Locality and evolution to equilibrium

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Physics Article

Keywords: quantum statistics, equilibrium, indistinguishable particles

Posted Date: August 12th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-737859/v1>

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Locality and evolution to equilibrium

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Abstract

Quantum statistics and non-locality are deeply rooted in quantum mechanics and go beyond our intuition reflected in classical physics. Quantum statistics can be derived using statistical methods for indistinguishable particles - particles of quantum mechanics. Violation of strong locality - colloquially called *the ghostly action at a distance* - is one of the most amazing properties of nature derived from quantum mechanics. An intriguing question is whether the non-local evolution of indistinguishable particles is needed to reach the equilibrium state given by quantum statistics.

Motivated by the above and similar questions, we developed a simple framework that allows us to follow space-time evolution of assembly of particles. It is based on a discrete-time Markov chain on countable space for indistinguishable particles. We summarise well-known and introduced new constraints on the transition matrix that grant space-time symmetries, locality of particle-transport, strong locality, and equilibrium state. Then, within the framework, several important cases are considered.

First, we show that the simplest transition matrix leads to equilibrium but violates particle transport and strong localities. Furthermore, we construct a simple matrix that leads to equilibrium obeying particle-transport locality and violating strong locality. This resembles the properties of quantum mechanics. Finally, we demonstrate that it is also possible to reach equilibrium by obeying both particle-transport and strong localities. Thus, within this framework, the violation of a strong locality is not needed to reach the equilibrium of indistinguishable particles. However, to obey strong locality, a complex structure of the transition matrix is needed. In addition, we comment on distinguishable particles and, in particular, show that their evolution seen by an observer blind to particle differences may look like the evolution of indistinguishable particles with the properties of quantum mechanics.

We hope that this work may help to study the relation between symmetries, localities and the evolution to equilibrium for indistinguishable and distinguishable particles.

PACS numbers:

Keywords:

1 The indistinguishable nature of particles and non-locality are deeply rooted in the world
 2 modelled by quantum mechanics. Concerning the former one, the spin-statistics theorem [1–
 3 3] relates the intrinsic spin of an indistinguishable particle to the quantum statistics it obeys.
 4 The integer spin particles follow the Bose-Einstein statistics, whereas the half-integer the Fermi-
 5 Dirac one. The latter property, violation of strong locality [4], indicates some sort of faster-
 6 than-light influence between remote events. The strong locality principle embeds a weaker but
 7 more intuitive property - matter, in general, conserved quantities cannot be transported faster
 8 than the speed of light in the vacuum. Recent measurements [5] give evidence of a violation
 9 of strong locality in nature, in the form of violation of the Bell's inequalities [6]. Referring to
 10 Fig. 1 (a), the experiments show correlations between an event E and events θ . even when
 11 possible correlations due to the common past of E and θ are removed. Despite the violation of
 12 strong locality, no signals can be sent faster than light velocity.

Figure 1: Sketches illustrating the principle of strong locality: (a) the sketch in continuous space-time and (b) the sketch within the Cell Model with six cells, $\Delta = 1$ and periodic boundary conditions, see text for details.

13 An intriguing question is whether the dynamics leading to the equilibrium of indistinguish-
 14 able particles requires violation of strong locality. In this paper, motivated by this and similar
 15 questions, we introduce a simple framework that addresses the relation between localities,

symmetries and equilibration. It is based on a discrete-time Markov chain on a finite set of microstates for indistinguishable particles. Then, within the framework several important cases are considered.

The paper is organized as follows. Section I introduces the Markov chain for indistinguishable particles. It also presents requirements on the transition matrix needed to obey space-time symmetries, particle-number transport and strong localities, as well as reach an equilibrium-steady state. Different types of transition matrices concerning localities and equilibrium are introduced and discussed in Sec. II. In Sec. III we consider the relationship between evaluations of indistinguishable and distinguishable particles and quantum mechanics. Summary presented in Sec. IV closes the paper.

I. DISCRETE-TIME MARKOV CHAIN FOR INDISTINGUISHABLE PARTICLES

The simplest way to address the locality of evolution and approach to equilibrium in a dynamic model is to consider a 1+1 dimensional discrete-time Markov chain with a conserved number of particles. The model is introduced in this section.

The *space* is assumed to be a set of V discrete cells arranged in a one-dimensional ring - the periodic boundary conditions are assumed. All cells are equivalent. For illustration a sketch for $V = 6$ is presented in Fig. 1 (b).

The total number of indistinguishable particles N is constant. For simplicity, only spin zero particles are assumed. Thus, a microstate of the system is defined entirely by particle multiplicities in cells, $(n_1, n_2, \dots, n_V)$. Thus the total number of microstates (see e.g. Ref. [7]) is

$$W(N, V) = \frac{(N + V - 1)!}{N!(V - 1)!} . \quad (1)$$

The system's evolution in *time* t is assumed to be discrete. Thus the evolution is given by sequential transitions between its microstates. The time steps at which microstates of the system are realized are equally distant. Transition probability from a microstate X at t to a microstate Y at $t + 1$ is assumed to depend only on the initial microstate X

$$B(X \rightarrow Y|X) , \tag{2}$$

and it is independent of microstates preceding X in time and the time step t . The above assumptions, together with the probability transition matrix (2) define a discrete-time Markov chain on a finite set of states [8]. As in Refs. [7, 9] we refer to this model as the Cell Model.

A. Conditions for space-time symmetries

Here conditions required for space-time symmetries within the Cell Model are introduced:

- (i) *Time-reversal symmetry*. The time-reversal symmetry is equivalent to detailed balance. That is, in the steady state frequency of a transition from X to Y , $\tilde{P}(X)B(X \rightarrow Y|X)$ is equal to the one of the reverse transition from Y to X , $\tilde{P}(Y)B(Y \rightarrow X|Y)$. Here $\tilde{P}(X)$ and $\tilde{P}(Y)$ denote probabilities to find the system in microstates X and Y , respectively.
- (ii) *Space-translation symmetry*. Given a space translation T , the transition probability from a microstate X to a microstate Y is the same as the transition probability from the microstate $T(X)$ to the microstate $T(Y)$.
- (iii) *Space-reversal symmetry*. Given any cell in the system, a reflection R of cell multiplicities with respect to the cell does not change transition probabilities. The transition probability from X to Y is the same as $R(X)$ to $R(Y)$.

The Cell Model with the space-time symmetries obeyed by the transition matrix is referred to as a *Symmetric Cell Model*.

B. Conditions for the equilibrium state

The *equilibrium state* is defined as a steady state of maximum entropy. Here we present the two conditions required to reach the equilibrium state within a Symmetric Cell Model. These are ergodicity and transition-matrix symmetry, as we will explain in this section.

The *steady state* refers to an ensemble of probabilities for the microstates to appear $\tilde{P}(X)$, which is left invariant by the probability transition matrix. In other words

$$0 \leq \tilde{P}(X) \leq 1 \quad \forall X, \quad (3)$$

$$\sum_X \tilde{P}(X) = 1, \quad (4)$$

$$\sum_X \tilde{P}(X) B(X \rightarrow Y|X) = \tilde{P}(Y) \quad \forall Y. \quad (5)$$

In the absence of other conditions than those defining the accessible states, as we will assume in this work, it corresponds to the ensemble in which all microstates appear with equal probability, $\tilde{P}(X) = \tilde{P}(Y)$ for any X and Y .

Ergodicity. The requirements for ergodicity in a finite Markov chain are [8]:

(a) *Irreducibility.* Each microstate is reachable from any other microstate by a sequence of transitions with non-vanishing probability.

(b) *Aperiodicity.* The maximum common divisor of the number of transitions of each sequence linking one microstate to itself is one.

Ergodicity is sufficient to ensure that the system asymptotically goes to a unique steady state. It also grants that time averages converge to the steady-state ensemble average in the limit of the infinite number of time steps.

Transition matrix symmetry. The equilibrium-steady state definition, $\tilde{P}(X) = \tilde{P}(Y)$, together with the time-reversal symmetry, $\tilde{P}(X) B(X \rightarrow Y|X) = \tilde{P}(Y) B(Y \rightarrow X|Y)$ implies the symmetry of the transition matrix:

$$B(X \rightarrow Y|X) = B(Y \rightarrow X|Y) . \quad (6)$$

We refer to a *Symmetric Cell Model* as *equilibrating*, if it is ergodic and it has the equilibrium-steady state. Equivalently, its probability transition matrix is ergodic and symmetric.

C. Conditions for transport and strong localities

Here the conditions needed for the locality of the system evolution within the Cell Model are introduced. These are the conditions for transport and strong locality. The *transport locality* denotes the locality of particle-number transport. The *strong locality* encompasses the transport one, and, in addition, it includes a more subtle locality of correlations between remote events.

Transport locality. Particle trajectories and velocities are defined for distinguishable particles and the transport locality requirement corresponds to the well-known requirement that particle velocities do not exceed the speed of light. Within the Cell Model, it can be formulated as follows. Let Δ be the maximum number of cells that particles can be displaced in a one-time step. Then a particle at the time step $t + 1$ has to be in a neighbourhood $\pm\Delta$ of its position at the time step t .

In the case of indistinguishable particles, the maximum speed of particle-number transport, coupled to the particle number conservation implies the following. During a single time-step the particle number in any interval of cells cannot be transported beyond an interval by Δ cells longer on the left and right. And it cannot be squeezed to an interval by Δ cells shorter on the left and right. This provides two transport-locality conditions:

$$\begin{aligned} \sum_{j=i}^{i+k} n_j^X &\leq \sum_{l=i-\Delta}^{i+k+\Delta} n_l^Y, \\ \sum_{l=i}^{i+k} n_l^Y &\leq \sum_{j=i-\Delta}^{i+k+\Delta} n_j^X, \end{aligned} \tag{7}$$

where $k = 0, 1, \dots$ and n_j^X, n_l^Y are particle numbers in cells j, l of initial (X) and final (Y) microstates, respectively.

Strong locality. According to the strong locality principle, the probability of an event E at time $t + 1$ can be influenced only by events within its past light cone. Thus E has to be independent of events outside its light cone when possible correlations due to the common past are removed. Referring to Fig. 1, given an initial configuration at the time-step t , any event E at the time-step $t + 1$ can depend only on the configurations of the system at time t , within the same cell,

and in the $\pm\Delta$ neighbours (the dark orange line in the plot (a)). The event E (for example particle number in cell "2" at $t + 1$) is independent of all events θ in gray areas (cells)

$$P(E \mid (F, \theta)) = P(E \mid F) \quad (8)$$

for all possible events F and θ . Note that events in the yellow regions can be correlated with E because possible common-past correlations are not removed.

Violation of strong locality can be quantified by introducing the measure:

$$\mathcal{V}_{SL} \equiv \sum_E \sum_F \sum_{\theta} \left(P(E \mid (F, \theta)) - \overline{P(E \mid (F, \theta))} \right)^2, \quad (9)$$

where $\overline{P(E \mid (F, \theta))}$ is obtained by averaging $P(E \mid (F, \theta))$ over all possible θ for given E and F . Obviously, for an evolution obeying strong locality $\overline{P(E \mid (F, \theta))} = P(E \mid (F, \theta)) = P(E \mid F)$, and thus $\mathcal{V}_{SL} = 0$.

Figure 1 (b) show an example sketch for the Cell Model with $V = 6$ cells and $\Delta = 1$. In this case the test of strong locality can be conducted by setting $E = n_2^{(t+1)}$, $F = (n_1^{(t)}, n_2^{(t)}, n_3^{(t)})$ and $\theta = (n_5^{(t+1)}, n_4^{(t)}, n_5^{(t)}, n_6^{(t)})$.

II. DIFFERENT TYPES OF TRANSITION MATRICES

This section discusses four types of transition matrices which satisfy the assumptions of the Symmetric Cell Model but differ with respect of equilibrium (EQ), transport locality (TL) and strong locality (SL). They are labelled as $[\text{EQ} \pm \text{TL} \pm \text{SL} \pm]$, where "+" indicates that a matrix has given property and "-" that it does not have it. For simplicity, we assume $\Delta = 1$ when considering locality conditions, see Sec. I C. To characterise the complexity of the transition matrix, its information entropy is calculated as

$$H[B] \equiv - \sum_{k=1}^K \mathbf{P}(B_k) \cdot \ln(\mathbf{P}(B_k)), \quad (10)$$

120 where $\mathbf{P}(B_k)$ is a normalised to unity number of transition-matrix elements with $B(X \rightarrow$
 121 $Y|X) = B_k$ and the summation runs over all K non-vanishing values of the transition-matrix
 122 elements. By definition $H[B]$ ranges between 0 and $\ln(W^2)$, where W is the total number of
 123 microstates given by Eq. (1).

Figure 2: Examples of transition matrices of the Symmetric Cell Model for four indistinguishable particles in six cells. (a): The matrix which leads to equilibrium violating transport and strong locality. (b) The matrix which leads to equilibrium obeying transport locality and violating strong locality. (c): The matrix which leads to non-equilibrium steady state obeying transport and strong locality. (d): The matrix which leads to equilibrium obeying transport and strong locality.

Microstates are ordered according to their sequential number $S(X)$ defined as a position of the microstate in a vector of “human-friendly” microstates labels calculated as $L(X) = \sum_{i=1}^V n_{V-i} \cdot 10^{i-1}$. The colour scale indicates values of transition probabilities, with the white colour denoting zero probability. Note different colour scales adopted to underline qualitative differences between the matrices.

A. Transition-matrix type [EQ+ TL- SL-]

The simplest transition matrix is the one which has all elements equal:

$$B(X \rightarrow Y|X) = 1/W , \quad (11)$$

where W is the total number of microstates given by Eq. (1). The matrix has no free parameters, and its information entropy is at minimum, $H[B] = 0$.

The matrix is ergodic because the transition probability between any two microstates in a single time-step is non-zero. By construction it is symmetric $B(X \rightarrow Y|X) = B(Y \rightarrow X|Y)$. Thus, it corresponds to an equilibrating Symmetric Cell Model. The matrix violates transport locality. It allows for the transfer of any number of particles by an arbitrarily large distance in a single time step. Consequently, it violates strong locality, $\mathcal{V}_{SL} \approx 0.011$. In summary, the transition matrix defined by Eq. (11) is of the type [EQ+ TL- SL-]. An example matrix of this type calculated for $V = 6$ and $N = 4$ is shown in Fig. 2 (*top-left*).

135 B. Transition-matrix type [EQ+ TL+ SL-]

136 The simplest matrix leading to equilibrium and obeying transport locality is constructed as
 137 follows. All off diagonal elements corresponding to transport-local transitions are equal and
 138 non-zero. Elements for transitions violating transport locality (teleportation) are set to zero.
 139 Then diagonal elements are given by unitarity. This procedure leaves one free parameter.

140 The matrix is ergodic; see Methods. By construction, the matrix is symmetric. Thus, it
 141 corresponds to an equilibrating Symmetric Cell Model.

142 By definition, the matrix obeys transport locality. However, it violates strong locality. This
 143 is because the number of possible transport-local transitions depends on the initial microstate
 144 X .

145 In summary, the transition matrix defined above is of the type [EQ+ TL+ SL-]. We note
 146 that these are properties of quantum mechanics. An example matrix of this type calculated for
 147 $V = 6$ and $N = 4$ is shown in Fig. 2 (*top-right*). Here the free parameter was selected to get a
 148 minimal value of $\mathcal{V}_{SL} \approx 0.010$. The information entropy of this matrix is ≈ 0.67 .

149 C. Transition-matrix type [EQ- TL+ SL+]

150 The simplest transition matrix, which obeys transport and strong locality, can be constructed
 151 as follows. One assumes that during a transition, the number of particles in the cell i at t is
 152 redistributed in the cells $i - 1, i, i + 1$ at $t + 1$, such as that all the possible configurations
 153 have an equal probability. This redistribution is performed for all cells of an initial microstate
 154 independently. Then the final microstate is constructed by summing contributions to each cell
 155 of the final microstate from all cells of the initial microstate. The matrix has no free parameters.

156 By construction - the particle number is redistributed locally and independently cell-by-cell
 157 - the matrix obeys transport and strong locality, $\mathcal{V}_{SL} = 0$.

158 It can be shown that the transition matrix is ergodic. However, it is not symmetric despite
 159 being time-reversible. The rather technical, analytic proof for that is presented in Methods.
 160 Thus, the steady state is not the equilibrium one.

161 In summary, the transition matrix defined above is of the type [EQ- TL+ SL+]. An

example matrix of this type calculated for $V = 6$ and $N = 4$ is shown in Fig. 2 (*bottom-left*).
The matrix-information entropy is $H[B] \approx 1$.

D. Transition-matrix type [EQ+ TL+ SL+]

Here, we show that it is possible to construct a transition matrix that leads to equilibrium as well as obeys transport and strong locality. The construction utilises the idea introduced in Sec. II C of independent cell-by-cell redistribution of particles from a given cell into itself and two closest neighbour cells. However, instead of postulating equal probabilities of all redistributions, one assumes that probabilities of redistributions can be different, and they depend only on the initial number of particles in a given cell. Then the final microstate is constructed as a sum of the contributions from all cells. By construction, the evolution obeys transport and strong localities, $\mathcal{V}_{SL} = 0$.

One can find probabilities for the redistributions that result in an ergodic and symmetric transition matrix if the symmetry is ensured order by order. In this way, all the probabilities except the ones for n particles to move together are constrained recursively by the fixed set of probabilities for $n - 1, n - 2 \dots 1$ particles to move. Thus, the matrix that obeys strong locality and leads to equilibrium has a number of free parameters which increases with the number of particles in the system.

In Methods we present details of the procedure and calculate several redistribution probabilities explicitly. We give a general recursive formula for the probabilities, but we do not prove that it always remains positive for an arbitrarily large N . Therefore, for any N larger than the one computed in Methods, one needs to verify that higher-order probabilities are positive. If the above is true, the constructed transition matrix is a well-defined transition matrix. It is symmetric and ergodic. Thus it corresponds to an equilibrating Symmetric Cell Model. Moreover, it obeys transport and strong localities. We note that a matrix of this type can be easily constructed for distinguishable particles - the simplest model assumes that particles can move by at most one cell (transport locality) independently of each other (strong locality).

In summary, the transition matrix defined above is of the type [EQ+ TL+ SL+]. An example matrix of this type calculated for $V = 6$ and $N = 4$ is shown in Fig. 2 (*top-right*). The

190 matrix-information entropy is $H[B] \approx 1.84$.

191 III. DISTINGUISHABLE PARTICLES

192 Needless to say, that any evolution of distinguishable particles looks like an evolution of
 193 indistinguishable particles for an observer blind to particle differences. In this section, we will
 194 first show that, given a transition matrix for indistinguishable particles, $B(X \rightarrow Y|X)$, it is
 195 possible to construct a transition matrix for distinguishable particles, $B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X})$, that goes
 196 to a steady state resembling the equilibrium-steady state for indistinguishable particles. Here $\mathcal{X}$
 197 and $\mathcal{Y}$ denote initial and final microstates of distinguishable particles. In particular, the evolu-
 198 tion of distinguishable particles given by $B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X})$ fulfills the requirements of a Symmetric
 199 Cell Model and preserves transport or strong locality even if the matrix for indistinguishable
 200 particles $B(X \rightarrow Y|X)$ does it. However, it does not lead to the equilibrium-steady state for
 201 distinguishable particles. We will further show that the resulting evolution of distinguishable
 202 particles cannot stem from a quantum model of these particles.

203 *Distinguishable and indistinguishable particles.* Let us denote as $\mathcal{P}(X \rightarrow Y)$ the num-
 204 ber of distinguishable-particle microstates $\mathcal{Y}$ having the same occupation numbers as Y ,
 205 $(n_1^{\mathcal{Y}}, n_2^{\mathcal{Y}}, \dots, n_V^{\mathcal{Y}}) = (n_1^Y, n_2^Y, \dots, n_V^Y)$, that can be reached by transport-local transitions from
 206 a microstate $\mathcal{X}$ having the same occupation numbers as X . We refer to $\mathcal{P}(X \rightarrow Y)$ as the
 207 number of permutations. Here we follow the usual convention for distinguishable particles -
 208 only permutations of particles in different cells are considered as different microstates. We can
 209 write then the probability transition matrix for distinguishable particles as

$$\begin{aligned} B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X}) &= 0 && \text{if } \mathcal{X} \rightarrow \mathcal{Y} \text{ violates transport locality,} \\ B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X}) &= \frac{B(X \rightarrow Y|X)}{\mathcal{P}(X \rightarrow Y)} && \text{otherwise,} \end{aligned} \tag{12}$$

210 where $X = [\mathcal{X}]$, and $Y = [\mathcal{Y}]$ are equivalence classes of the microstates for distinguishable
 211 particles having the same occupation numbers as $\mathcal{X}$ and $\mathcal{Y}$, respectively. It can be easily
 212 proven that $\mathcal{P}(X \rightarrow Y)$ depends only on the equivalence classes X and Y , and not on $\mathcal{X}$ and
 213 $\mathcal{Y}$ starting from microstate $\mathcal{X}' \in X$, and noting that exchanging particles in the microstate,

one can use the same transitions as for the original one. The initial and final microstates will be different, but the transport locality preserving transitions are equivalent.

By construction, the transition probabilities in (12) are normalised to one. The sum over the final microstates would correspond to the sum over the permutations $\mathcal{P}(X \rightarrow Y)$ for each $B(X \rightarrow Y|X)$, simplifying with the denominator, and then a sum over the different Y , which sums up to 1. By construction, if the original matrix is ergodic, the resulting one for distinguishable particles is also ergodic. If the original matrix is a Symmetric Cell Model for indistinguishable particles, the resulting one is a Symmetric Cell Model for distinguishable ones. Most of the proofs stem directly from the properties of the matrix for indistinguishable particles and the fact that in the distinguishable case, each permutation of particles does not change the transition probabilities. The proof that the construction preserves the time-reversibility of the evolution is less trivial, and we show it in detail below.

Let us denote as $\tilde{P}(X)$ the steady-state for the $B(X \rightarrow Y|X)$ matrix and calculate the total number of permutations in the equivalence class X as

$$\mathcal{P}(X = [\mathcal{X}]) = N! \prod_{i=1}^V \frac{1}{n_i^{\mathcal{X}}!}. \quad (13)$$

Then, it is possible to prove that

$$P(\mathcal{X}) = \frac{\tilde{P}(X)}{\mathcal{P}(X)} \quad (14)$$

is a steady-state for the distinguishable particles, and it fulfills the time-reversal condition

$$P(\mathcal{X})B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X}) = P(\mathcal{Y})B(\mathcal{Y} \rightarrow \mathcal{X}|\mathcal{Y}), \quad (15)$$

This, together with ergodicity, shows that $\tilde{P}(X)$ is the unique-steady state of the system.

Using (12) and (14), one gets

$$P(\mathcal{X})B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X}) = \frac{\tilde{P}(X)}{\mathcal{P}(X)} \frac{B(X \rightarrow Y|X)}{\mathcal{P}(X \rightarrow Y)} = \frac{\tilde{P}(X)B(X \rightarrow Y|X)}{\mathcal{P}(X)\mathcal{P}(X \rightarrow Y)} = \frac{\tilde{P}(Y)B(Y \rightarrow X|Y)}{\mathcal{P}(X)\mathcal{P}(X \rightarrow Y)}. \quad (16)$$

232 In the last equality we have used the time reversibility of the transition matrix for indistin-
 233 guishable particles. In order to complete the proof, one notes that

$$\mathcal{P}(X)\mathcal{P}(X \rightarrow Y) = \mathcal{P}(Y)\mathcal{P}(Y \rightarrow X). \quad (17)$$

234 The left-hand side in Eq. (17) is the number of permutations of X times the number of per-
 235 mutations within Y accessible by a single element $\mathcal{X} \in [\mathcal{X}] = X$ of the equivalence class. This
 236 exhausts all the permutations within Y since it is always possible to use the same transitions
 237 and change the permutation within X so that the final permutation in Y is the wanted one.
 238 Moreover, all transitions $\mathcal{X} \rightarrow \mathcal{Y}$ can be repeated in the opposite direction. Exhausting all the
 239 initial permutations in Y and final ones in X .

240 Finally, one gets

$$P(\mathcal{X})B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X}) = \frac{\tilde{P}(Y)B(Y \rightarrow X|Y)}{\mathcal{P}(X)\mathcal{P}(X \rightarrow Y)} = \frac{\tilde{P}(Y)B(Y \rightarrow X|Y)}{\mathcal{P}(Y)\mathcal{P}(Y \rightarrow X)} = P(\mathcal{Y})B(\mathcal{Y} \rightarrow \mathcal{X}|\mathcal{Y}). \quad (18)$$

241 This is the time-reversibility condition for the distinguishable particle case. From the unitarity
 242 of $B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X})$ follows that $P(\mathcal{X})$ is a steady state.

243 Thus we conclude that one can construct a transition matrix for distinguishable particles,
 244 leading to the evolution that for an observer blind to differences between particles obeys space-
 245 time symmetries and transport or strong locality, and leads to equilibrium. In particular,
 246 a matrix leading to an evolution compatible with quantum mechanics, like in Sec. IIB and
 247 Sec. IID, for the observer blind to particle differences can be obtained.

248 *Distinguishable particles and quantum mechanics.* We note that, in general,

$$\sum_{\mathcal{X}} B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X}) \neq 1 = \sum_{\mathcal{Y}} B(\mathcal{X} \rightarrow \mathcal{Y}|\mathcal{X}), \quad (19)$$

249 that is, the transpose of the probability transition matrix is not normalised to one. This is, in
 250 fact, enough to prove that it cannot stem from quantum dynamics. Before proving the latter,
 251 we check that (19) is correct. The simplest way to do this is to select a final microstate $\mathcal{Y}_0$
 252 as a microstate with all particles in the same cell. Thus $\mathcal{Y}_0$ is invariant under permutation of

particle. Therefore, independently on the initial state $\mathcal{X}$, because of the definition (12), one gets

$$B(\mathcal{X} \rightarrow \mathcal{Y}_0|\mathcal{X}) = \frac{B(X \rightarrow [\mathcal{Y}_0]|X)}{\mathcal{P}(X \rightarrow [\mathcal{Y}_0])} \equiv B(X \rightarrow [\mathcal{Y}_0]|X), \quad (20)$$

for any $\mathcal{X}$ that can go to $\mathcal{Y}_0$, since, no matter the initial microstate, there is only one permutation of particles in the final one.

By assumption, one has

$$\sum_X [B(X \rightarrow Y|X) = B(Y \rightarrow X|Y)] = 1, \quad \forall Y. \quad (21)$$

However, the sum over the $\mathcal{X}$ microstates repeats probabilities for each equivalence class $X = [\mathcal{X}]$ a number of times equal to the number of permutations $\mathcal{P}(X)$, therefore

$$\begin{aligned} \sum_{\mathcal{X}} B(\mathcal{X} \rightarrow \mathcal{Y}_0|\mathcal{X}) &= \sum_X \mathcal{P}(X) B(X \rightarrow [\mathcal{Y}_0]|X) = \\ &= \sum_X B(X \rightarrow [\mathcal{Y}_0]|X) + \sum_X (\mathcal{P}(X) - 1) B(X \rightarrow [\mathcal{Y}_0]|X) = \\ &= 1 + \sum_X (\mathcal{P}(X) - 1) B(X \rightarrow [\mathcal{Y}_0]|X) > 1, \end{aligned} \quad (22)$$

where the inequality follows from the fact that (at least some of) the microstates of distinguishable particles change under permutation of particles.

Regarding the constraints imposed by quantum mechanics, we understand that the set of microstates is a complete basis of the system. At each time step, a measurement is performed, and the system collapses in one of the microstates of the basis, with a probability given by the unitary-time evolution between the time t_i and t_{i+1} . The probability of starting from the microstate $|I\rangle$ and ending up in the microstate $|J\rangle$ is then equal to the square of the modulus of the amplitude

$$B(I \rightarrow J|I) = \left| \langle J|U(t_{i+1}, t_i)|I \rangle \right|^2 = \left| \langle J|e^{-i(t_{i+1}-t_i)\hat{H}}|I \rangle \right|^2, \quad (23)$$

for a generic Hamiltonian $\hat{H}$, and using the natural units. Assuming that $\delta t = t_{i+1} - t_i$ is independent of the iteration i , Eq. (23) defines a transition matrix for a Markov Chain. This matrix is normalized to 1, both summing over the initial microstates or the final ones. In other words, the transpose matrix is still a transition matrix

$$\begin{aligned}\sum_I B(I \rightarrow J|I) &= \sum_I \left| \langle J|U|I \rangle \right|^2 = \sum_I \langle J|U|I \rangle \langle I|U^\dagger|J \rangle = \langle J|UU^\dagger|J \rangle = \langle J|J \rangle = 1, \\ \sum_J B(I \rightarrow J|I) &= \sum_J \left| \langle J|U|I \rangle \right|^2 = \sum_J \langle I|U^\dagger|J \rangle \langle J|U|I \rangle = \langle I|U^\dagger U|I \rangle = \langle I|I \rangle = 1.\end{aligned}\tag{24}$$

Consequently, the evolution of distinguishable particles, which leads to the equilibrium-steady state of indistinguishable particles, cannot stem from a quantum model of distinguishable particles.

IV. SUMMARY

The paper introduces a simple framework that allows studying the time evolution of the assembly of indistinguishable particles. It is based on a discrete-time Markov chain on countable space. We summarize well-known and introduced new constraints on the transition matrix that grant space-time symmetries, particle-transport and strong localities, and equilibrium steady state.

Several types of evaluations which fulfil space-time symmetries are considered. The transition matrix with zero information entropy leads to the equilibrium-steady state by violating particle transport and strong localities. Furthermore, we construct system evolution that leads to an equilibrium-steady state obeying particle-transport locality and violating strong locality. This resembles the properties of quantum mechanics. Finally, we show that it is also possible to reach equilibrium by obeying both particle-transport and strong localities. Thus, within the framework, the violation of a strong locality is not required to reach the equilibrium of indistinguishable particles. However, to obey strong locality, a high information entropy of the transition matrix is needed.

290 Additionally, we show that one can define a transition matrix for distinguishable parti-
291 cles, leading to the evolution that for an observer blind to particle differences has “quantum-
292 mechanical” properties - obeys space-time symmetries and particle-transport locality violates
293 strong locality and leads to equilibrium.

294 We hope that this work opens new possibilities for the study of the relation between space-
295 time symmetries, particle-transport and strong localities and evolution to equilibrium for in-
296 distinguishable and distinguishable particles. In particular, future studies may address the role
297 of space-time symmetries and include processes of particle creation and annihilation.

Acknowledgments

We are thankful to F. Giacosa, St. Mrowczynski and A.A. Semenov for comments, and M. Gazdzicka and M. Prior for help in preparation of the manuscript. This work is partially supported by the Polish National Science Centre grant 2018/30/A/ST2/00226, 2016/21/D/ST2/01983, the German Research Foundation grant GA1480/8-1, Collaborative Research Center CRC-TR 211 project number 315477589 - TRR 211 and the Target Program of Fundamental Research of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine (N 0120U100857).

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