Connectivity Index of M-polar Fuzzy Graph with Application

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Research Article

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Posted Date: October 5th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-737239/v1>

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Connectivity index of m -polar fuzzy graph with application

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Abstract

This paper brings in connectivity index of a m -polar fuzzy graph (m PFG) with its boundedness. We investigate how the connectivity index of a m PFG changes when a vertex or an edge is deleted. Some special types of vertices include m -polar fuzzy connectivity enhancing node(m PFCEN), m -polar fuzzy connectivity neutral node (m PFCNN), m -polar fuzzy connectivity reducing node (m PFCRN) with their properties are discussed.

Keywords: m PFG, Connectivity index of a m PFG, Boundedness of connectivity index of a m PFG, Average connectivity index of a m PFG.

1 Introduction

Graph theory plays a major part, to connect some values with respect to some parameters by the use of an appropriate connection in many disciplines like IT, operation research, the network of wireless sensors, network routing, engineering and medical science etc. In 1965, Zadeh [26] replace the classical set by fuzzy set which gives better exactness in both theory and application. In 1975, Rosenfeld [21] initiate the concept of fuzzy graph and in various field it has manifold application. In 1994, Zhang [27] established the notion of bipolar fuzzy sets. Pal et al.[16] introduced more new concept on fuzzy graph on their book. In various connective network fields the bipolar fuzzy graph offers more exact results if there are a positive thinking

and negative thinking side, such as: effect and side effect, forward and backward, cooperation and competition, loss and gain etc. The concepts of m -polar fuzzy (m PF) sets was established by Chen et al. [1] in 2014. Based on this concept, Ghorai and Pal [13] defined m -polar fuzzy graph(m PFG) and Presented several new result. Several new results, theories and applications on m PFGs are studied by Ghorai and Pal [14, 10, 15, 12]. Binu M. [5] introduced the connectivity index in fuzzy graphs. The idea of strong arc in fuzzy graph was given by Bhutani and Rosenfeld [7] and types of arc in fuzzy graph was given by Mathew and Sunitha [20]. The notion of bridge, trees, cycles, cut node, end node were introduced by Rosenfeld [21]. The concepts of strength of connectedness in m PFG, m PF tree, m PF cut node are established by Mandal et al. [18]. Different types fuzzy graphs with operations and applications are explained in the references [2, 3, 4, 17, 22, 23, 24].

In this paper, we will describe the connectivity index for m PFG. The upper and lower boundary of connectivity index for m PFG are discussed. If we delete an edge from a m PFG then the effects of the connectivity index in m PFG are given. The average connectivity index in m PFG are described.

2 Preliminaries

Firstly, we define m PFGs and other related terms.

In this paper, for a natural number m , m -power of $[0,1]$ or $[0,1]^m$ is considered as a poset with point-wise order $\leq$. “ $\leq$ ” is defined by $x' \leq y' \Leftrightarrow p_i(x') \leq p_i(y')$ for each $i = 1, 2, \dots, m$, where $x', y' \in [0,1]^m$ and $p_i : [0,1]^m \rightarrow [0,1]$ be the i th projection mapping.

Definition 2.1. [11] An m -polar fuzzy graph (m PFG) of a graph $G^* = (V, E)$ is a pair $G = (V, A, B)$ where $B : \widetilde{V}^2 \rightarrow [0,1]^m$ and $A : V \rightarrow [0,1]^m$ are an m PF set in $\widetilde{V}^2$ and an m PF set in V respectively such that $p_i \circ B(a, b) \leq \min\{p_i \circ A(a), p_i \circ A(b)\}$ for all $(a, b) \in \widetilde{V}^2$, for each $i = 1, 2, \dots, m$ and $B(a, b) = \mathbf{0}$ for all $(a, b) \in (\widetilde{V}^2 - E)$, (The smallest element in $[0,1]^m$ is $\mathbf{0} = (0, 0, \dots, 0)$).

Definition 2.2. [10] If an m PFG $G = (V, A, B)$ satisfies the relation

$$p_i \circ B(x, z) = \min\{p_i \circ A(x), p_i \circ A(z)\}, \text{ for all } x, z \in V, i = 1, 2, 3, \dots, m.$$

Definition 2.3. [18] A path $u' = v_1, v_2, \dots, v_n = v'$ in m PFG G is said to be an m PF path if

Figure 1: A 2PFG

this path satisfies the relation $p_i \circ B(v_j, v_{j+1}) > 0$, ($j = 1, 2, \dots, n-1$) for at least one i and all the vertices are distinct except v_1 which may be the same as v_n .

Definition 2.4. [18] The strength of the mPF path $P : u' = v_1, v_2, \dots, v_n = v'$ in mPFG G is defined as

$$S(P) = (B_1^n(u', v'), B_2^n(u', v'), \dots, B_m^n(u', v')),$$

where, $B_k^n(u', v') = \min_{1 \leq i < j \leq n} (p_k \circ B(v_i, v_j))$, $k = 1, 2, \dots, m$.

$CONN_G(u', v')$ is the strength of connectedness between u' and v' and is defined as

$$CONN_G(u', v') = ((\max_{n \in N} (B_1^n(u', v'))), (\max_{n \in N} (B_2^n(u', v'))), \dots, (\max_{n \in N} (B_m^n(u', v')))).$$

Definition 2.5. [18] An mPFG is said to be mPF connected graph if $(p_i \circ B(a', b'))^\infty > 0$, for at least one $i = 1, 2, 3, \dots, m$.

Definition 2.6. [18] A $u' - v'$ path $P : u' = v_1, v_2, \dots, v_n = v'$ in mPFG G is said to be a strongest mPF $u' - v'$ path if $S(P) = CONN_G(u', v')$.

Definition 2.7. [18] An edge (a', b') of an mPFG G is said to be strong mPF arc if $B(a', b') \geq CONN_{G-(a', b')}(a', b')$.

Definition 2.8. [18] A path $P : x = x_1, x_2, \dots, x_n = y$ from x to y is called strong mPF path if (x_i, x_{i+1}) is strong mPF arc for all $1 \leq i \leq n-1$.

Definition 2.9. [18] A vertex y is an mPF cut vertex of G if removal of it from G reduces the connectedness strength between some other pair of nodes G .

Definition 2.10. [18] An mPFG G is called an mPF tree if it has a spanning mPF subgraph H' which is an m-polar F-tree and such that for all i , $p_i \circ B'(x, y) = 0$ implies $p_i \circ B(x, y) < p_i \circ CONN_{H'}(x, y)$.

Definition 2.11. [19] A maximum spanning mPF tree of a connected mPFG $G = (V, A, B)$ is an mPF spanning subgraph T of G , which is a m polar F-tree, such that $CONN_G(u, v)$ is the strength of the unique strongest uv mPF path in T for all $u, v \in G$.

3 Connectivity index of mPFG

In a network system, connectivity is a major factor. Here, we have presented a mPFG connectivity index.

Definition 3.1. The connectivity index of mPFG is characterised by $CI_{mPF}(G)$, defined as $CI_{mPF}(G) = (p_1 \circ CI_{mPF}(G), p_2 \circ CI_{mPF}(G), \dots, p_m \circ CI_{mPF}(G)) = (\sum_{s,t \in V} (p_1 \circ A(s))(p_1 \circ A(t))(p_1 \circ CONN_G(s, t)), \sum_{s,t \in V} (p_2 \circ A(s))(p_2 \circ A(t))(p_2 \circ CONN_G(s, t)), \dots, \sum_{s,t \in V} (p_m \circ A(s))(p_m \circ A(t))(p_m \circ CONN_G(s, t)))$ where, $p_i \circ CI_{mPF} = \sum_{s,t \in V} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s, t))$ is the i th component of connectivity index of mPFG G and $p_i \circ CONN_G(s, t)$ is the i -th component of strength of connectedness between s and t .

Example 3.2. Here we take the connected mPFG $G = (V, A, B)$ in Figure 2 where $V = \{q, r, s, t, u, v, w, x\}$. We now think that each vertex's membership value is $(1, 1, 1)$. Then the connectivity index of mPFG G is $CI_{mPF}(G) = (20.3, 14.7, 11.9)$.

Figure 2: An mPFG G .

In the above example, we calculated the connectivity index of a 3mPFG. Depended on this, we established the following Theorems. Different property on connectivity index of a mPFG is discussed below.

Theorem 3.3. *Let $G' = (V', A', B')$ be a partial mPFSG of a mPFG $G = (V, A, B)$ then $p_i \circ CI_{mPF}(G') \leq p_i \circ CI_{mPF}(G) \forall i = 1, 2, \dots, m$.*

Proof. Here, G' is partial mPFSG of G . So, $p_i \circ A'(s) \leq p_i \circ A \forall s \in G'$ and $p_i \circ B'(s, t) \leq p_i \circ B(s, t) \forall s, t \in G'$ and $\forall i$.

Again $CONN_{G'}(x, y)$ is connectedness between s and t in G' where G' is the partial mPF subgraph of G and $p_i \circ B'(s, t) \leq p_i \circ B(s, t) \forall (s, t) \in G'$, so we have $p_i \circ CONN_{G'}(s, t) \leq p_i \circ CONN_G(s, t) \forall (s, t) \in G'$ and $i = 1, 2, \dots, m$. Next we know $(p_i \circ A(s))(p_i \circ A(t)) \geq 0$ and $(p_i \circ A'(s))(p_i \circ A'(t)) \geq 0 \forall i$. So, $(p_i \circ A'(s))(p_i \circ A'(t))p_i \circ (CONN_{G'}(s, t)) \leq (p_i \circ A(s))(p_i \circ A(t))p_i \circ CONN_G(s, t)$ because $p_i \circ A'(s) \leq p_i \circ A(s)$. This implies $\sum_{s, t \in V'} (p_i \circ A'(s))(p_i \circ A'(t))p_i \circ (CONN_{G'}(s, t)) \leq \sum_{s, t \in V'} (p_i \circ A(s))(p_i \circ A(t))p_i \circ CONN_G(s, t) \forall i$.

Hence, $p_i \circ CI_{mPF}(G') \leq p_i \circ CI_{mPF}(G) \forall i$ i.e. $CI_{mPF}(G') \leq CI_{mPF}(G)$.

An mPFSG is a partial mPFG. Then the Theorem 3.3 holds for a mPFSG that is given in the next note.

Note 1. *For a mPFSG $G' = (V, A', B')$ of a mPFG $G = (V, A, B)$ $CI_{mPF} \leq CI_{mPF}(G)$.*

Theorem 3.4. *Let $G = (V, A, B)$ be a connected mPFG. If $G' = (V', A', B')$ is a mPFSG of G , where $V' = V - \{t\} \ t \in V$, then $p_i \circ CI_{mPF}(G') < p_i \circ CI_{mPF}(G) \forall i = 1, 2, \dots, m$.*

Proof. Suppose n is the total number of nodes in G i.e. $|V| = n$ and $s \in V$. Then, $V' = V - \{s\}$ which implies $|V'| = n - 1$. Next we consider $V = \{s = s_1, s_2, \dots, s_n\}$. So, $V' = \{s_2, \dots, s_n\}$. Again G' is a mPFSG then G' is also a partial mPFSG of G . Then $p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G') + \sum_{j=2}^n (p_i \circ A(s_1))(p_i \circ A(s_j))(p_i \circ CONN_G(s_1, s_j)) \forall i = 1, 2, \dots, m$.

From the above relation we easily say that $p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G')$ for all i .

Hence $CI_{mPF}(G) > CI_{mPF}(G')$.

4 Boundedness of connectivity index of m Polar fuzzy graph

In this section we described boundedness implies lower and upper boundary of connectivity index of a mPFG. Next, we introduced some Theorems on connectivity index of mPFG.

Theorem 4.1. Let n be the number of vertices in a connected m PFG $G = (V, A, B)$. If $G' = (V', A', B')$ is the complete m PFG spanned by the nodes of G , then $0 \leq p_i \circ CI_{mPF}(G) \leq p_i \circ CI_{mPF}(G') \forall i = 1, 2, \dots, m$.

Proof. If G has only one vertex. Then also G' has one vertex. Then, $CONN_G(s, t) = 0 = CONN_{G'}(s, t)$ which implies $p_i \circ CONN_G(s, t) = 0 = p_i \circ CONN_{G'}(s, t) \forall i$ and $s, t \in V$.

So, $p_i \circ CI_{mPF}(G) = 0 = p_i \circ CI_{mPF}(G') \forall i$ which means $CI_{mPF}(G) = 0 = CI_{mPF}(G')$.

Let G contain more than one vertex. Then the complete m PFG $G' = (V', A', B')$ with $|V'| = n$ and $p_i \circ A'(s) = p_i \circ A(s) \forall i$ and $s \in V$ because G' is complete m PFG spanned by the node set of G .

Then for every edge (s, t) of G we have $p_i \circ B(s, t) \leq p_i \circ B'(s, t) \forall i$.

From the above relation, we say that $p_i \circ CONN_G(s, t) \leq p_i \circ CONN_{G'}(s, t) \forall i = 1, 2, \dots, m$.

$$\begin{aligned} & \text{So } \forall i (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s, t)) \leq (p_i \circ A'(s))(p_i \circ A'(t))(p_i \circ CONN_{G'}(s, t)) \\ \Rightarrow & \sum_{s, t \in V} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s, t)) \leq \sum_{s, t \in V} (p_i \circ A'(s))(p_i \circ A'(t))(p_i \circ CONN_{G'}(s, t)) \\ \Rightarrow & p_i \circ CI_{mPF}(G) \leq p_i \circ CI_{mPF}(G') \end{aligned}$$

Hence $CI_{mPF}(G') \geq CI_{mPF}(G)$.

Thus, we have $0 \leq CI_{mPF} \leq CI_{mPF}$.

Theorem 4.2. If G is complete m PFG with vertex set $V = \{u_1, u_2, \dots, u_n\}$. The membership value of each vertex u_j is u_j is $A(u_j) = (k_j^1, k_j^2, \dots, k_j^m)$ i.e. $p_i \circ A(u_j) = k_j^i$ for all $i = 1, 2, \dots, n$, where $k_1^i \leq k_2^i \leq \dots \leq k_n^i$ for all i . Then $p_i \circ CI_{mPF}(G) = \sum_{s=1}^{n-1} (k_s^i)^2 \sum_{j=s-1}^{n-2} (k_{j+2}^i) \forall i = 1, 2, \dots, m$.

proof. Let u_1 be an node in G . Here $p_i \circ A(u_1) = k_1^i \forall i$.

Again we know, G is complete m PFG, then $p_i \circ CONN_G(u_s, u_j) = p_i \circ B(u_s, u_j) \forall i$ and for every $u_s, u_j \in V$.

Thus $p_i \circ B(u_1, u_j) = k_1^j \forall i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

So, $(p_i \circ A(u_1))(p_i \circ A(u_j))(p_i \circ CONN_G(u_1, u_j)) = k_1^i k_j^i k_1^i = (k_1^i)^2 k_j^i \forall i$ and $j = 2, 3, \dots, n$

This implies $\sum_{j=2}^n (p_i \circ A(u_1))(p_i \circ A(u_j))(p_i \circ CONN_G(u_1, u_j)) = \sum_{j=2}^n (k_1^i)^2 k_j^i$

Then, $\sum_{s=1}^n \sum_{j=2}^n (p_i \circ A(u_1))(p_i \circ A(u_j))(p_i \circ CONN_G(u_1, u_j)) = \sum_{s=1}^n \sum_{j=2}^n (k_1^i)^2 k_j^i \forall i$.

Then, $p_i \circ CI_{mPF}(G) = \sum_{s=1}^{n-1} (k_s^i)^2 \sum_{j=s-1}^{n-2} k_{j+1}^i \forall i = 1, 2, \dots, m$.

Example 4.3. Here G is a complete $3m$ PFG in Figure 3, where $V = \{s_1, s_2, s_3, s_4\}$.

Figure 3: An $mPFG$ G .

Here $A(u_j) = (k_j^1, k_j^2, k_j^3)$ for $j = 1, 2, 3, 4$.

$$A(u_1) = (k_1^1, k_1^2, k_1^3) = (0.1, 0.4, 0.2)$$

$$A(u_2) = (k_2^1, k_2^2, k_2^3) = (0.2, 0.5, 0.3)$$

$$A(u_3) = (k_3^1, k_3^2, k_3^3) = (0.4, 0.7, 0.5)$$

$$A(u_4) = (k_4^1, k_4^2, k_4^3) = (0.4, 0.7, 0.5).$$

Here we see that, $k_4^1 \geq k_3^1 \geq k_2^1 \geq k_1^1$, $k_4^2 \geq k_3^2 \geq k_2^2 \geq k_1^2$ and $k_4^3 \geq k_3^3 \geq k_2^3 \geq k_1^3$.

Now,

$$\begin{aligned}
& p_1 \circ CI_{mPF}(G) \\
&= \sum_{s=1}^4 (k_s^1)^2 \sum_{j=s-1}^{n-2} k_{j+2}^1 \\
&= (k_1^1)^2 k_2^1 + (k_1^1)^2 k_3^1 + (k_1^1)^2 k_4^1 + (k_2^1)^2 k_3^1 + (k_2^1)^2 k_4^1 + (k_3^1)^2 k_4^1 \\
&= (0.1)^2(0.2) + (0.1)^2(0.4) + (0.1)^2(0.4) + (0.2)^2(0.4) + (0.2)^2(0.4) + (0.4)^2(0.4) \\
&= 0.106.
\end{aligned}$$

$$\begin{aligned}
& p_2 \circ CI_{mPF}(G) \\
&= \sum_{s=1}^4 (k_s^2)^2 \sum_{j=s-1}^{n-2} k_{j+2}^2 \\
&= (k_1^2)^2 k_2^2 + (k_1^2)^2 k_3^2 + (k_1^2)^2 k_4^2 + (k_2^2)^2 k_3^2 + (k_2^2)^2 k_4^2 + (k_3^2)^2 k_4^2 \\
&= (0.4)^2(0.5) + (0.4)^2(0.7) + (0.4)^2(0.7) + (0.5)^2(0.7) + (0.5)^2(0.7) + (0.7)^2(0.7) \\
&= 0.997.
\end{aligned}$$

$$\begin{aligned}
& p_3 \circ CI_{mPF}(G) \\
= & \sum_{s=1}^4 (k_s^2)^3 \sum_{j=s-1}^{n-2} k_{j+2}^3 \\
= & (k_1^3)^2 k_2^3 + (k_1^3)^2 k_3^3 + (k_1^3)^2 k_4^3 + (k_2^3)^2 k_3^3 + (k_2^3)^2 k_4^3 + (k_3^3)^2 k_4^3 \\
= & (0.2)^2(0.3) + (0.2)^2(0.5) + (0.2)^2(0.5) + (0.3)^2(0.5) + (0.3)^2(0.5) + (0.5)^2(0.5) \\
= & 0.262.
\end{aligned}$$

Therefore, $CI_{mPF}(G) = (0.106, 0.997, 0.262)$.

5 Find connectivity index of edge deleted m -polar fuzzy sub-graph

In a m PFG, if a vertex delete from G then the connectivity index reduces [from Theorem]. But if we remove an edge from G , it could decrease or stay the same connectivity index. In this section we will present the connectivity index of edge deleted m PFG.

Next, we find out the connectivity index of edge deleted m PFSG of m PFG using the following example.

Example 5.1. Consider an m PFG G (see in fig 4). $G' = G - (s_1, s_5)$, $G'\iota = G - (s_3, s_4)$ be two edge deleted m PFSG of G . Here membership value of all vertices is $(1, 1, 1)$.

Here,

$$\begin{aligned}
CI_{mPF}(G) &= (p_1 \circ CI_{mPF}(G), p_2 \circ CI_{mPF}(G), p_3 \circ CI_{mPF}(G)) \\
&= (7.5, 6.5, 4.5).
\end{aligned}$$

$$\begin{aligned}
CI_{mPF}(G') &= (p_1 \circ CI_{mPF}(G'), p_2 \circ CI_{mPF}(G'), p_3 \circ CI_{mPF}(G')) \\
&= (7.5, 6.5, 4.5).
\end{aligned}$$

$$\begin{aligned}
CI_{mPF}(G'\iota) &= (p_1 \circ CI_{mPF}(G'\iota), p_2 \circ CI_{mPF}(G'\iota), p_3 \circ CI_{mPF}(G'\iota)) \\
&= (7.3, 6.3, 4.3).
\end{aligned}$$

Figure 4: A Connected 3PF graph G with its $mPFSG$ G' and G'' .

So, Here we see that $\forall i = 1, 2, 3, p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G')$ for the arc deleted $mPFSG$ G' of G .

Again, $\forall i, p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G'')$ for the arc deleted $mPFSG$ G'' of G .

Theorem 5.2. Let $G' = G - (a, b)$ be a mPF subgraph of $mPFG$ G where (a, b) be an edge in G . Then for all $i, p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G')$ iff (a, b) is a mPF bridge of G .

proof. Let (s, t) be a $mPFB$ of G . Using the concept of $mPFB$, there exist one edge $(s_1, t_1) \in G$ s.t. $p_i \circ CONN_G(s_1, t_1) > p_i \circ CONN_{G'}(s_1, t_1) \forall i$.

Then, $\forall i$

$$(p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ CONN_G(s_1, t_1)) > (p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ CONN_{G'}(s_1, t_1))$$

which implies,

$$\sum_{s_1, t_1 \in V} (p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ \text{CONN}_G(s_1, t_1)) > \sum_{s_1, t_1 \in V} (p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ \text{CONN}_{G'}(s_1, t_1))$$

Hance, $p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G'), \forall i$.

Conversely, suppose $p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G')$ for all i . Next we suppose that, the edge (s, t) is not a mPF bridge. Then for each pair of vertices (s_1, t_1) in G , we have, $p_i \circ \text{CONN}_G(s_1, t_1) \leq p_i \circ \text{CONN}'_G(a_1, b_1) \forall i$.

This implies

$$\sum_{s_1, t_1 \in V} (p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ \text{CONN}_G(s_1, t_1)) \leq \sum_{s_1, t_1 \in V} (p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ \text{CONN}_{G'}(s_1, t_1))$$

Hence, $p_i \circ CI_{mPF}(G) \leq p_i \circ CI_{mPF}(G'), \forall i$. This is a contradiction.

Hence (s, t) is $mPFB$ of G .

Example 5.3. In the above example, $p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G'), \forall i$. So the edge (s_3, s_4) is a $mPFB$ of G .

Theorem 5.4. Let $G' = G - (s, t)$ be a spanning $mPFSG$ of $mPFG$ G , where (s, t) is an edge of G . Then G is a mPF tree iff $p_i \circ CI_{mPF} > p_i \circ CI_{mPF}(G')$.

proof. Suppose, G be a $mPFT$. Then the arcs of spanning $mPFSG$ G' of G are the $mPFB$ of G . So by Theorem 5.2, we have $p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G') \forall i$.

Conversely, let $p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G') \forall i$.

Then from the Theorem 5.2, we have the edge (s, t) is a $mPFB$ and also $p_i \circ B(s, t) = p_i \circ \text{CONN}_G(s, t) \forall i$. Therefore G is a $mPFT$.

Example 5.5. From Example 4, we see that $\forall i, p_i \circ CI_{mPF}(G) > p_i \circ CI_{G''}$. Then G' is a spanning of G . So, G is a $mPFT$.

Corollary 5.5.1. Let G be $mPFG$ and $G' = G - (s, t)$ be its $mPFSG$, where (s, t) is an arc of G . Then $p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G')$ iff $\forall i, p_i \circ B(x, y) \leq p_i \circ \text{CONN}_{G'}(s, t)$.

Example 5.6. From Example 4, we see that for the $mPFSG$ G' of G , $B(s_2, s_5) = (0.6, 0.5, 0.3)$ and $\text{CONN}_{G'}(s_2, s_5) = (0.7, 0.6, 0.4)$. This implies $B(s_2, s_5) < \text{CONN}_{G'}$. Also, $CI_{mPF}(G) = (7.5, 6.5, 4.5) = CI_{mPF}(G')$. So corollary 5.5.1 is satisfied.

Example 5.7. From Example 4, we see that for the mPFSG G'' of G . Here $B(s_3, s_4) = (0.8, 0.7, 0.5)$ and $CONN_{G''}(s_3, s_4) = (0.7, 0.6, 0.4)$. This implies that, $B(s_3, s_4) > CONN_{G''}(s_3, s_4)$. So $CI_{mPF}(G) = (7.5, 6.5, 4.5) \neq (7.3, 6.3, 4.3) = CI_{mPF}(G'')$.

Theorem 5.8. Let $G = (V, A, B)$ and $G' = (V', A', B')$ be two isomorphic mPFGs. Then $p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G') \forall i$.

proof. Here, Two isomorphic mPFGs are G and $G' = (V', A', B')$. Then, there a bi-jection mapping $h : G \rightarrow G'$ exists and so $\forall i, p_i \circ A(s) = p_i \circ A'(h(s)) \forall s \in V$ and $p_i \circ B(s, t) = p_i \circ B'(h(s), h(t)) \forall (s, t) \in \tilde{V}^2$. Since G is isomorphic to G' , so we have $CONN_G(s, t) = CONN'_G(h(s), h(t)), \forall s, t \in V$. Therefore,

$$\sum_{s, t \in V} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s, t)) = \sum_{h(s), h(t) \in V'} (p_i \circ A'(h(s)))(p_i \circ A'(h(t)))(p_i \circ CONN'_G(h(s), h(t))).$$

Hence, $CI_{mPF}(G) = CI_{mPF}(G')$.

Theorem 5.9. Let G be a mPFG. Then $CI_{mPF}(G^{r_1}) \geq CI_{mPF}(G^{r_2})$ where $0 \leq r_1 \leq r_2 \leq 1$.

proof. Let G be mPFG. Since $0 \leq r_1 \leq r_2 \leq 1$, G^{r_1} and G^{r_2} are the r_1 and r_2 cuts of G respectively. So G^{r_1} is a partial mPFSG of G^{r_2} . Then $CONN_{G^{r_1}}(s, t) \geq CONN_{G^{r_2}}(s, t)$. Then by Theorem , we have $CI_{mPF}(G^{r_1}) \geq CI_{mPF}(G^{r_2})$.

6 Average connectivity index of a mPFG

In this section, we explain average connectivity index of mPFG. In Theorem , we see that, if a node is removed from a connected mPFG, then its connectivity index reduces. But if a vertex is exclude from a connected mPFG then its average connectivity index may reduces or increases or same. Here certain types nodes mPFCRN, mPFCEN, mPFCNN are introduced and some properties on those nodes has been established.

Definition 6.1. The average connectivity index of mPFG G denoted by $ACI_{mPF}(G)$ is defined as $ACI_{mPF}(G) = (p_1 \circ ACI_{mPF}(G), p_2 \circ ACI_{mPF}(G), \dots, p_m \circ ACI_{mPF}(G)) = \frac{1}{\binom{n}{2}} (\sum_{s, t \in V} (p_1 \circ A(s))(p_1 \circ A(t))(p_1 \circ CONN_G(s, t)), \sum_{s, t \in V} (p_2 \circ A(s))(p_2 \circ A(t))(p_2 \circ CONN_G(s, t)), \dots, \sum_{s, t \in V} (p_m \circ A(s))(p_m \circ A(t))(p_m \circ CONN_G(s, t)))$

Here, $p_i \circ ACI_{mPF}(G)$ is the i th component of average mPF Connectivity index and $|V| = n$. Also, $0 \leq p_i \circ ACI_{mPF}(G) \leq 1 \forall i$.

Example 6.2. From Example 6.2.1, for the m PFG G in (see Fig. 2), then $ACI_{mPF}(G) = (\frac{20.3}{\binom{11}{2}}, \frac{14.7}{\binom{11}{2}}, \frac{11.9}{\binom{11}{2}}) = (0.369, 0.267, 0.216)$.

Definition 6.3. Let $G = (V, A, B)$ be a connected m PFG. A node v is said to be a

- i) m polar fuzzy Connectivity reducing node ($mPFCRN$) if $p_i \circ ACI_{mPF}(G-v) < p_i \circ ACI_{mPF}(G)$
 $\forall i = 1, 2, 3, \dots, m$.
- ii) m polar fuzzy Connectivity enhancing node ($mPFCEN$) if $p_i \circ ACI_{mPF}(G-v) > p_i \circ ACI_{mPF}(G)$
 $\forall i = 1, 2, 3, \dots, m$.
- iii) m polar fuzzy Connectivity neutral node ($mPFCNN$) if $p_i \circ ACI_{mPF}(G-v) = p_i \circ ACI_{mPF}(G)$
 $\forall i = 1, 2, 3, \dots, m$.

Figure 5: A Connected 3PFG G .

Example 6.4. For the m PFG in fig5. we have,

$$ACI_{mPFG}(G) = (\frac{0.844}{10}, \frac{.603}{10}, \frac{0.328}{10}) = (0.0844, 0.0603, 0.0328)$$

$$ACI_{mPFG}(G - s_1) = (\frac{0.59}{6}, \frac{.396}{6}, \frac{0.154}{6}) = (0.098, 0.066, 0.0256)$$

$$ACI_{mPFG}(G - s_2) = (\frac{0.484}{6}, \frac{.279}{6}, \frac{0.247}{6}) = (0.0806, 0.0465, 0.0412)$$

$$ACI_{mPFG}(G - s_3) = (\frac{0.444}{6}, \frac{.351}{6}, \frac{0.202}{6}) = (0.074, 0.0585, 0.0336)$$

$$ACI_{mPFG}(G - s_4) = (\frac{0.615}{6}, \frac{.303}{6}, \frac{0.122}{6}) = (0.102, 0.0505, 0.0203)$$

$$ACI_{mPFG}(G - s_5) = (\frac{0.509}{6}, \frac{.42}{6}, \frac{0.202}{6}) = (0.0848, 0.07, 0.0336)$$

Thus the node s_5 is $mPFCEN$ of G .

Theorem 6.5. Let $G = (V, A, B)$ be a connected $mPFG$ and $|V| = n \geq 3$ and $r_i = \frac{p_i \circ CI_{mPF}(G)}{p_i \circ CI_{mPF}(G-s)}$, $s \in V$. The node s is a

(i) $mPFCEN$ iff $r_i < \frac{n}{n-2} \forall i$.

(ii) $mPFCRN$ iff $r_i > \frac{n}{n-2} \forall i$.

(iii) $mPFCNN$ iff $r_i = \frac{n}{n-2} \forall i$.

Proof. (i) Suppose, s is a $mPFCEN$. Then, $\forall i$,

$$\begin{aligned} & p_i \circ ACI_{mPF}(G) < p_i \circ ACI_{mPF}(G - s) \\ \Rightarrow & \frac{p_i \circ ACI_{mPF}(G)}{\binom{n}{2}} < \frac{p_i \circ ACI_{mPF}(G - s)}{\binom{n-1}{2}} \\ \Rightarrow & \frac{p_i \circ ACI_{mPF}(G)}{p_i \circ ACI_{mPF}(G - s)} < \frac{\binom{n}{2}}{\binom{n-1}{2}} \\ \Rightarrow & r_i < \frac{n}{n-2}. \end{aligned}$$

Conversely, let $\forall i$,

$$\begin{aligned} & r_i < \frac{n}{n-2} \\ \Rightarrow & \frac{p_i \circ ACI_{mPF}(G)}{p_i \circ ACI_{mPF}(G - s)} < \frac{\binom{n}{2}}{\binom{n-1}{2}} \\ \Rightarrow & \frac{p_i \circ ACI_{mPF}(G)}{\binom{n}{2}} < \frac{p_i \circ ACI_{mPF}(G - s)}{\binom{n-1}{2}} \\ \Rightarrow & p_i \circ ACI_{mPF}(G) < p_i \circ ACI_{mPF}(G - s) \\ \Rightarrow & s \text{ is a } mPFCEN \text{ of } G. \end{aligned}$$

Similarly, (ii) and (iii) can be proved. □

Corollary 6.5.1. From the above theory, we get,

(i) If s is a $mPFCNN$ then $r_1 = r_2 = \dots = r_m$.

Theorem 6.6. Let G be a connected $mPFG$, where $|V| = n \geq 3$. Let $p_i \circ = \sum_{s \in V-y} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G)(s, t) \forall i$,

(i) $p_i \circ k < \frac{2}{n-2}$ iff t is an $mPFCEN$.

(ii) $p_i \circ k > \frac{2}{n-2}$ iff t is an $mPFCRN$.

(iii) $p_i \circ k = \frac{2}{n-2}$ iff t is a $mPFCNN$.

Proof. Let $p_i \circ k < \frac{2}{n-2}$. Now

$$\begin{aligned}
p_i \circ CI_{mPF}(G) &= p_i \circ CI_{mPF}(G - t) \\
&\quad + \sum_{s \in V-t} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s, t)) \\
\Rightarrow p_i \circ CI_{mPF}(G) &= p_i \circ CI_{mPF}(G - t) + p_i \circ k \\
\Rightarrow \frac{p_i \circ CI_{mPF}(G)}{\binom{n}{2}} &= \frac{p_i \circ CI_{mPF}(G - t)}{\binom{n}{2}} + \frac{p_i \circ k}{\binom{n}{2}} \\
\Rightarrow p_i \circ ACI_{mPF}(G) &< \frac{p_i \circ CI_{mPF}(G - t)}{\binom{n-1}{2}} \frac{n-2}{n} + \frac{\frac{2}{n-2}}{\binom{n}{2}} \\
\Rightarrow p_i \circ ACI_{mPF}(G) &< p_i \circ CI_{mPF}(G - t) - \frac{2}{n} [p_i \circ ACI_{mPF}(G - y) - \frac{2}{(n-1)(n-2)}] \\
\Rightarrow p_i \circ ACI_{mPF}(G) &< p_i \circ CI_{mPF}(G - t).
\end{aligned}$$

Similarly, (ii) and (iii) can be proved. $\square$

Definition 6.7. Let G be a $mPFG$.

(i) If G has at least one $mPFCEN$, then G is called *Connectivity enhancing m -polar fuzzy graph* ($CEmPFG$).

(ii) If G has no $mPFCRN$, then G is called *Connectivity reducing m -polar fuzzy graph* ($CRmPFG$).

(iii) If every nodes are of G $mPFCNN$, then G is called *neutral m -polar fuzzy graph* ($NmPFG$).

Example 6.8. For the $mPFG$, G in Example 6.4, we show that s_5 is an $mPFCEN$ of G . Hence G is a $CEmPFG$ and G is not contain any $mPFCRN$ vertex then G is also called $CRmPFG$ but G is not a $NmPFG$.

Theorem 6.9. For the number $(k_1, k_2, \dots, k_m)$ where k_i is a positive real number $\forall i$, there always a $mPFG$ G exists s.t. $p_i \circ CI_{mPF}(G) = k_i$, $|V| = n$

Proof. Let $p_i \circ A(s) = 1$, $\forall x \in V$ and $i = 1, 2, \dots, m$. Now we think of a path in G s.t. $p_i \circ B(s, t) = \frac{k_i}{\binom{n}{2}}$ $\forall$ edges $(s, t) \in G$. Then, $p_i \circ CI_{mPF}(G) = \sum_{s, t \in V} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s, t)) = \binom{n}{2} \frac{k_i}{\binom{n}{2}} = k_i$. $\square$

Theorem 6.10. Let G be a $mPFT$ and $G - x = G_1 \cup G_2$, where $G_1 = (V, A, B)$ and $G_2 = (V_2, A_2, B_2)$ are the connected components of $G - s$, where s is not an $mPFEN$. Let $p_i \circ K = \sum_{t \in V-s} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s, t))$. Then

- (i) $\sum_{k=1}^2 \sum_{t, r \in V_i} (p_i \circ A_k(s))(p_i \circ A_k(t))(p_i \circ CONN_G(s, t)) > (\frac{n-2}{2}(p_i \circ K))$ iff the node s is a $mPFCEN$.
- ii) $\sum_{k=1}^2 \sum_{t, r \in V_i} (p_i \circ A_k(s))(p_i \circ A_k(t))(p_i \circ CONN_G(s, t)) < (\frac{n-2}{2}(p_i \circ K))$ iff the node s is a $mPFCRN$.
- iii) $\sum_{k=1}^2 \sum_{t, r \in V_i} (p_i \circ A_k(s))(p_i \circ A_k(t))(p_i \circ CONN_G(s, t)) = (\frac{n-2}{2}(p_i \circ K))$ iff the node s is a $mPFCNN$.

Proof. Since G is a $mPFT$ and x is not an $mPFEN$ of G , $G - s$ is disconnected. Again, as G_1 and G_2 are connected components of $G - s$, So $G_1 \cup G_2$ and $G_1 \cap G_2 = \phi$. First assume that the node x is a $mPFCEN$. Then $p_i \circ ACI_{mPF}(G) < p_i \circ ACI_{mPF}(G - s) \forall i$. This implies that

$$\begin{aligned}
& \frac{1}{\binom{n}{2}} \left[\sum_{k=1}^2 \sum_{t, r \in V_k} (p_i \circ A_k(t))(p_i \circ A_k(r))(p_i \circ CONN_{G_i}(t, r)) \right] + (p_i \circ K) \\
& < \frac{1}{\binom{n-1}{2}} \left[\sum_{k=1}^2 \sum_{t, r \in V_k} (p_i \circ A_k(t))(p_i \circ A_k(r))(p_i \circ CONN_{G_i}(t, r)) \right] \\
& \Rightarrow \sum_{k=1}^2 \sum_{t, r \in V_k} (p_i \circ A_k(t))(p_i \circ A_k(r))(p_i \circ CONN_{G_i}(t, r)) + (p_i \circ K) \\
& < \frac{n}{n-2} \left[\sum_{k=1}^2 \sum_{t, r \in V_k} (p_i \circ A_k(t))(p_i \circ A_k(r))(p_i \circ CONN_{G_i}(t, r)) \right] \\
& \Rightarrow \sum_{k=1}^2 \sum_{t, r \in V_k} (p_i \circ A_k(t))(p_i \circ A_k(r))(p_i \circ CONN_{G_i}(t, r)) > \frac{n-2}{2}(p_i \circ K).
\end{aligned}$$

The converse part is shown when all these steps are reversed. Similarly, (ii) and (iii) can be proved. $\square$

7 Application

We are living in the age of the mobile world. Mobile is a wireless device. Other wireless technologies include tablets, laptops, and a variety of other devices. In our daily lives, we utilize mobile devices. Our mobile phones have become a necessity in our lives. The mobile network allows us to communicate with our friends and family members. A mobile communication system

connects wireless devices (such as an iphone, ipad, laptop, or netbook) to other wireless devices via mobile infrastructure. Images, audio, videos, texts, emails and other data can be shared via the mobile device.

Everywhere we go, we see mobile towers. This includes long antennas at top of the mobile towers. Transceivers are another term for mobile towers. Transceivers are situated inside specified geographical regions, which are referred to as cells. While most people are pleased with the increased number of mobile phone towers in their neighborhoods, the National Rights Commission (NHRC) is worried about the negative consequences of radiation released from these towers in residential areas. The negative impact of having argued. Now we will look at some of the negative effects on human life.

1. If you take expert advise, they will warn you that staying in a mobile tower within 50m is like staying in a microwave oven the whole day. Electromagnetic radiations are carcinogenic components that are unquestionably harmful to one's health.
2. Although cancer is the worst hazard to those who live near a mobile phone tower, also other health problems can occur. Sleeping difficulties, memory loss, tiredness, skin problems, headache, depression, hearing issues, joint aches and even cardiovascular illnesses are experienced by those in the region.
3. The birds have been most noticeably affected by mobile tower radiation. The number of birds such as sparrows has begun decreasing from the residential areas where mobile towers have been installed. The mobile towers are known to emit microwaves that harm birds' eggs and embryos, as they thin the skulls of chicks and eggshells.
4. Children have thinner skulls and are thus known to be impacted more by cell tower radiation. It is the same for an underdeveloped child bearing pregnant women. It is simpler to penetrate radiation and the impact might be extremely catastrophic if not controlled. Children these days are known to have less concentration power – one impact of putting cell towers close to schools and hospitals in residential areas.

So, mobile tower is very harmful for our environment. For this reason, we want to reduce the number of tower. Here we can give an idea for reducing the tower from a city (or country). Using the concept of Average connectivity index, we reduce the number of mobile tower.

Figure 6: A Connected 2PFG G .

Fig. 6 shows a model of the road network which is represented by a 2PF graph $G = (V, A, B)$, where vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_2v_5, v_2v_4, v_1v_3\}$. In this case, the vertex represents the location of the mobile tower in a city and edge represent the communication between corresponding two vertex. The membership value of each vertex is characterized by two criteria :{tower's ability, cost of unit} and The membership value of each edge is characterized by two criteria :{demand of better network accessibility, transfer cost}. If we consider the vertex v_1 , the first membership value of v_1 indicates the capability of the tower that is present at that location and second membership value of v_1 indicates the unit price. For the $mPFG$, G (see Fig.6) we have, $ACI_{mPFG}(G) = (0.08, 0.03)$, $ACI_{mPFG}(G - v_1) = (0.08, 0.03)$, $ACI_{mPFG}(G - v_2) = (0.09, 0.02)$, $ACI_{mPFG}(G - v_3) = (0.08, 0.04)$, $ACI_{mPFG}(G - v_4) = (0.07, 0.03)$, $ACI_{mPFG}(G - v_5) = (0.1, 0.02)$.

Here we see that $ACI_{mPFG}(G - v_4) < ACI_{mPFG}(G)$. From here it is clear that, if we eliminate

the mobile network that is present at the v_4 location, the average network speed in each device in the city remains the same. In this way we reduce the mobile tower in a city(or country).

8 Conclusion

In this paper, we have shown how to calculate the connection index in $mPFG$. The boundary of positive and negative connectivity index of a $mPFG$ are described. The characteristics of the connectivity index in vertex and edge deleted $mPFG$ have been studied. The average connectivity index in $mPFG$ and the especial nodes $mPFCEN$, $mPFCRN$, $mPFCNN$ are recount with their properties.

Conflict of interest

The authors declare that they have no conflict of interest.

Authors' contributions

All the authors contribute equally in this work.

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