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Fixed Point Result in Controlled Fuzzy Metric Spaces with Application to Dynamic Market Equilibrium

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Abstract. In this paper, inspired by the work of Müzeyyen Sangurlu Sezen [22] and H. Saleh Nasr et al. [20], we introduce Θ_f -type controlled fuzzy metric spaces and establish some fixed point results in controlled fuzzy metric spaces. We provide suitable examples to validate our result. We also employ an application to substantiate the utility of our established result for finding the unique solution of an integral equation emerging in the dynamic market equilibrium aspects to economics.

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Keywords: Fixed point, Fuzzy metric spaces, Controlled fuzzy metric spaces, Fuzzy Θ_f -contractive mapping, Dynamic market equilibrium.

1 Introduction

In 1922, S. Banach [6] provided an important result to the fixed point theory. This topic has been studied, presented and generalized by many researchers in many different spaces. Firstly, the work of Bakhtin [5], Bourbaki [7] and Czerwik [8] expanded the theory of fixed points for b-metric spaces. Also, many authors proved some important fixed point theorems in b-metric spaces ([1], [2], [3]). Controlled metric spaces introduced by Nabil Mlaiki et al. [18] and proved some fixed point theorems.

On the other hand, important theoretical development in the fuzzy sets theory introduced by Zadeh [25]. Fuzzy sets theory is the way of defining the concept of fuzzy metric spaces by Kramosil and Michalek [14], which can be regarded as a generalization of the statistical metric spaces. Subsequently, M. Grabiec [11] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces. Following Grabiec's work, George and Veeramani [10] modified the notion of M -complete fuzzy metric spaces with the help of continuous t-norms.

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Nadaban [19] introduced the concept of fuzzy b-metric spaces. Kim et al. [13] established some fixed point results in fuzzy b-metric spaces. Recently, Mehmood et al. [15] has defined a new concept called extended fuzzy b-metric spaces, which is the generalization of fuzzy b-metric spaces. Müzeyyen Sangurlu Sezen [22] introduced controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

Many authors introduced and generalized the numerous types of fuzzy contractive mappings ([4], [16], [17], [23], [24]) and investigated some fixed point theorems in fuzzy metric spaces. H. Saleh Nasr et al. [20] introduced Fuzzy Θ_f - contractive mapping establish some fixed point theorems.

2 Preliminaries

Now, we begin with some basic concepts, notations and definitions. Let $\mathbb{R}$ represent the set of real numbers, $\mathbb{R}_+$ represent the set of all non-negative real numbers and $\mathbb{N}$ represent the set of natural numbers.

We start with the following definitions of a fuzzy metric space.

Definition 2.1. [10]. *An ordered triple $(X, M, *)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t -norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions, for all $x, y, z \in X, s, t > 0$.*

(FM-1) $M(x, y, t) > 0$.

(FM-2) $M(x, y, t) = 1$ iff $x = y$.

(FM-3) $M(x, y, t) = M(y, x, t)$.

(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$.

(FM-5) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is left continuous.

In 2017, Nădăban [19] introduced the idea of a fuzzy b-metric space to generalize the notion of a fuzzy metric spaces introduced by Kramosil and Michalek [14].

Definition 2.2. [19]. *Let X is a non-empty set and $b \geq 1$ be a given real number and $*$ be a continuous t -norm. A fuzzy set M in $X^2 \times (0, \infty)$ is called fuzzy b-metric on X if for all $x, y, z \in X$ and $s, t > 0$ the following conditions hold.*

(FbM-1) $M(x, y, t) = 0$.

(FbM-2) $M(x, y, t) = 1$ iff $x = y$.

(FbM-3) $M(x, y, t) = M(y, x, t)$.

(FbM-4) $M(x, z, b(t + s)) \geq M(x, y, t) * M(y, z, s)$.

(FbM-5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is left continuous and $\lim_{t \rightarrow \infty} M(x, y, t) = 1$.

Mehmood et al. [15] introduced the notion of an extended fuzzy b-metric space following the approach of Grabiec [11].

Definition 2.3. [15]. *Let X be a non-empty set, $\alpha : X \times X \rightarrow [1, \infty)$, $*$ is a continuous t -norm and M_α is a fuzzy set on $X^2 \times (0, \infty)$, is called extended fuzzy b-metric on X if for all $x, y, z \in X$ and $s, t > 0$, satisfying the following conditions.*

$(FbM_\alpha 1) \ M_\alpha(x, y, 0) = 0.$
 $(FbM_\alpha 2) \ M_\alpha(x, y, t) = 1$ iff $x = y.$
 $(FbM_\alpha 3) \ M_\alpha(x, y, t) = M_\alpha(y, x, t).$
 $(FbM_\alpha 4) \ M_\alpha(x, z, \alpha(x, z)(t + s)) \geq M_\alpha(x, y, t) * M_\alpha(y, z, s).$
 $(FbM_\alpha 5) \ M_\alpha(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.
 Then $(X, M_\alpha, *, \alpha(x, y))$ is an extended fuzzy b-metric space.

In [22], Sezen introduced the controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

Definition 2.4. [22]. Let X be a non-empty set, $\lambda : X \times X \rightarrow [1, \infty)$, $*$ is a continuous t -norm and M_λ is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions, for all $a, c, d \in X, s, t > 0$:

$(FM-1) \ M_\lambda(a, c, 0) = 0.$
 $(FM-2) \ M_\lambda(a, c, t) = 1$ iff $a = c.$
 $(FM-3) \ M_\lambda(a, c, t) = M_\lambda(c, a, t).$
 $(FM-4) \ M_\lambda(a, d, t + s) \geq M_\lambda(a, c, \frac{t}{\lambda(a, c)}) * M_\lambda(c, d, \frac{s}{\lambda(c, d)}).$
 $(FM-5) \ M_\lambda(a, c, \cdot) : [0, \infty) \rightarrow [0, 1]$ is continuous.

Then, the triple $(X, M_\lambda, *)$ is called a controlled fuzzy metric space on X .

Definition 2.5. [21]. Let $F : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. Then F is an α -admissible mapping, if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(F(x), F(y)) \geq 1, \quad x, y \in X. \quad (2.1)$$

Definition 2.6. [20]. $\Theta_f : (0, 1) \rightarrow (0, 1)$, such that Θ_f is non decreasing continuous and satisfying condition for each sequence $\{\beta_n\} \subset (0, 1)$,

$$\lim_{n \rightarrow \infty} \Theta_f(\beta_n) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} \beta_n = 1. \quad (2.2)$$

Definition 2.7. [22]. Let $(X, M_\lambda, *)$ be a controlled fuzzy metric spaces. Then

1. A sequence $\{x_n\}$ in X is said to be G -convergent to x in X , if and only if $\lim_{n \rightarrow \infty} M_\lambda(x_n, x, t) = 1$ for any $n > 0$ and for all $t > 0$.
2. A sequence $\{x_n\}$ in X is said to be G -Cauchy sequence if and only if $\lim_{n \rightarrow \infty} M_\lambda(x_n, x_{n+m}, t) = 1$ for any $m > 0$ and for all $t > 0$.
3. The controlled fuzzy metric space is called G -complete if every G -Cauchy sequence is convergent.

The objective of this work is to prove a Banach type fixed point theorem in controlled fuzzy metric spaces using fuzzy Θ_f - contractive mapping, which is an extension of [10], [15], [19], [20], [22]. Our result generalizes many recent fixed point theorems in the literature ([13], [15], [19], [20]). We furnish an example to validate our result. Application is also provided to show the utility of our result.

3 Main Result

In this section, we introduce some new definitions and establish a fixed point theorem in controlled fuzzy metric spaces.

Definition 3.1. Let Y be a non-empty set, $\lambda : Y \times Y \rightarrow [1, \infty)$, $*$ is a continuous t -norm and M_λ is a fuzzy set on $Y^2 \times (0, \infty)$, for all $a, c, d \in Y, s, t > 0$, $\Theta_f : [0, 1] \rightarrow [0, 1]$, satisfying the following conditions,

- ($\Theta_f FM_\lambda$ -1) $\Theta_f(M_\lambda(a, c, 0)) = 0$.
- ($\Theta_f FM_\lambda$ -2) $\Theta_f(M_\lambda(a, c, t)) = 1$ iff $a = c$.
- ($\Theta_f FM_\lambda$ -3) $\Theta_f(M_\lambda((a, c, t))) = \Theta_f(M_\lambda(c, a, t))$.
- ($\Theta_f FM_\lambda$ -4) $\Theta_f(M_\lambda(a, d, t + s)) \geq \Theta_f(M_\lambda(a, c, \frac{t}{\lambda(a, c)}) * M_\lambda(c, d, \frac{s}{\lambda(c, d)}))$.
- ($\Theta_f FM_\lambda$ -5) $\Theta_f(M_\lambda(a, c, \cdot)) : [0, \infty) \rightarrow [0, 1]$ is continuous.

Then, the triple $(Y, \Theta_f M_\lambda, *)$ is called a Θ_f -type controlled fuzzy metric spaces on Y .

Now, we display an example to verify our definition.

Example 3.2. Let $Y = A \cup C$ where $A = (0, 2)$ and $C = [2, \infty)$. Define M_λ is a fuzzy set on $Y^2 \times (0, \infty)$, as

$$M_\lambda(a, c, t) = \begin{cases} 1 & \text{if } a = c \\ e^{-\frac{3}{ct}} & \text{if } a \in A \text{ and } c \in C \\ e^{-\frac{3}{at}} & \text{if } a \in C \text{ and } c \in A \\ e^{-\frac{3}{t}} & \text{otherwise.} \end{cases}$$

With the continuous t -norm $*$ such that $t_1 * t_2 = t_1 t_2$. Taking $\Theta_f(\beta) = e^{1-\frac{1}{\beta}}$ and define $\lambda : Y \times Y \rightarrow [1, \infty)$, as

$$\lambda(a, c) = \begin{cases} 1 & \text{if } a, c \in A \\ \max\{a, c\} & \text{otherwise.} \end{cases}$$

Let us show that $(Y, \Theta_f M_\lambda, *)$ is a Θ_f - type controlled fuzzy metric space on Y . It is easy to prove conditions ($\Theta_f FM_\lambda$ -1), ($\Theta_f FM_\lambda$ -2) and ($\Theta_f FM_\lambda$ -3). We have to examine the following cases to show that condition ($\Theta_f FM_\lambda$ -4) holds.

Case I. If $d = a$ or $d = c$, ($\Theta_f FM_\lambda$ -4) is satisfied.

Case II. If $d \neq a$ and $d \neq c$, ($\Theta_f FM_\lambda$ -4) holds when $a = c$.

Suppose that $a \neq c$. Then, we get $a \neq c \neq d$. Now, we can see that ($\Theta_f FM_\lambda$ -4) is satisfied in all the cases below:

1. Let $a, c, d \in A$ and $a, c, d \in C$, choose $t = 1, s = 1$ and $a = 1, c = \frac{1}{2}, d = \frac{3}{2}$ and $\lambda(a, c) = 1, \lambda(a, d) = 1$, then we get $\Theta_f(e^{-1.5}) \geq \Theta_f(e^{-6})$.
2. Let $a, d \in A$ and $c \in C$, choose $t = 1, s = 1$ and $a = \frac{1}{2}, c = 3, d = \frac{3}{2}$ and $\lambda(a, c) = \max\{a, c\} = 3, \lambda(a, d) = \frac{3}{2}$, then we get $\Theta_f(e^{-0.5}) \geq \Theta_f(e^{-1})$.
3. Let $a, d \in C$ and $c \in A$, choose $t = 1, s = 1$ and $a = 3, c = \frac{3}{2}, d = 4$ and $\lambda(a, c) = \max\{a, c\} = 3, \lambda(a, d) = 1$, then we get $\Theta_f(e^{-0.5}) \geq \Theta_f(e^{-1.3})$.

4. Let $a \in C$ and $c, d \in A$, choose $t = 1, s = 1$, and $a = 4, c = \frac{1}{2}, d = \frac{3}{2}$ and $\lambda(a, c) = \max\{a, c\} = 4$, $\lambda(a, d) = 1$, then we get $\Theta_f(e^{-0.37}) \geq \Theta_f(e^{-0.93})$.
5. Let $a, c \in A$ and $d \in C$, choose $t = 1, s = 1$ and $a = \frac{1}{2}, c = \frac{3}{2}, d = 4$ and $\lambda(a, c) = 1$, $\lambda(a, d) = 1$, then we get $\Theta_f(e^{-1.5}) \geq \Theta_f(e^{-6})$.
6. Let $a, c \in C$ and $d \in A$, choose $t = 1, s = 1$, and $a = 3, c = 4, d = \frac{1}{2}$ and $\lambda(a, c) = \max\{a, c\} = 4$, $\lambda(a, d) = 3$, then we get $\Theta_f(e^{-1.5}) \geq \Theta_f(e^{-1.75})$.
7. Let $c, d \in C$ and $a \in A$, choose $t = 1, s = 1$, and $a = \frac{3}{2}, c = 3, d = 4$ and $\lambda(a, c) = \max\{a, c\} = 3$, $\lambda(a, d) = 4$, then we get $\Theta_f(e^{-0.5}) \geq \Theta_f(e^{-0.6})$.

Consequently, $(Y, \Theta_f M_\lambda, *)$ is a Θ_f type controlled fuzzy metric space on X . Also, for the same functions λ using by $(\Theta_f FM_\lambda-4)$, we get

$$\Theta_f(M_\lambda(a, d, t + s)) \geq \Theta_f(M_\lambda(a, d, \frac{t}{\lambda(a, c)}) * M_\lambda(c, d, \frac{t}{\lambda(c, d)})).$$

Definition 3.3. Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f type controlled fuzzy metric space on X with $\lambda : Y \times Y \rightarrow [1, \infty)$, A mapping $S : Y \rightarrow Y$ is called a fuzzy Θ_f weak contractive with respect to $\Theta_f \in \Omega$, and Y is α -admissible. There exist $l \in (0, 1)$ such that

$$M_\lambda(Sa, Sc, t) < 1 \Rightarrow \alpha(a, c)\Theta_f(M_\lambda(Sa, Sc, t)) \geq [\Theta_f(N(a, c, t))]^l \quad (3.1)$$

for all $a, c \in Y$ and $t > 0$, where

$$N(a, c, t) = \min\{M_\lambda(a, c, t), M_\lambda(a, Sa, t), M_\lambda(c, Sc, t), \frac{M_\lambda(a, Sa, t)M_\lambda(c, Sc, t)}{M_\lambda(a, c, t)}\}. \quad (3.2)$$

Theorem 3.4. Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f type controlled fuzzy metric space on Y . A mapping $S : Y \rightarrow Y$ is a fuzzy Θ_f weak contractive and Y is α -admissible, then S admits a unique fixed point.

Proof. Let a_0 is an arbitrary point in Y ,

$$\alpha(a_0, Sa_0) \geq 1.$$

We define a sequence $\{a_n\}$ in Y by

$$a_{n+1} = Sa_n \text{ for all } n \in \mathbb{N}.$$

Obviously if, there exists $n_0 \in \mathbb{N}$ such that

$$a_{n_0} = Sa_{n_0+1},$$

then $Sa_{n_0} = a_{n_0}$ and the proof is finished. Suppose that $a_n \neq a_{n+1}$ for all $n \in \mathbb{N}$, that is, $\alpha(S_n, S_{n+1}) \geq 1$.

$$M_\lambda(Sa_{n-1}, Sa_n, t) < 1 \text{ for all } n \in \mathbb{N} \text{ and } s > 0.$$

Using (3.1), since S is a (α, Θ_f) type contraction, so for all $n \in \mathbb{N}$, we can write

$$\begin{aligned}
 1 > M_\lambda(Sa_{n-1}, Sa_n, t) &\Rightarrow \alpha(a_n, a_{n+1})\Theta_f(M_\lambda(Sa_{n-1}, Sa_n, t)) \\
 &\geq [\Theta_f(N(a_{n-1}, a_n, t))]^l \\
 &\geq [\Theta_f(\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t), \\
 &\quad \frac{M_\lambda(a_{n-1}, a_n, t)M_\lambda(a_n, a_{n+1}, t)}{M_\lambda(a_{n-1}, a_n, t)}\})]^l \\
 &\geq [\Theta_f(\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\})]^l.
 \end{aligned}$$

Thus

$$\Theta_f(M_\lambda(a_n, a_{n+1}, t)) \geq [\Theta_f(\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\})]^l. \quad (3.3)$$

If there exists $n \in \mathbb{N}$ such that

$$\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\} = M_\lambda(a_n, a_{n+1}, t),$$

so

$$\Theta_f(M_\lambda(a_n, a_{n+1}, t)) \geq [\Theta_f(M_\lambda(a_n, a_{n+1}, t))]^l > M_\lambda(a_n, a_{n+1}, t).$$

Which is contradiction. Therefore

$$\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\} = M_\lambda(a_{n-1}, a_n, t),$$

so

$$\Theta_f(M_\lambda(a_n, a_{n+1}, t)) \geq [\Theta_f(M_\lambda(a_{n-1}, a_n, t))]^l > M_\lambda(a_{n-1}, a_n, t),$$

for all $n \in \mathbb{N}$. Thus (3.3), we get

$$\begin{aligned}
 \Theta_f(M_\lambda(a_n, a_{n+1}, t)) &\geq [\Theta_f(M_\lambda(a_{n-1}, a_n, \frac{t}{k}))]^l \\
 &\geq [\Theta_f^2(M_\lambda(a_{n-2}, a_{n-1}, \frac{t}{k^2}))]^{l^2} \\
 &\geq [\Theta_f^3(M_\lambda(a_{n-3}, a_{n-2}, \frac{t}{k^3}))]^{l^3} \\
 &\vdots \\
 &\geq [\Theta_f^n(M_\lambda(a_0, a_1, \frac{t}{k^n}))]^{l^n}.
 \end{aligned} \quad (3.4)$$

Thus by (3.3), we have

$$\geq [\Theta_f(M_\lambda(a_0, a_1, \frac{t}{k^n}))]^{l^n} > M_\lambda(a_0, a_1, \frac{t}{k^n}). \quad (3.5)$$

Consider the triangle inequality, using the condition $(\Theta_f FM_\lambda-4)$, we have

$$\begin{aligned}
\Theta_f(M_\lambda(a_n, a_{n+m}, t)) &\geq \Theta_f(M_\lambda(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}) * M_\lambda(a_{n+1}, a_{n+m}, \frac{t}{2\lambda(a_{n+1}, a_{n+m})})) \\
&\geq \Theta_f(M_\lambda(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}) \\
&\quad * M_\lambda(a_{n+1}, a_{n+2}, \frac{t}{(2)^2\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+1}, a_{n+2})}) \\
&\quad * M_\lambda(a_{n+2}, a_{n+m}, \frac{t}{(2)^2\lambda(a_{n+1}, a_{n+m})\mu(a_{n+2}, a_{n+m})})) \\
&\geq \Theta_f(M_\lambda(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}) \\
&\quad * M_\lambda(a_{n+1}, a_{n+2}, \frac{t}{(2)^2\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+1}, a_{n+2})}) \\
&\quad * M_\lambda(a_{n+2}, a_{n+3}, \frac{t}{(2)^3\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+2}, a_{n+m})\lambda(a_{n+2}, a_{n+3})}) \\
&\quad * M_\lambda(a_{n+3}, a_{n+m}, \frac{t}{(2)^3\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+2}, a_{n+m})\lambda(a_{n+3}, a_{n+m})})) \\
&\quad \vdots \\
&\geq \Theta_f^n(M_\lambda(a_0, a_1, \frac{t}{2k^{n-1}\lambda(a_n, a_{n+1})})) \\
&\quad * [\ast_{i=n+1}^{n+m-2} \Theta_f^i(M_\lambda(a_0, a_1, \frac{t}{(2)^{m-1}k^{i-1}(\prod_{j=n+1}^i \mu(a_j, a_{n+m}))\lambda(a_i, a_{i+1})}))] \\
&\quad * [\Theta_f^{n+m-1}(M_\lambda(a_0, a_1, \frac{t}{(2)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1} \mu(a_i, a_{n+m}))}))].
\end{aligned} \tag{3.6}$$

Θ_f is non decreasing continuous and satisfying condition for each sequence $\{\beta_n\} \subset (0, 1)$

$$\lim_{n \rightarrow \infty} \Theta_f(\alpha_n) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} \alpha_n = 1. \tag{3.7}$$

Therefore, by taking limit as $n \rightarrow \infty$ in (3.6), from (3.5) together with (3.1) we have

$$\lim_{n \rightarrow \infty} \Theta_f(M_\lambda(a_n, a_{n+m}, t)) \geq (1 * 1 * \dots * 1) = 1,$$

for all $t > 0$ and $n, m \in \mathbb{N}$. Thus, $\{a_n\}$ is a G-Cauchy sequence in X . From the completeness of $(X, M_\lambda, *)$, there exists $u \in X$ such that

$$\lim_{n \rightarrow \infty} M_\lambda(a_n, u, t) = 1, \tag{3.8}$$

for all $t > 0$. Now we show that u is a fixed point of h . For any $t > 0$ and from the condition $(FM_\lambda-4)$, we have

$$\begin{aligned}
\Theta_f(M_\lambda(u, Su, t)) &\geq \Theta_f(M_\lambda(u, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}) * M_\lambda(a_{n+1}, Su, \frac{t}{2\lambda(a_{n+1}, Su)})) \\
&= \Theta_f(M_\lambda(u, a_{n+1}, \frac{t}{2\lambda(u, a_{n+1})}) \\
&\quad * M_\lambda(Sa_n, Su, \frac{t}{2\lambda(a_{n+1}, Su)})) \\
&\geq \Theta_f(M_\lambda(u, a_{n+1}, \frac{t}{2\lambda(u, a_{n+1})})) \\
&\quad * \Theta_f(M_\lambda(a_n, u, \frac{t}{2\lambda(a_{n+1}, Su)})).
\end{aligned} \tag{3.9}$$

Letting $n \rightarrow \infty$ in (3.9) and using (3.8), we get

$$\lim_{n \rightarrow \infty} \Theta_f(M_\lambda(u, Su, t)) = 1,$$

by definition of Θ_f , we have

$$\lim_{n \rightarrow \infty} \Theta_f(M_\lambda(u, Su, t)) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} M_\lambda(u, Su, t) = 1,$$

for all $t > 0$, that is, $u = Su$.

Let $w \in X$ is an another fixed point of S and there exists $t > 0$ such that $u \neq w$, then it follows from (3.1) that

$$\begin{aligned}
\Theta_f(M_\lambda(u, w, t)) &= \Theta_f(M_\lambda(Su, Sw, t)) \\
&\geq \Theta_f(M_\lambda(u, w, \frac{t}{k})) \\
&\geq \Theta_f(M_\lambda(u, w, \frac{t}{k^2})) \\
&\vdots \\
&> \Theta_f(M_\lambda(u, w, \frac{t}{k^n})),
\end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N}$. By taking limit as $n \rightarrow \infty$ in (3.9), $M_\lambda(u, w, t) = 1$ for all $t > 0$, that is, $u = w$. This completes the proof. $\square$

Remark 3.5. Putting $\Theta_f(\beta) = \beta$, $\alpha(a, c) = 1$ and

$$\begin{aligned}
N(a, c, t) &= \min\{M_\lambda(a, c, t), M_\lambda(a, Sa, t), M_\lambda(c, Sc, t), \frac{M_\lambda(a, Sa, t)M_\lambda(c, Sc, t)}{M_\lambda(a, c, t)}\} \\
&= M_\lambda(a, c, t)
\end{aligned}$$

in Theorem 3.4, we obtain the following result.

Corollary 3.6. *Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f -type controlled fuzzy metric space on Y . If $S : Y \rightarrow Y$ is a mapping such that for all $a, c \in Y$, and $\lambda : Y \times Y \rightarrow [1, \infty)$, $t > 0$ for some $l \in (0, 1)$*

$$M_\lambda(Sa, Sc, t) \geq [M_\lambda(a, c, t)]^l,$$

then S admits a unique fixed point.

Remark 3.7. *By taking $l = 1$, in corollary 3.6, we infer the Theorem 2 in [22].*

We furnish an example to validate our main result.

Example 3.8. *Let $Y = A \cup C$ where $A = \{2^{2^n} : n \in \mathbb{N}\}$ and $C = \{2\}$. Define $M_\lambda : Y \times Y \times [0, \infty) \rightarrow [0, 1]$ as*

$$M_\lambda(a, c, t) = \begin{cases} 1 & \text{if } a = c \\ \frac{t}{t + \frac{1}{a}} & \text{if } a \in A \text{ and } c \in C \\ \frac{t}{t + \frac{1}{c}} & \text{if } a \in C \text{ and } c \in A \\ \left(\frac{t}{t+1}\right)^{\frac{1}{3}} & \text{otherwise.} \end{cases}$$

With the continuous t -norm $$ such that $t_1 * t_2 = t_1 \cdot t_2$. Define $\lambda : Y \times Y \rightarrow [1, \infty)$, as*

$$\lambda(a, c) = \begin{cases} 1 & \text{if } a, c \in A \\ \max\{a, c\} & \text{otherwise.} \end{cases}$$

Clearly $(Y, \Theta_f FM_\lambda, *)$ is a Θ_f type controlled fuzzy metric space and we take $\Theta_f(\beta) = \beta$, $\beta \in (0, 1]$, $\alpha(a, c) \geq 1$. Consider $S : Y \rightarrow Y$ by

$$S(a) = \begin{cases} \sqrt{2} & \text{if } a \in A \\ 2^{2^{n+1}} & \text{if } a \in C. \end{cases}$$

Now let us show that $(Y, \Theta_f FM_\lambda, *)$ is a Θ_f controlled fuzzy metric space. It is easy to prove conditions $(\Theta_f FM_\lambda-1)$, $(\Theta_f FM_\lambda-2)$ and $(\Theta_f FM_\lambda-3)$. We have to examine the following cases to show that condition $(\Theta_f FM_\lambda-4)$ holds.

Case I. If $a = c$ then we have $Sa = Sc$. In this case:

$$\Theta_f(M_\lambda(Sa, Sc, t)) = 1 = \Theta_f(N_\lambda(a, c, t))^{\frac{1}{2}}.$$

Case II. Let $a \in A$ and $c \in C$, then we have $Sa \in A$ and $Sc \in C$. In this case:

$$\begin{aligned} \Theta_f(M_\lambda(Sa, Sc, t)) &= \Theta_f\left(\frac{t}{t + \frac{1}{S(a)}}\right) \\ &= \Theta_f\left(\frac{t}{t + \frac{1}{2^{2^{n+1}}}}\right) \\ &\geq \Theta_f\left[\frac{t}{t + \frac{1}{2^{2^n}}}\right]^{\frac{1}{2}} \\ &= \Theta_f[N_\lambda(a, c, t)]^l. \end{aligned} \tag{3.11}$$

Figure 1: Variation of $L.H.S = \Theta_f(M_\lambda(Sa, Sc, t))$ with $R.H.S = \Theta_f[N(a, c, t)]^l$ of Example 3.8, case-II on 2D and 3D view, for:

(a) $\Theta_f(M_\lambda(Sa, Sc, t))$ vs $\Theta_f[N(a, c, t)]^l$ at $t, n \in (1, 10)$.

Table 1 and 2 show the variation between $\Theta_f(M_\lambda(Sa, Sc, t))$ and $\Theta_f[N(a, c, t)]^l$ as a function of n with relative to t . This table justifies inequality (3.11), which observed in both the curves for the value of t is a higher than 50 as a function of n .

Value of t	Value of n	$\Theta_f(M_\lambda(Sa, Sc, t))$	$\Theta_f[N(a, c, t)]^l$
1	1	0.9412	0.4000
	2	0.9961	0.4706
	5	1	0.5000
	10	1	0.5000
50	1	0.9988	0.4975
	2	0.9999	0.4994
	20	1.0000	0.5000
	50	1.0000	0.5000

Table 1: Variation of $\Theta_f(M_\lambda(Sa, Sc, t))$ with $\Theta_f[N(a, c, t)]^l$ of inequality (3.11), as a function of n with fixed value of $t = 1$ and $t = 50$.

Value of n	Value of t	$\Theta_f(M_\lambda(Sa, Sc, t))$	$\Theta_f[N(a, c, t)]^l$
1	1	0.9412	0.4000
	2	0.9697	0.4444
	50	0.9988	0.4975
	100	0.9994	0.4988
50	1	1.000	0.5000
	2	1.000	0.5000
	50	1.000	0.5000
	100	1.000	0.5000

Table 2: Variation of $\Theta_f(M_\lambda(Sa, Sc, t))$ with $\Theta_f[N(a, c, t)]^l$ of inequality (3.11) as a function of n with fixed value of $n = 1$ and $n = 50$.

Case III. Let $a \in C$ and $c \in A$, then we have $Sa \in C$ and $Sc \in A$. In this case:

$$\begin{aligned}\Theta_f(M_\lambda(Sa, Sc, t)) &= \Theta_f\left(\frac{t}{t + \frac{1}{S(c)}}\right) \\ &= \Theta_f\left(\frac{t}{t + \frac{1}{2^{2^{n+1}}}}\right) \\ &\geq \Theta_f\left[\frac{t}{t + \frac{1}{2^{2^n}}}\right]^{\frac{1}{2}} \\ &= \Theta_f[N(a, c, t)]^l.\end{aligned}$$

Case IV. For other states of a , c and similarly Sa , Sc , we have,

Figure 2: Variation of $L.H.S = \Theta_f(M_\lambda(Sa, Sc, t))$ with $R.H.S. = \Theta_f[N(a, c, t)]^l$ of Example 3.8, case-IV on 2D and 3D view, for:

(a) $\Theta_f(M_\lambda(Sa, Sc, t))$ vs $\Theta_f[N(a, c, t)]^l$ at $t, n \in (1, 10)$.

$$\begin{aligned}\Theta_f(M_\lambda(Sa, Sc, t)) &= \Theta_f\left(\frac{t}{t + 1}\right)^{\frac{1}{3}} \\ &\geq \Theta_f\left[\frac{t}{t + 1}\right]^{\frac{1}{6}} \\ &= \Theta_f[N(a, c, t)]^l.\end{aligned}$$

Therefore, all the conditions of Theorem 3.4 hold and S has a unique fixed point $y = 1$.

Remark 3.9. Putting $\Theta_f(\beta) = e^{\beta-1}$, $\alpha(a, c) = 1$,

$$\begin{aligned}N(a, c, t) &= \min\{M_\lambda(a, c, t), M_\lambda(a, Sa, t), M_\lambda(c, Sc, t), \frac{M_\lambda(a, Sa, t)M_\lambda(c, Sc, t)}{M_\lambda(a, c, t)}\} \\ &= M_\lambda(a, c, t)\end{aligned}$$

and $\alpha(a, c) = 1$ in Theorem 3.4, we obtain the following result.

Corollary 3.10. Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f -type controlled fuzzy metric space on Y . If $S : Y \rightarrow Y$ is a mapping such that for all $a, c \in Y$, $t > 0$. Define $\lambda : Y \times Y \rightarrow [1, \infty)$, for some $l \in (0, 1)$.

$$e^{M_\lambda(Sa, Sc, t)-1} \geq [e^{M_\lambda(a, c, t)-1}]^l,$$

then S admits a unique fixed point.

Similarly, by choosing different values of $\Theta_f(\beta)$, we can obtain several results in the literature.

4 Application

We also provide an application to substantiate the utility of our established result to find the unique solution of an integral equation appearing in the dynamic market equilibrium aspects to economics.

Supply Q_s and demand Q_d are influenced by current prices and price trends (i.e., whether prices are rising or falling and whether they are rising or falling at an increasing or decreasing rate) in many markets [9]. The economist, therefore, needs to know the current price $P(t)$, the first derivative $\frac{dP(t)}{dt}$, and the second derivative $\frac{d^2P(t)}{dt^2}$. Assume

$$\begin{aligned} Q_s &= g_1 + u_1 P(t) + e_1 \frac{dP(t)}{dt} + c_1 \frac{d^2P(t)}{dt^2} \\ Q_d &= g_2 + u_2 P(t) + e_2 \frac{dP(t)}{dt} + c_2 \frac{d^2P(t)}{dt^2}. \end{aligned}$$

g_1, g_2, u_1, u_2, e_1 and e_2 are constants. Comment on the dynamic stability of the market, if price clears the market at each point in time. In equilibrium, $Q_s = Q_d$. Therefore,

$$g_1 + u_1 P(t) + e_1 \frac{dP(t)}{dt} + c_1 \frac{d^2P(t)}{dt^2} = g_2 + u_2 P(t) + e_2 \frac{dP(t)}{dt} + c_2 \frac{d^2P(t)}{dt^2}$$

since

$$(c_1 - c_2) \frac{d^2P(t)}{dt^2} + (e_1 - e_2) \frac{dP(t)}{dt} + (u_1 - u_2) P(t) = -(g_1 - g_2).$$

Letting $c = c_1 - c_2$, $e = e_1 - e_2$, $u = u_1 - u_2$ and $g = g_1 - g_2$ in above, we have

$$c \frac{d^2P(t)}{dt^2} + e \frac{dP(t)}{dt} + u P(t) = -g,$$

dividing through by c , $P(t)$ is governed by the following initial value problem

$$\begin{cases} P'' + \frac{e}{c} P' + \frac{u}{c} P(t) = -\frac{g}{c} \\ P(0) = 0 \\ P'(0) = 0, \end{cases} \quad (4.1)$$

where $\frac{e^2}{c} = \frac{4u}{c}$ and $\frac{u}{c} = \mu$ is a continuous function. It is easy to show that the problem (4.1) is equivalent to the integral equation:

$$P(t) = \int_0^T \zeta(t, r) F(t, r, P(r)) dr. \quad (4.2)$$

where $\zeta(t, r)$ is Green's function given by

$$\zeta(t, r) = \begin{cases} re^{\frac{\mu}{2}(t-r)} & \text{if } 0 \leq r \leq t \leq T \\ te^{\frac{\mu}{2}(r-t)} & \text{if } 0 \leq t \leq r \leq T. \end{cases} \quad (4.3)$$

In this section, by using Corollary 3.6, we will show the existence of a solution to the integral equation:

$$P(t) = \int_0^T G(t, r, P(r))dr. \quad (4.4)$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$. For $t > 0$, we define

$$M_\lambda(a, c, t) = \sup_{t \in [0, T]} \frac{\min\{a, c\} + t}{\max\{a, c\} + t} \quad (4.5)$$

for all $a, c \in Y$, with the continuous t-norm $*$ such that $t_1 * t_2 = t_1 t_2$. Taking $\Theta_f(\beta) = \beta$ and define $\lambda : Y \times Y \rightarrow [1, \infty)$, as

$$\lambda(a, c) = \begin{cases} 1 & \text{if } a, c \in A \\ \max\{a, c\} & \text{otherwise.} \end{cases}$$

It is easy to prove that $(Y, M_\lambda, *)$ is a controlled fuzzy metric spaces. Consider the mapping $S : Y \rightarrow Y$ defined by

$$SP(t) = \int_0^T G(t, r, P(r))dr. \quad (4.6)$$

Theorem 4.1. *Consider equation (4.4) and suppose that*

1. $G : [0, T] \times [0, T] \rightarrow \mathbb{R}^+$ is continuous function,
2. There exist a continuous function $\zeta : [0, T] \times [0, T] \rightarrow \mathbb{R}^+$ such that $\sup_{t \in [0, T]} \int_0^T \zeta(t, r)dr \geq 1$,
3. $\max\{G(t, r, a(r)) - G(t, r, c(r))\} \geq \zeta(t, r)\max\{a(r), c(r)\}$,
and $\min\{G(t, r, a(r)) - G(t, r, c(r))\} \geq \zeta(t, r)\min\{a(r), c(r)\}$,
4. $\max\{a(r), c(r)\} + t \geq (\max\{a(r), c(r)\} + t)^l$,
and $\min\{a(r), c(r)\} + t \geq (\min\{a(r), c(r)\} + t)^l$,

for all $l \in (0, 1)$. Then, the integral equation (4.4) has a unique solution.

Proof. For $a, c \in Y$, by using of assumptions (1)–(4), we have

$$\begin{aligned}
M_\lambda(Sa, Sc, t) &= \sup_{t \in [0, T]} \frac{\min\{\int_0^T G(t, r, a(r))dr, \int_0^T G(t, r, c(r))dr.\} + t}{\max\{\int_0^T G(t, r, a(r))dr, \int_0^T G(t, r, c(r))dr.\} + t} \\
&= \sup_{t \in [0, T]} \frac{\int_0^T \min\{G(t, r, a(r)), G(t, r, c(r))\}dr + t}{\int_0^T \max\{G(t, r, a(r)), G(t, r, c(r))\}dr + t} \\
&\geq \sup_{t \in [0, T]} \frac{\int_0^T \zeta(t, r) \min\{a(r), c(r)\}dr + t}{\int_0^T \zeta(t, r) \max\{a(r), c(r)\}dr + t} \\
&\geq \sup_{t \in [0, T]} \frac{\min\{a(r), c(r)\} \int_0^T \zeta(t, r)dr + t}{\max\{a(r), c(r)\} \int_0^T \zeta(t, r)dr + t} \\
&\geq \frac{(\min\{a(r), c(r)\} + t)^l}{(\max\{a(r), c(r)\} + t)^l} \\
&\geq \left(\frac{\min\{a(r), c(r)\} + t}{\max\{a(r), c(r)\} + t} \right)^l \\
&\geq (M_\lambda(a, c, t))^l.
\end{aligned}$$

Therefore all the conditions of Corollary 3.6 are satisfied. As a result, the mapping S has a unique fixed point $y \in Y$, which is a solution of the integral equation (4.4). $\square$

Conclusions. In this article, motivated and inspired by the work of Müzeyyen Sangurlu Sezen [22] and H. Saleh Nasr et al. [20], we generalize the controlled fuzzy metric spaces using fuzzy Θ_f - contractive mapping. We obtain a fixed point result by using generalized contractive conditions in the framework of controlled fuzzy metric spaces. Our investigations and results obtained were supported by suitable examples. We also provide an application of our result to the existence of a solution to an integral equation, which determine dynamic market equilibrium aspects to economics problems. This work provides a new path for researchers in the concerned field.

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