) and Eq.(2.13). Although one could use a minimum square error between *input*, Eq.(2.14), and *output*, Eq.(2.8), as the third condition, it would take considerable computational time with the available search method. Instead, we evaluated input and output at 20 evenly spaced points and manually selected c_2 to produce the smallest maximum absolute error over the range $0 \leq \bar{x} \leq \bar{x}_1$. For $q=1$ there is only one undetermined parameter c_1 (with $c_0=0$, $c_2=1.5$ and $\bar{x}_0=0$) which was found by making $\bar{v} = \bar{v}_c$ at $\bar{x} = \bar{x}_1$; condition Eq.(2.12) was then found to be closely satisfied. The “Find” program in Mathcad was used in the other cases, which took less than one second per case on a laptop computer for each guessed c_2 ; the trial and error method to select c_2 took very little time because the acceptable range of c_2 was small.

The trial solution Eq.(2.14) is used in Eq.(2.8) to predict the solution. If they agree, then the result is the exact solution. If they are very close we consider Eq.(2.14) an acceptable representation. Fig.3 for $\bar{v}_c=1$ shows that Eq.(2.14) is indeed acceptable; not shown are the

results for $q = 2$ ($\bar{v}_c = 3$) and $q=3$ ($\bar{v}_c = 1.5$), in which the input and output agree like that for $q=3$, $\bar{v}_c=1$.

Table 1 lists the constants; for completeness, fitting constants for $q=0$ are included as well as constants defined in Section 3. Note that the exponent c_2 decreases as q and $\bar{v}_c$ increase; with increasing $\bar{v}_c$ this exponent approaches that for the far- field, 0.5. The values listed for $\bar{\alpha} \equiv \bar{x}_1$ and I are for the critical state. The displacement for $q=3$ $\bar{v}_c=1$ agrees graphically with that in Greenwood and Johnson (1981).

The largest $\bar{v}_c$ for which Eq.(2.14) remains valid is determined primarily by the length of the traction tail; i.e. the value of $\sigma_f(\bar{v}_c)/\sigma_m$. Viscoelasticity solutions, using the method in Section 3, start to degrade significantly when this ratio is less than 6% (which is the value for $q=2$ ($\bar{v}_c = 3$) and $q=3$ ($\bar{v}_c = 1.5$)). The elastic solutions do not begin to degrade until this ratio is somewhat smaller. However, a long tail with a limiting value of 6% may be unrealistic because of microscopic nonhomogeneity with cohesive cracks or surface roughness with adhesive cracks. In view of this, in the next section we omit predictions for $q=3$ ($\bar{v}_c = 1.5$) but include those for $q=2$ ($\bar{v}_c = 3$) to illustrate the effect of a long tail.

3.0 Viscoelasticity.

The viscoelastic opening displacement is from (Schapery 1975a),

$$v_v = (1/2\pi) \int_{t_0}^t C_v(t-\tau) \left[\partial \left[\int_0^a \sigma_f[v_v(x',\tau)] F(x'/x(X,\tau)) dx' \right] / \partial \tau d\tau \right] \quad (3.1)$$

where $t \geq t_0$ and t_0 is the time when the crack tip reaches the generic point X . Additionally,

$$x = x(X, \tau) = a(\tau) - X \quad (3.2)$$

3.1 Continuous crack growth

In this section we address only the problem of continuous growth, not the transient initiation of growth. The latter is briefly covered in the Appendix. Our prediction is simplified by the

realistic assumption that the crack speed $\dot{a}$ is essentially constant during the time it takes for growth equal to the FZ length α . Considering how small α is assumed to be (i.e. inside the singularity zone), this is not a serious restriction. Over the entire amount of growth, the speed $\dot{a}$ may vary widely. Without loss in generality for a locally homogeneous body, there is no dependence on X , and we can set $t_0 = 0$ for notational simplicity. Thus, in Eq.(3.1)

$$x(X, \tau) = x(\tau) = \dot{a}\tau \quad (3.3)$$

A realistic creep compliance $C_v(t)$ will be used in the form of a modified power law,

$$\bar{C}(\bar{t}) = 1 + C_n \left(\frac{\bar{t}}{1 + \bar{t}} \right)^n \quad (3.4)$$

in which normalized quantities have been introduced,

$$\bar{C} \equiv C_v / C_g, \quad C_n \equiv \frac{C_e - C_g}{C_g}, \quad \bar{t} \equiv t / t_e \quad (3.5)$$

where C_g and C_e are the so-called glassy and equilibrium (elastic) compliances, respectively. The time t_e determines when the equilibrium compliance is approached and accounts for environmental effects, such as temperature and moisture, if the material is *rheologically simple*; these effects are assumed to be constant during the time taken for the crack to grow the current length of the FZ, α .

Elastic ($n=0$), viscous ($n=1$) and power law behavior ($\bar{t} \gg 1$) appear as special cases. The creep compliance for polyurethane rubber in (Mueller and Knauss (1971) and in Schapery 1975c) is well-characterized with the values $n = 0.5$, $C_n = 145.7$, $t_e = 0.84s$ ($0^\circ c$). It is drawn in Fig 4 along with the other two compliances used in the examples in this paper. However, the theory will be developed using the general form of Eq.(3.4).

After using normalizations, Eq.(3.1) becomes,

$$\bar{v}_v = \int_0^{\bar{t}} \left[1 + C_n \left(\frac{\bar{t} - \tau}{1 + \bar{t} - \tau} \right)^n \right] \left[\partial \left[\int_0^{\bar{x}_1} (1 + \bar{v}_v(\bar{y}))^{-q} F(\bar{y} / \bar{a}\tau) d\bar{y} \right] / \partial \tau d\tau \right] \quad (3.6)$$

where we have defined a normalized crack speed, $\bar{\dot{a}}$,

$$\bar{\dot{a}} \equiv k_g t_e \dot{a} / v_0 \quad (3.7)$$

which changes Eq.(3.3) to,

$$\bar{x} = \bar{\dot{a}} t \quad (3.8)$$

Eq.(3.6) becomes,

$$\bar{v}_v = \int_0^{\bar{x}} \left[1 + C_n \left(\frac{\bar{x} - z}{\bar{\dot{a}} + \bar{x} - z} \right)^n \right] \left[\partial \left[\int_0^{\bar{x}_1} (1 + \bar{v}_v(y))^{-q} F(y/z) dy \right] / \partial z dz \right] \quad (3.9)$$

The problem is to find the $\bar{v}_v$ that satisfies the double integral Eq.(3.9) when $\bar{x} \leq \bar{x}_1$.

This will be done by first defining the *auxiliary displacement*,

$$v_w(\bar{x}) \equiv \int_0^{\bar{x}_1} (1 + \bar{v}_v(x'))^{-q} F(x'/\bar{x}) dx' \quad (3.10)$$

and introducing an effective compliance,

$$C_{eff}(\bar{x}, \bar{\dot{a}}) \equiv 1 + C_n \frac{\int_0^{\bar{x}} \left[\left(\frac{\bar{x} - z}{\bar{\dot{a}} + \bar{x} - z} \right)^n \right] [\partial v_w / \partial z] dz}{v_w} \quad (3.11)$$

Rewrite Eq.(3.9) using the effective compliance,

$$\bar{v}_v(\bar{x}, \bar{\dot{a}}) = C_{eff}(\bar{x}, \bar{\dot{a}}) \int_0^{\bar{x}_1} [1 + \bar{v}_v(x', \bar{\dot{a}})]^{-q} F(x'/\bar{x}) dx' \quad (3.12)$$

To motivate the development of a solution $\bar{v}_v(\bar{x}, \bar{\dot{a}})$ to Eq.(3.12), change variables,

$$y = C_{eff}(\bar{x}, \bar{\dot{a}}) \bar{x}', \quad y_1 = C_{eff}(\bar{x}, \bar{\dot{a}}) \bar{x}_1 \quad (3.13)$$

Thus,

$$\bar{v}_v(\bar{x}, \bar{\dot{a}}) = \int_0^{y_1} [1 + \bar{v}_v(y, \bar{\dot{a}})]^{-q} F(y / C_{eff}(\bar{x}, \bar{\dot{a}}) \bar{x}) dy \quad (3.14)$$

The solution to Eq.(3.14) would be the elastic solution with argument $\bar{x}$ replaced by $C_{eff}(\bar{x}, \bar{a})\bar{x}$ if the upper limit were $y_1 = C_{eff}(\bar{x}_1, \bar{a})\bar{x}_1$. Nevertheless, it motivates us to propose a solution for $0 \leq \bar{x} \leq \bar{x}_1$,

$$\bar{v}_v = f(\bar{a})\bar{v}_A(C_{eff}(\bar{x}, \bar{a})\bar{x}) \quad (3.15)$$

where $f(0) = f(\infty) = 1$, and $\bar{v}_A$ is the elastic displacement, Eq.(2.14). The coefficient $f(\bar{a})$ is found by using Eq.(3.12) to match $\bar{v}_v$ inside the integral (the *input*) to $\bar{v}_v$ outside (the *output*) at $\bar{x}_1$; this matching is done after C_{eff} is derived. Eq.(3.15) is found to be an accurate solution for $0 \leq \bar{x} \leq \bar{x}_1$, given the FZ models in Table 1.

We next observe that the displacement for a propagating crack is needed only at $\bar{x} = \bar{x}_1$. Let the viscoelastic solution at $\bar{x}_1$ be written as,

$$\bar{v}_{1v} = f(\bar{a})\bar{v}_A(C_{eff}(\bar{x}_1, \bar{a})\bar{x}_1) \quad (3.16)$$

Using approximation Eq.(2.14), the solution is simply,

$$\bar{v}_{1v} = f(\bar{a})c_1[C_{eff}(\bar{x}_{1v}, \bar{a})\bar{x}_{1v}]^{c_2} \quad (3.17)$$

where $\bar{x}_{1v} = \bar{x}_{1v}(\bar{a})$ is the root of the equation,

$$\bar{v}_c - f(\bar{a})c_1[C_{eff}(\bar{x}, \bar{a})\bar{x}]^{c_2} = 0 \quad (3.18)$$

which selects $\bar{x}_{1v}$ to be at the end of the FZ as the crack propagates at speed $\bar{a}$ with $\bar{v} = \bar{v}_c$.

The end of the cohesive zone is designated here by the speed dependent parameter $\bar{x}_{1v}$ to distinguish it from that for elastic behavior, $\bar{x}_1$

Note that $f(\bar{a}) \equiv 1$ for the Dugdale model because Eq.(3.14) must reduce to $\bar{v}_{1v} = \bar{v}_A(C_{eff}(\bar{x}_{1v}, \bar{a})\bar{x}_{1v})$ when $q=0$.

3.2 Characterization of C_{eff} and f .

The problem has been reduced to finding $C_{eff}(\bar{x}, \bar{a})$ and $f(\bar{a})$. We will show that C_{eff} is well-approximated by the creep compliance itself but shifted by a constant along the $\log(\bar{a})$ axis; this simplicity greatly aids the determination of $f(\bar{a})$. The process is further simplified by neglecting the low speed portion of the compliance *for now*, using

$$\bar{C} = 1 + C_n \left(\frac{\bar{x}}{\bar{a}} \right)^n \quad (3.19)$$

As the first case, consider the Dugdale traction, for which $q=0$. The auxiliary displacement Eq.(3.10) is the elastic displacement, Eq.(2.8), although it is dependent on the crack speed through $\bar{x}_1$. Fig.5 shows that it is well-approximated by a simple power law, with $p=2$ (regardless of speed),

$$v_w = c\bar{x}^p \quad (3.20)$$

in which the speed-dependent coefficient obviously has no effect on C_{eff} . Upon using Eq.(3.20) in Eq.(3.11) we find the effective compliance,

$$\hat{C}_{eff} = 1 + C_n \left(\frac{\bar{x}}{s_f \bar{a}} \right)^n \quad (3.21)$$

where s_f is a shift factor on the $\log(\bar{a})$ axis). It can be expressed in terms of gamma functions,

$$s_f = \left[\frac{\Gamma(p+n+1)}{\Gamma(p+1)\Gamma(n+1)} \right]^{1/n} \quad (3.22)$$

This is the same shift factor reported in (Schapery 1975, Eq.28). Fig. 6 covers the range of n values, as well as p values found later, used in this paper. Fig.5 is plotted using $n=0.5$ and $\bar{a}=1$; but all three n and all $\bar{a}$ give the same agreement. The difference between the two curves in Fig.5 produces only a 2% difference in $\hat{C}_{eff}$ at $\bar{x}_{1v}$; the difference was found by evaluating Eq.(3.11) at $\bar{x}_{1v}$ using Eq.(3.20), and then comparing it with that predicted using the exact auxiliary displacement in Eq.(3.10) with $q=0$.

The complete effective compliance will be used to obtain f and make all subsequent predictions so that the long-time equilibrium value is approached smoothly,

$$\bar{C}_{eff} = 1 + C_n \left(\frac{\bar{x}}{s_f \bar{a} + \bar{x}} \right)^n \quad (3.23)$$

which may be expressed more concisely as

$$\bar{C}_{eff} = \bar{C} \left(\frac{\bar{x}}{s_f \bar{a}} \right) \quad (3.24)$$

where $\bar{C}$ is the normalized creep compliance, Eq.(3.4).

For the other q -values we find that the auxiliary function is also well-approximated by Eq.(3.20) in which p is essentially independent of crack speed. The value of p also has negligible dependence on f for its actual range of variation ($1 \leq f < 1.35$) and vice-versa. Thus, Eq.(3.15), with $f = 1$, is substituted into Eq.(3.10), using Eq.(3.24) for compliance; p and c are selected such that Eq.(3.20) and Eq.(3.10) agree closely in shape and match at $\bar{x}_{lv}$, respectively; all resulting p -values are in Table 1.

After selecting the p -value using $f=1$, $f(\bar{a})$ is found by matching the associated input, Eq.(3.15), and output, Eq.(3.12), at $\bar{x}_{lv}$, given a set of $\bar{a}$ spaced one-decade apart, and then fitting it to a sixth order polynomial in $(\log(\bar{a}) + \text{constant})$.

Figs.7 and 8 show f and input-output pairs for two selected cases. The results for the other cases provide even closer agreement between input and output, and therefore are not shown. For $q=1$ and $q=2$ ($v_c = 1$) the deviation of $f(\bar{a})$ and I_R (Eq.(3.31)) from unity is less than that in Fig. 7. The speed chosen for Fig.8 is close to the maximum of the f -curves in Fig.7, which produces the largest input-output difference, although barely graphically distinguishable.

Although s_f was found without the low speed term, we shall now show that the approximate effective compliance Eq.(3.24) is quite good for all $\bar{a}$, compared to the use of the complete

creep compliance, Eq.(3.4), in deriving C_{eff} . It is sufficient here to express the latter using the single variable $\hat{t} \equiv \bar{x} / \bar{\dot{a}}$:

$$cC_{eff}(\hat{t}) \equiv 1 + C_n \frac{\int_0^{\hat{t}} \left[\frac{\hat{t} - \tau}{1 + \hat{t} - \tau} \right]^n \left[\frac{dv_w(\tau)}{d\tau} \right] d\tau}{v_w(\hat{t})} \quad (3.25)$$

where v_w is in Eq.(3.20), but with τ replacing $\bar{x}$. The value of p is affected only slightly at low speeds when Eq.(3.25) is used in Eq.(3.10).

Fig.9 shows, in the approach to equilibrium, the ratio R_c of Eq.(3.24) to Eq. (3.25) for the minimum and maximum values of p and three values of n . In this near-equilibrium period the difference in compliances is quite small, especially for the common range $n \leq 0.5$. The case $n=1$ only characterizes a viscous fluid, which has no equilibrium compliance; but it is used here to complete the upper end of the thermodynamically allowable range, $0 \leq n \leq 1$. When Eq.(3.25) is used instead of Eq.(3.24) to find $f(\bar{a})$ there is negligible change because of a countering change in $\bar{x}_{1v}$.

3.3 Prediction of crack growth.

Eqs.(2.9) and (3.18) enable the prediction of the crack speed vs. stress intensity factor. First, we note that Eqs.(2.9) and (2.10) are valid for elastic and viscoelastic materials. In the former case one uses Eq.(2.16) for $\bar{x}_1$ while in the latter case $\bar{x}_1 = \bar{x}_{1v}$, where $\bar{x}_{1v}$ is found from Eq.(3.18). Let us define

$$\bar{\alpha} \equiv \bar{x}_{1v} = \frac{k_g}{v_0} \alpha \quad (3.26)$$

and use the stress intensity factor using Eq.(2.9) and fracture energy Eq.(2.12). Finally,

$$\bar{\alpha} = \frac{K^2 I_{1g}^2 C_g}{8 I_1^2 \Gamma} \left(\frac{\bar{v}_c}{c_1} \right)^{\frac{1}{c_2}} \quad (3.27)$$

where

$$I_1 \equiv \int_0^1 [1 + \bar{v}_v(\eta \bar{\alpha})]^{-q} \frac{d\eta}{\sqrt{\eta}} \quad (3.28)$$

and I_{1g} is for the elastic state,

$$I_{1g} \equiv \int_0^1 \left[1 + \bar{v} \left(\eta \left(\frac{\bar{v}_c}{c_1} \right)^{\frac{1}{c_2}} \right) \right]^{-q} \frac{d\eta}{\sqrt{\eta}} \quad (3.29)$$

whose values are those of I_1 in Table 1.

Next substitute Eq.(3.27) into Eq.(3.18) and use Eq.(3.24) to find,

$$\Gamma = \frac{1}{8} \left(\frac{K}{I_R} \right)^2 C_g \bar{C} \left(\frac{\bar{\alpha}}{s_f \bar{a}} \right) \quad (3.30)$$

where,

$$I_R \equiv \frac{I_1}{I_{1g} f^{\frac{1}{2c_2}}} \quad (3.31)$$

Observe that

$$\frac{\bar{\alpha}}{\bar{a}} = \frac{\alpha}{t_e \dot{a}} \quad (3.32)$$

is the normalized time for the crack to propagate the length of the FZ. The shift factor s_f decreases this time to an *effective time for creep softening*. For an elastic body $I_R = 1$, which reduces the right side of Eq.(3.30) to the energy release rate.

Introduce a normalized stress intensity factor,

$$\bar{K} \equiv \sqrt{\frac{C_g}{\Gamma}} K \quad (3.33)$$

so that Eq.(3.30) becomes,

$$\frac{1}{8} \left(\frac{\bar{K}}{I_R} \right)^2 \bar{C} \left(\frac{\bar{\alpha}}{s_f \bar{a}} \right) = 1 \quad (3.34)$$

Next, use Eqs.(3.31) and (3.33) in Eq.(3.27) to express $\bar{\alpha}$ in terms of quantities in Eq.(3.34),

$$\bar{\alpha} = \frac{1}{s_q f^{\frac{1}{c_2}}} \left(\frac{\bar{K}}{I_R} \right)^2 \quad (3.35)$$

where

$$s_q \equiv 8 \left(\frac{c_1}{v_c} \right)^{\frac{1}{c_2}} \quad (3.36)$$

Values of this factor are in Table 1. Eq.(2.16) shows that

$$s_q = 8 / \bar{\alpha}_g \quad (3.37)$$

where $\bar{\alpha}_g$ is the critical glassy (elastic) FZ length.

Eq.(3.35) can be simplified to

$$\bar{\alpha} = \frac{1}{s_q} \left(\frac{I_{1g}}{I_1} \right)^2 \bar{K}^2 \quad (3.38)$$

which is drawn in Fig.10. The dotted lines are for $I_R = f = 1$, which shows that these functions have only a small effect on $\bar{\alpha}$. As expected, $\bar{\alpha} \rightarrow 8/s_q$ as $\bar{a} \rightarrow \infty$ in view of Eq.(3.38) and Eq.(3.44) below.

Finally, the crack growth equation can be written in the simplified form,

$$\frac{1}{8} (\bar{K}_R)^2 \bar{C} \left(\frac{\bar{K}_R^2}{s_q s_f A} \right) = 1 \quad (3.39)$$

where

$$\bar{K}_R \equiv \frac{\bar{K}}{I_R}, \quad A \equiv f^{\frac{1}{c_2}} \bar{a} \quad (3.40)$$

For the Dugdale model $I_R = 1$ because $I_1 = 2$ and $f = 1$, regardless of crack speed.

It is observed that s_q appears as a shift factor for crack speed in Eq.(3.39). Eq.(3.37) shows that it decreases with increasing length of the glassy FZ. With a long traction tail, the portion of the tail with a very small traction is not effective in its relationship to K . This is accounted for in Eq.(3.38), which can be written in the form,

$$\frac{\bar{\alpha}}{\bar{\alpha}_g} = \frac{1}{8} \left(\frac{I_{1g}}{I_1} \right)^2 \bar{K}^2 \quad (3.41)$$

Thus, a long tail with viscoelasticity is reduced by the long tail for the glassy state.

Eq.(3.39) provides a relationship between K_R and $s_q s_f A$ that is independent of the FZ properties and depends only on the normalized creep compliance $\bar{C}$. It may be solved explicitly for A , given the normalized compliance in Eq.(3.4). Thus,

$$A = \frac{K_R^2}{s_q s_f} \frac{1-S}{S} \quad (3.42)$$

where,

$$S \equiv \left(\frac{8}{C_n K_R^2} - \frac{1}{C_n} \right)^{1/n} \quad (3.43)$$

Let *glassy* $\bar{K}_g$ and *equilibrium* (elastic) $\bar{K}_e$ define the normalized stress intensity factors corresponding to the high ($S=0$) and low($S=1$) speed limits, respectively. Since $I_R = 1$ at these limits, we find,

$$\bar{K}_g = \sqrt{8} \quad \text{and} \quad \bar{K}_e = \sqrt{\frac{8}{C_n + 1}} \quad (3.44)$$

Except for these limits, I_R and A are rather involved functions of crack speed. However, this behavior can be easily taken into account by using $\bar{a}$ as the independent variable.

Eq.(3.34) is plotted in Fig.11 with solid lines while Eq.(3.39) is plotted using dotted lines; the dotted lines can be interpreted, of course, as plots of $\log(\bar{K})$ vs. $\log(\bar{a})$ when $I_R = f = 1$. Because $I_R = f = 1$ and $s_q = 16$ for the Dugdale model, the dotted lines in Fig.11a are also the solution for this model except for a small difference in s_f and s_q .

In an intermediate speed range Eq.(3.42) obeys the power law,

$$K_R = \left(\frac{8s_q^n s_f^n}{C_n} \right)^{\frac{1}{2(1+n)}} A^{\frac{n}{2(1+n)}} \quad (3.45)$$

which appears as a light straight line in Fig.11. The slope of this line for $n=0.5, 1/6$, is in agreement with the experimental data in Schapery (1975c) (neglecting the small effect of the speed dependence of I_1 and f), which was obtained by Mueller and Knauss (1971). As Fig. 11 shows, the effect of this speed dependence is quite small on logarithmic axes in a realistic range of n for polymeric solids.

Except for $q=2$, $\bar{v}_c=3$, there is very little difference in the $\bar{a}$ position for each q because all are shifted on the log scale by approximately the same amount relative to the creep compliance. In the former case the relative shift is noticeable because of the small value of s_q in Table 1. The associated long tail in Fig.2 produced this small s_q .

3.4 Direct comparison of traction models.

It is important to recall that $\bar{\alpha}$, $\bar{x}$ and $\bar{a}$ have been defined in terms of physical variables using σ_m , Eq.(2.7). Thus, to directly compare the physical response of different models (assuming Γ, C_v and v_0 are the same for all models) it is necessary to re-normalize for the different values of σ_m using Eq.(2.6),

$$\sigma_m = \frac{\Gamma}{v_0} cf \quad (3.46)$$

where the correction factor, cf , relative to the Dugdale model is,

$$cf \equiv \frac{q-1}{1-(\bar{v}_c+1)^{1-q}} \quad (3.47)$$

It's values are listed in Table 1. Thus, for direct comparison of the models, they should all be plotted using common variables, say:

:

$$\bar{x}_D = \frac{\bar{x}}{cf}, \quad \bar{\alpha}_D = \frac{\bar{\alpha}}{cf}, \quad \bar{a}_D = \frac{\bar{a}}{cf} \quad (3.48)$$

The compliance is unaffected by this change because its argument is a ratio of these corrections.

This correction will change the relative horizontal position of each curve in Fig.11, but not the shape. Unless v_0 and $\bar{v}_c$ are known, the specific model cannot be established from experimental data on K vs. $\dot{a}$

3.5 Comparison of CODs.

Although there may not be enough difference in $\bar{K}(\bar{a})$ for the different models to identify the applicable model without precise experimental data, we will see if displacement measurements from $\bar{\alpha}$ out through the far-field (singularity) zone will help. This zone is defined roughly by $5\bar{\alpha} < \bar{x}$ normalized distance from the crack tip to the nearest geometric feature.

We find from Eqs.(2.2) and (3.12) that far from the crack tip,

$$\bar{v}_v \approx \bar{v}_{ff} \equiv 2\bar{C} \left(\frac{\bar{x}}{s_f \bar{a}} \right) \sqrt{\bar{x} \bar{\alpha}} I_1 = 2cf\bar{C} \left(\frac{\bar{x}_D}{s_f \bar{a}_D} \right) \sqrt{\bar{x}_D \bar{\alpha}_D} I_1 \quad \text{for } \bar{x}_D > 5\bar{\alpha}_D. \quad (3.49)$$

where I_1 is in Eq.(3.28); the values for s_f and I_1 do not change in the conversion to a common length measure for model comparisons. This displacement for large $\bar{x}$ may not behave as the elastic singular displacement, $\sqrt{\bar{x}}$. For example, when the compliance obeys a power law,

$$v_v \propto \bar{x}^{n+0.5} \quad \text{for } \bar{x} > 5\bar{\alpha}. \quad (3.50)$$

showing that the displacement shape can deviate greatly from elastic behavior.

Fig.12 for $n=0.5$ illustrates the difference between the displacements for the Dugdale ($q=0$) and $q=3$ models. For both (a) elastic and (b) strong viscoelastic effects, the Dugdale COD joins the far-field solution at approximately five times the length of the FZ (0.7 decade). The $q=3$ model blends more quickly at approximately three times (0.5 decade) its FZ length. Note in Fig.12 that the slope of the line for the far field singular zone is 0.5 for elastic behavior, as expected, and approximately unity when $\bar{a}_D = 1$, as predicted from Eq.(3.50). Thus, in this latter case, the COD shape is essentially that of a flat-faced wedge. We have also found that unless measurements of the COD are made within a short distance from the FZ they cannot be used to identify the model—even when there is a large difference in the FZ traction shape. (In Schapery, 1975c, the FZ for the rubber is submicroscopic.)

It should be added that the finite element method may be needed for realistic COD comparisons of theory and experiment due to the large local deformations near the crack tip.

As a final observation, we mention that the *stress* from the crack tip out through the singularity zone ($x < 0$) is affected by the creep compliance due to its dependence on the traction in the FZ (Schapery 1975a, Eq.(17)).

4.0 Discussion and Conclusions

We have shown that crack growth in a linear viscoelastic body is defined almost entirely by the creep compliance when the FZ traction is constant or a monotonically decreasing function of the crack opening displacement. The different FZ tractions have only a small effect on $\log(\bar{K})$ vs. $\log(\bar{a})$, except through speed shift factors; generally, the combined shift, $s_f s_q$ (Fig.6 and Table 1), decreases with increasing concavity of the FZ traction. Because the shift factors are combined with FZ properties, they cannot uniquely establish the form of the underlying function $\sigma_f(u)$ from only experimental data on crack growth. However, very precise measurements of K vs. $\dot{a}$ and COD close to the crack tip may be helpful in providing information on the underlying traction model.

We have also examined unreported cases, using a partial sine function for $\sigma_f(v)$, in which the traction vs. COD gradually decreases and may increase near the critical value, with both cases

causing the traction to increase as the critical value is reached (cf. Fig.1); the methods used in this paper were found to work well.

All examples utilized a creep compliance that was characterized in an intermediate time period by a single power law. For some materials the intermediate period is very broad, covering many decades, with a gradually changing log-log slope. As shown in Schapery (1975b), the *local* log-log slope of the creep compliance can be used in place of a single, constant exponent, n to predict the effective compliance because it has a narrow-band weight function.

Crack growth Eq.(3.30) is identical to that in Schapery (1975b) if I_1 is constant and $f=1$. Here, the model enabled prediction of the crack speed dependence of I_1 and f .

The present model predicts unstable crack growth only when the glassy compliance is reached, or if one or more of the FZ properties decrease with speed at some point. Instabilities at speeds lower than glassy speed in filled elastomers have been reported and explained by continuum nonlinearities (e.g. Morishita M et al. 2016). As shown in Schapery (1984), instabilities at these lower speeds are also predicted by embedding an FZ inside a thin *process* zone with viscoelastic properties (different from the continuum) and using the so-called pseudo J -integral as the crack driving force; the craze zone in some plastics could be a candidate fitting this description. This J -integral and crack growth theory for the shear-mode has been used to predict rubber friction with Schallamach waves (Schapery 2020a,b).

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6.0 Appendix- Initiation of crack growth.

We continue to use Eq.(3.1), but now have a transient problem. Referring to Fig.1, the crack tip in the *initial* unloaded state is located at the coordinate system's origin, $X=0$. Thus, the left end of the FZ is at the point,

$$x_1(t) = \alpha(t) \quad (5.1)$$

Other points in the FZ are located at,

$$x(X, t) = \alpha(t) - X > 0. \quad (5.2)$$

With this notation Eq.(3.1) becomes,

$$v(X, t) = (1 / 2\pi) \int_0^t C_v(t - \tau) [\partial v_w(X, \tau) / \partial \tau] d\tau \quad (5.3)$$

where v_w is defined as an auxiliary displacement,

$$v_w(X, \tau) \equiv \left[\int_0^{\alpha(\tau)} \sigma_f[v(X', \tau)] F \left[\frac{\alpha(\tau) - X'}{\alpha(t) - X} \right] dX' \right] \quad (5.4)$$

The problem of predicting the time at which crack propagation starts is somewhat analogous to that of continuous growth. However, it is greatly simplified if the so-called quasi-elastic approximation (Schapery 1965,1967) or if the Dugdale model is applicable. The former requires that both the creep compliance and $v_w(X,t)$ (or their logarithms) exhibit small curvature on the logarithmic time scale. This approximation for the present problem reduces Eq.(5.3) to

$$v(X,t) = \frac{C_v(t)}{2\pi} v_w(X,t) \quad (5.5)$$

which is the elastic solution, but with creep compliance in place of the elastic compliance. Thus, the fracture initiation time t_c is the root of

$$\Gamma_t - \frac{C_v(t_c)}{8} K(t_c)^2 = 0, \quad (5.6)$$

where Γ_t is not necessarily the elastic fracture energy.

The Dugdale model, under the assumption that $K(t)$ is nondecreasing, provides a simple, exact solution for initiation time t_c (Schapery 1975b), which is the root of

$$\Gamma_t - \frac{C_{ief}(t_c)}{8} K(t_c)^2 = 0 \quad (5.7)$$

where

$$C_{ief}(t_c) = \frac{1}{K(t_c)^2} \int_0^{t_c} C_v(t_c - \tau) \frac{dK(\tau)^2}{d\tau} d\tau \quad (5.8)$$

is an effective (or secant) compliance. When $K(t)$ is a step function of time applied at $t=0$, this result obviously reduces to Eq.(5.6), which does not require Dugdale specialization.

We can now estimate the time t_c relative to the time taken for the crack to propagate the length of the FZ, $\frac{\alpha}{\dot{a}}$, immediately after growth initiates. Comparing Eqs.(3.30) and (5.6), assuming K is constant after it is applied stepwise, and that $\Gamma_t \approx \Gamma_e$ and $I_R \approx 1$,

$$t_c \approx \frac{\alpha}{s_f \dot{a}} \quad (5.9)$$

Referring to Fig.6, after growth begins it takes 3 – 4 times the initiation time for the crack to grow the length of the FZ for the Dugdale model, and slightly less for the other models. Thus, the time for the start of continuous growth may be negligible in many cases.

Table 1. Constants for elastic solutions (except for s_q , p , and cf).

q	$\bar{v}_c$	c_0	c_1	c_2	I_1		$\bar{x}_0$	$\bar{x}_1$		s_q	p	cf
0	1	2.10	4.56	2.19	2		0	0.5		16	2	1.00
1	1	0	2.88	1.50	1.67		0	0.494		16.2	1.5	1.44
2	1	3.59	2.25	1.23	1.39		0.177	0.517		15.5	1.3	2.00
3	1	4.11	1.77	1.00	1.15		0.186	0.564		14.2	1.2	2.67
3	1.5	3.80	1.54	0.830	0.931		0.260	0.969		8.24	1.0	2.38
2	3	3.00	1.61	0.819	0.836		0.401	2.14		3.74	1.0	1.33

Fig.1 Normal stresses acting along crack plane in the neighborhood of the crack tip **P**.

Fig. 2 Normalized failure zone tractions, $\bar{\sigma} = \sigma_f / \sigma_n$, vs. opening displacement.

Fig.3 Comparison of input (solid line) and output (dots) elastic CODs for $v_c = 1$.

Fig. 4 Creep compliances.

Fig.5. Predicted (dots) and power law ($p=2$, solid line) auxiliary displacements for the Dugdale model ($q=0$).

Fig. 6 Shift factor

Fig.7 The factor f and modified traction integral I_R vs. logarithmic crack speed..

Fig.8 Comparison of input (solid line) and output (dots) for auxiliary displacement and COD using $n=1$ and $\bar{a} = 0.1$.

Fig.9. Ratio of approximate-to-complete effective compliances in the transition to long-time elastic behavior.

Fig. 10. Failure zone length vs. $\log(\text{crack speed})$.

Fig.11. Log(stress intensity factor) vs. log(crack speed).

Fig. 12. COD comparisons for $q=3$ and Dugdale ($q=0$) models.

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